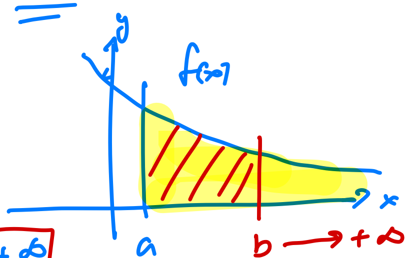
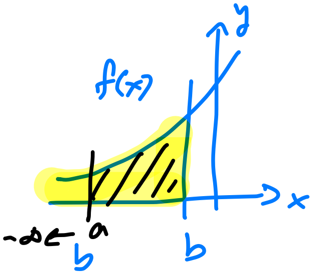
⇒ อินทิกรัลไม่ตรงแบบ:

- แบบ 1: ขอบเขตเป็น $\pm\infty$
- แบบ 2: $\int_a^b f(x)\,dx$ แล้วมี $c \in [a,b]$ ที่ $f(c) = \pm\infty$
- แบบ 3: แบบผสม ① + ②

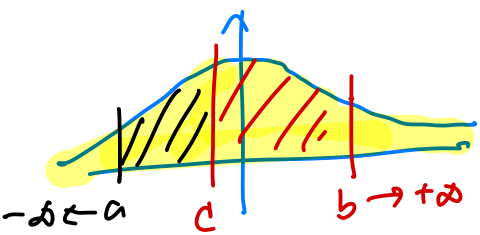
แบบ 1:



$$\int_a^{\boxed{+\infty}} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx$$



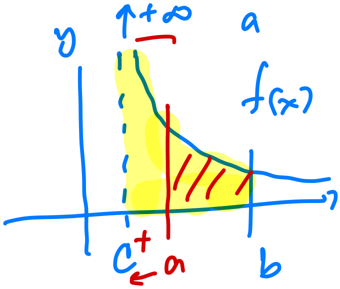
$$\int_{\boxed{-\infty}}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$



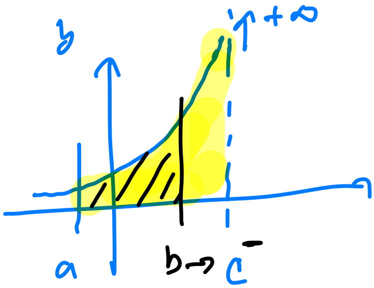
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx$$

$$= \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to+\infty} \int_c^b f(x)\,dx$$

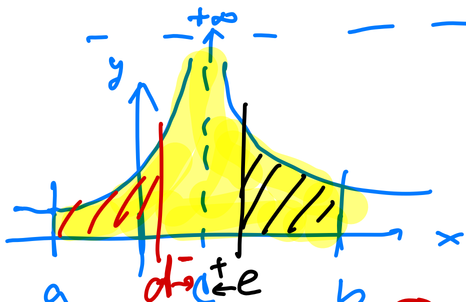
• **แบบที่ 2:** $\int_a^b f(x)\,dx$, $c \in [a,b]$ ที่ $f(c) = \pm\infty$



$$\int_c^b f(x)\,dx = \lim_{a\to c^+}\int_a^b f(x)\,dx$$



$$\int_a^c f(x)\,dx = \lim_{b\to c^-}\int_a^b f(x)\,dx$$

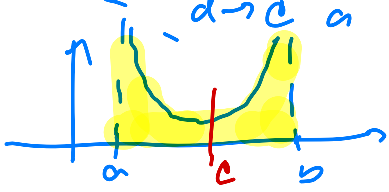


$\int_a^b f(x)\,dx$, $c \in (a,b)$ ที่ $f(c) = \pm\infty$

$$\Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{d\to c^-}\int_a^d f(x)\,dx + \lim_{e\to c^+}\int_e^b f(x)\,dx$$

(หรือ)

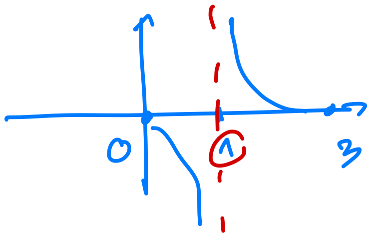


$$\Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Ex: $\displaystyle\int_0^3 \underbrace{\frac{1}{(x-1)^{2/3}}}_{f(x)}\,dx$ $\Rightarrow$ $x=1$ ทำให้ $f(x)=\boxed{\frac{1}{0}}=(\pm\infty)$

และ $1\in[0,3]$

ดังนั้น เป็นอินทิกรัลไม่ตรงแบบ.



$$\Rightarrow \int_0^3 \frac{1}{(x-1)^{2/3}}\,dx \quad , \left(f(1)=\pm\infty\right)$$

$$= \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\,dx$$

เขียนในรูปผลบวก เป็น

$$= \overbrace{\lim_{b\to1^-}\int_0^b \frac{1}{(x-1)^{2/3}}\,dx}^{1} + \overbrace{\lim_{a\to1^+}\int_a^3 \frac{1}{(x-1)^{2/3}}\,dx}^{2}$$

พิจารณา: $\displaystyle\int \frac{1}{(\underbrace{x-1}_{u})^{2/3}}\,dx = \frac{(x-1)^{-\frac{2}{3}+1}}{\left(-\frac{2}{3}+1\right)} + C$

$$= +3(x-1)^{\frac{1}{3}} + C$$

ส่วนที่ 1:

$$\lim_{b\to1^-}\int_0^b \frac{1}{(x-1)^{2/3}}\,dx = \lim_{b\to1^-}\left(+3(x-1)^{\frac{1}{3}}\right)\Big|_{x=0}^{x=b}$$

$$= \lim_{b\to1^-}\left[+3(b-1)^{\frac{1}{3}} - \left(+3(0-1)^{\frac{1}{3}}\right)\right]$$

$$= 0-(-3) = +3 \qquad ต$$

②:

$$\lim_{a\to 1^+}\int_a^3 \frac{1}{(x-1)^{2/3}}\,dx = \lim_{a\to 1^+}\left[3(x-1)^{\frac{1}{3}}\right]\Bigg|_{x=a}^{x=3}$$

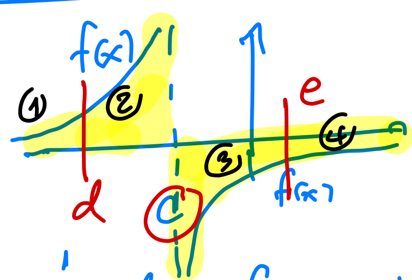
$$= \lim_{a\to 1^+}\left[\left(3(3-1)^{\frac{1}{3}}\right) - \left(3(a-1)^{\frac{1}{3}}\right)\right]$$

$$= 3\sqrt[3]{2} - 0 = 3\sqrt[3]{2} \quad \blacksquare$$

ดังนั้น $$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\,dx$$

$$= 3 + 3\sqrt[3]{2} \quad \blacksquare$$

$\Rightarrow$ **แบบผสม:** Ex:



ใน $\int_{-\infty}^{+\infty} f(x)\,dx$ กรณี $\lim_{x\to c^-} f(x) = +\infty$,

$\lim_{x\to c^+} f(x) = -\infty$.

$$\Rightarrow \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx$$

$$= \int_{-\infty}^{d} f(x)\,dx + \int_{d}^{c} f(x)\,dx + \int_{c}^{e} f(x)\,dx + \int_{e}^{+\infty} f(x)\,dx$$

เขียนในรูป lim

$$= \lim_{A \to -\infty} \int_{A}^{d} f(x)\,dx + \lim_{B \to c^-} \int_{d}^{B} f(x)\,dx$$

$$+ \lim_{C \to c^+} \int_{C}^{e} f(x)\,dx + \lim_{D \to +\infty} \int_{e}^{D} f(x)\,dx \quad \blacksquare$$

Ex: จงเขียนอินทิกรัลในรูปลิมิต ของ $\displaystyle\int_{-\infty}^{3} \underbrace{\frac{1}{(x-3)(x-4)}}_{f(x)}\,dx$

$f(x) = \pm\infty$ ที่ $x = 3, 4$

โดยที่ $3 \in (-\infty, 3]$, 4 ไม่อยู่ในโดเมน.

$$\int_{-\infty}^{3} \frac{1}{(x-3)(x-4)}\,dx = \int_{-\infty}^{0} \frac{1}{(x-3)(x-4)}\,dx + \int_{0}^{3} \frac{1}{(x-3)(x-4)}\,dx$$

$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{1}{(x-3)(x-4)}\,dx + \lim_{b \to 3^-} \int_{0}^{b} \frac{1}{(x-3)(x-4)}\,dx$$

⇒ แบบฝึกหัด Final ชุด 2 :

1.) จงหา อินทิเกรต:

1.1). $\displaystyle\int \frac{2}{\sqrt{x}} + \frac{5}{x^2} + \frac{6x}{x^2+1} + \frac{1}{x}\,dx$

$$= \frac{2x^{\left(-\frac{1}{2}+1\right)}}{\left(-\frac{1}{2}+1\right)} + \frac{5x^{(-2+1)}}{(-2+1)} + \int \frac{6x}{\underbrace{x^2+1}}\,dx + \ln|x| + C$$

$u = x^2+1$

$du = 2x\,dx$

$dx = \dfrac{du}{2x}$

$= 3\ln|x^2+1| + C$

1.2.) $\displaystyle\int \overbrace{(2x+18)}^{u}{}^5 + \overbrace{(2+18x)}^{v}{}^{1/5}\,dx$

$$= \frac{(2x+18)^6}{6\cdot 2} + \frac{(2+18x)^{\frac{1}{5}+1}}{\left(\frac{1}{5}+1\right)\cdot 18} + C$$

1.3). $\displaystyle\int e^{\sin x}\cos x - 2^{\cos x}\sin x\,dx$

$$= \int e^{\overbrace{\sin x}^{u=\sin x}}\cancel{\cos x}\,\underbrace{dx}_{\frac{du}{\cancel{\cos x}}} - \int 2^{\overbrace{\cos x}^{u=\cos x}}\cancel{\sin x}\,\underbrace{dx}_{\frac{du}{-\cancel{\sin x}}}$$

$$= e^{\sin x} + \frac{2^{\cos x}}{\ln 2} + C$$

1.4.) $\int \left[(1-\tan x)^{9/2} + \cot x \right] \sec^2 x \, dx$

เปลี่ยนรูป

$= \int \left[\left(1 - \underbrace{\tan x}_{u}\right)^{9/2} + \frac{1}{\underbrace{\tan x}_{u}} \right] \cancel{\sec^2 x} \, \underbrace{dx}_{= \frac{du}{\cancel{\sec^2 x}}}$

$u = \tan x$

$du = \sec^2 x \, dx$

$dx = \frac{du}{\sec^2 x}$

$= \int \left[(1-u)^{9/2} + \frac{1}{u} \right] du$

$= \frac{(1-u)^{\frac{9}{2}+1}}{(-1)\left(\frac{9}{2}+1\right)} + \ln|u| + \underline{\underline{C}}$ | แทน $u = \tan x$

(แทน $u = \tan x$)

$= \ldots$

$\boxed{}$

1.5.) $\int \frac{\cancel{4x}\ 2}{\sqrt{1 - (\underbrace{1-x^2}_{u})^2}} \, \underbrace{dx}_{\frac{du}{\cancel{-2x}}}$

$u = (1-x^2)$

$du = -2x \, dx$

$dx = \frac{du}{-2x}$

$= -2 \int \frac{1}{\sqrt{1-u^2}} \, du$

(!) ดูสูตร.

$= -2 \arcsin(u) + C$

แทน u กลับ.

$= -2 \arcsin(1-x^2) + \underline{\underline{C}}$ $\boxed{}$

1.6). $\int x^4 \operatorname{cosec}(5x^5+5) \cot(\underbrace{5x^5+5}_{u}) \, dx$

$u = 5x^5 + 5$

$du = 25x^4 \, dx$

$dx = \frac{du}{25x^4}$

$$= \int \cancel{x^4}\, \operatorname{cosec}(u)\cot(u)\, \frac{du}{25\cancel{x^4}}$$

$u = 5x^5 + 5$

$$= \frac{1}{25}\int \operatorname{cosec}(u)\cot(u)\,du = -\frac{1}{25}\operatorname{cosec}(u)\cot(u) + C$$

$= \dots$ $\boxed{}$

2.) จงหาอินทิกรัล โดยใช้ integration by parts

2.1) $$\int \frac{\overbrace{\ln(x^2+1)}^{u}\,\overbrace{dx}^{dv}}{2x^2}$$ | $\int u\,dv = uv - \int v\,du$

ให้ $u = \ln(x^2+1)$

$$\int dv = \int \frac{dx}{2x^2} \Rightarrow v = -\frac{1}{2x}$$

ใช้ by parts

$$= \ln(x^2+1)\cdot\left(-\frac{1}{2x}\right) - \int\left(-\frac{1}{\cancel{2x}}\right)\underbrace{d\ln(x^2+1)}_{\frac{1}{(x^2+1)}\cancel{(2x)}dx}$$

$$= -\frac{\ln(x^2+1)}{2x} + \int \frac{1}{(x^2+1)}\,dx$$

$$= -\frac{\ln(x^2+1)}{2x} + \arctan(x) + C \quad \boxed{}$$

2.2.) $\int \underbrace{(x^3+1)}_{u}\, \underbrace{x^2 e^{x^3} dx}_{dv}$

ให้ $u = (x^3+1)$

$$\int dv = \int x^2 e^{x^3} dx \Rightarrow v = \frac{e^{x^3}}{3}$$

by parts ; $(x^3+1)\left(\frac{e^{x^3}}{3}\right) - \int \frac{e^{x^3}}{3}\, d(\underbrace{x^3+1}_{u = x^3+1})$ $\quad x^3 \leftarrow u-1$

$x^3 = u-1$

$$= (x^3+1)\frac{e^{x^3}}{3} - \int \frac{e^{u} \cdot e^{-1}}{3}\, du$$

$$= (x^3+1)\frac{e^{x^3}}{3} - \frac{e^{-1}}{3} \cdot e^{(x^3)} + \underline{\underline{C}} \quad \blacksquare$$

3.) หาอินทิกรัล $\int \sin^3(2\pi x)\, dx$

$$u = \cos(2\pi x) \Rightarrow du = -2\pi \sin(2\pi x)\, dx$$

$$\Rightarrow dx = \frac{du}{-2\pi \sin(2\pi x)}$$

แทนค่า

$$\Rightarrow \int \underbrace{\sin^2(2\pi x)}_{1-\cos^2(2\pi x)} \cdot \cancel{\sin(2\pi x)} \frac{du}{-2\pi \cancel{\sin(2\pi x)}}$$

$$= \frac{1}{-2\pi}\int (1-u^2)\,du = \frac{1}{-2\pi}\left[u - \frac{u^3}{3}\right] + C$$

แทน $u = \cos(2\pi x)$

$$= \frac{1}{-2\pi}\left[\cos(2\pi x) - \frac{\cos(2\pi x)^3}{3}\right] + C$$

4.) จงหา $\int \underline{\cos(x)\cos(2x)}\cos(3x)\,dx$

$\sin(mx)\cos(nx) = \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)$

$\sin(mx)\sin(nx) = \frac{1}{2}(\cos(m-n)x - \cos(m+n)x)$

$\cos(mx)\cos(nx) = \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)$

จาก

$\cos(x)\cos(2x)$

$$= \frac{1}{2}\left[\cos(-1\cdot x) + \cos(3x)\right]$$

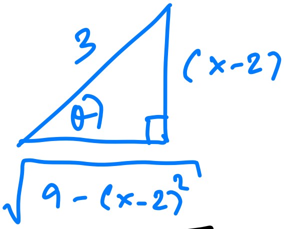
จะได้

$$\frac{1}{2}\int \cos(-x)\cos(3x) + \cos(3x)\cos(3x)\,dx$$

$$= \frac{1}{4}\int \cos(-4x) + \cos(2x) + \underbrace{\cos(0x)}_{=1} + \cos(6x)\,dx$$

$$= \frac{1}{4}\left[\frac{\sin(-4x)}{(-4)} + \frac{\sin(2x)}{2} + x + \frac{\sin(6x)}{6}\right] + C$$

5.) จงหา $\int \frac{x}{\sqrt{9-(x-2)^2}}\,dx$ โดยการแทนค่า f^n ตรีโกณมิติ.



$$\Rightarrow dx = 3\cos\theta \, d\theta$$

- $(x-2) = 3\sin\theta$
- $\sqrt{9-(x-2)^2} = 3\cos\theta$

$$\cos\theta = \frac{\sqrt{9-(x-2)^2}}{3}$$

[illegible]

$$\Rightarrow \int \left(\frac{3\sin\theta + 2}{\cancel{3\cos}\theta} \right) \cancel{3\cos\theta}\, d\theta$$

$$= -3\cos\theta + 2\theta + C$$

$$= -3\left(\frac{\sqrt{9-(x-2)^2}}{3} \right) + 2\arcsin\left(\frac{x-2}{3}\right) + \underline{C}$$