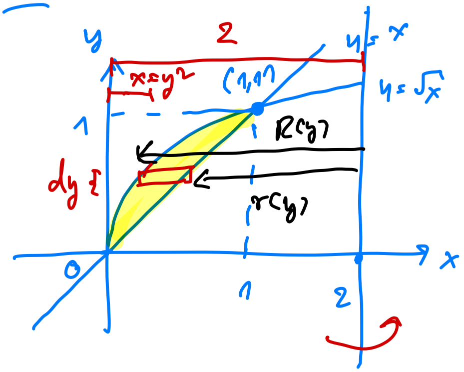
⇒ การบ้าน: แบบฝึกหัด 3.9.

2.3.) พท. ปิดล้อมด้วย $y=x$, $y=\sqrt{x}$ หมุนรอบเส้นตรง $x=2$



แบบ 1: Disk: ▯ ⊥ กับแกนหมุน

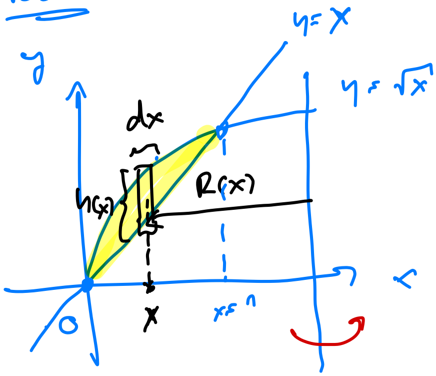
$$V=\int_{y=0}^{y=1}\pi\left(R^2(y)-r^2(y)\right)dy$$

- $R(y)=2-y^2$
- $r(y)=2-y$

ดังนั้น

$$\boxed{V=\int_{y=0}^{y=1}\pi\left[(2-y^2)^2-(2-y)^2\right]dy} \quad \text{— ①}$$

แบบ 2: แบบ เชลล์: ▯ // กับ แกนหมุน



$$V=\int_{x=0}^{x=1}2\pi R(x)h(x)\,dx$$

- $R(x)=2-x$
- $h(x)=\sqrt{x}-x$

ดังนั้น

$$\boxed{V=\int_{x=0}^{x=1}2\pi(2-x)(\sqrt{x}-x)\,dx} \quad \text{— ②}$$

วงแหวน: ①

$$V = \int_{y=0}^{y=1} \pi \left[(2-y^2)^2 - (2-y)^2 \right] dy$$

$$= \pi \int_{y=0}^{y=1} \left[\underbrace{(4 - 4y^2 + y^4) - (4 - 4y + y^2)}_{y^4 - 5y^2 + 4y} \right] dy$$

$$= \pi \left(\frac{y^5}{5} - \frac{5y^3}{3} + \frac{4y^2}{2} \right) \Bigg|_{y=0}^{y=1} = \pi \left(\frac{1}{5} - \frac{5}{3} + 2 \right)$$

$$= \pi \left(\frac{3 - 25 + 30}{15} \right) = \frac{8}{15}\pi \quad \square$$

แบบ ②

$$V = \int_{x=0}^{x=1} 2\pi (2-x)(\sqrt{x} - x) \, dx$$

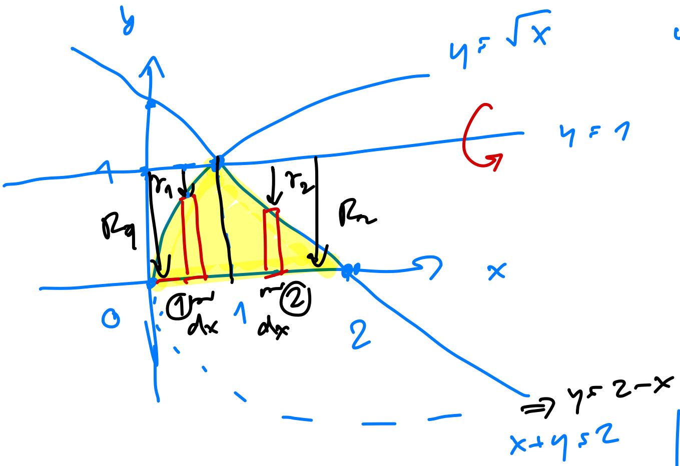
$$= 2\pi \int_{x=0}^{x=1} \left(2\sqrt{x} - 2x - x\sqrt{x} + x^2 \right) dx$$

$$= 2\pi \left(2 \cdot 2 \cdot \frac{x^{\frac{3}{2}}}{3} - \frac{2x^2}{2} - \frac{2 \cdot x^{\frac{5}{2}}}{5} + \frac{x^3}{3} \right) \Bigg|_{x=0}^{x=1}$$

$$= 2\pi \left(\frac{4}{3} - 1 - \frac{2}{5} + \frac{1}{3} \right) = 2\pi \left(\frac{20 - 15 - 6 + 5}{15} \right)$$

$$= \frac{8\pi}{15} \quad \square$$

2.4.) $y = \sqrt{x}$, $y = 0$, $x + y = 2$ หมุนรอบเส้น $y = 1$



หาจุดตัด. แกน $y = \sqrt{x}$

$\Rightarrow x + \sqrt{x} = 2$

$\sqrt{x} = 2 - x$

$x = 4 - 4x + x^2$

$\Rightarrow x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

โดยที่ $R_1(x) = 1$
$r_1(x) = 1 - \sqrt{x}$

$R_2(x) = 1$
$r_2(x) = 1 - (2 - x)$

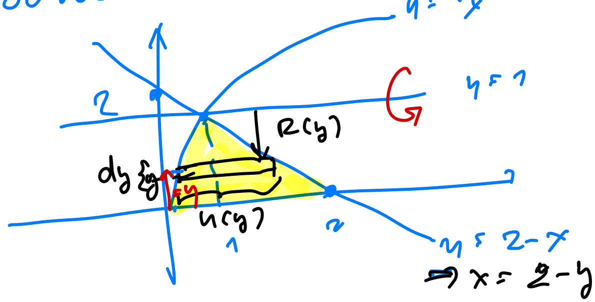
แบบ 1: ดิสก์ : ⫴ ⊥ กับแกนหมุน

$$V = V_1 + V_2$$

$$= \int_{x=0}^{x=1} \pi\left(R_1^2(x) - r_1^2(x)\right)dx + \int_{x=1}^{x=2} \pi\left(R_2^2(x) - r_2^2(x)\right)dx$$

$$V = \int_{x=0}^{x=1} \pi\left[1^2 - (1-\sqrt{x})^2\right]dx + \int_{x=1}^{x=2} \pi\left[1^2 - (1-(2-x))^2\right]dx$$

แบบเชลล์: ⫴ // กับแกนหมุน $y = \sqrt{x} \Rightarrow x = y^2$



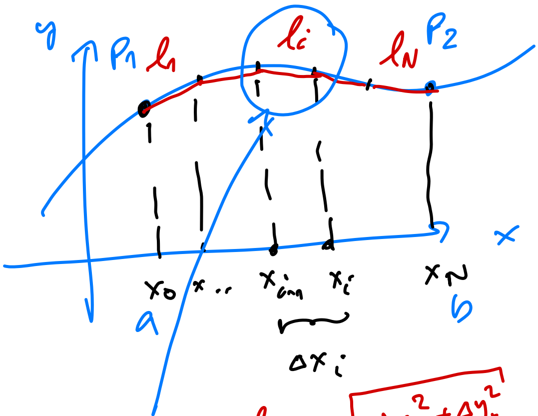
โดยที่ $R(y) = 1 - y$

$h(y) = (2 - y) - y^2$

$$V = \int_{y=0}^{y=1} 2\pi R(y) \cdot h(y)\, dy$$

$$\Rightarrow V = \int_{y=0}^{y=1} 2\pi (1-y)\left((2-y)-y^2\right) dy$$

$\Rightarrow$ ความยาวส่วนโค้ง.

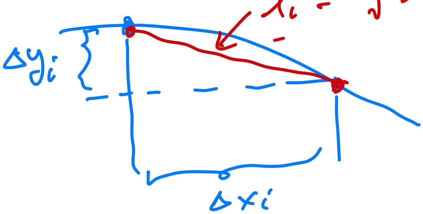


หาความยาวส่วนโค้ง จาก P_1 ถึง P_2 ของ $f(x)$

$$L_{P_1P_2} \approx \sum_{i=1}^{N} l_i$$

$$L_{P_1P_2} = \lim_{\substack{N\to\infty \\ (\Delta x_i \to 0)}} \sum_{i=1}^{N} l_i$$

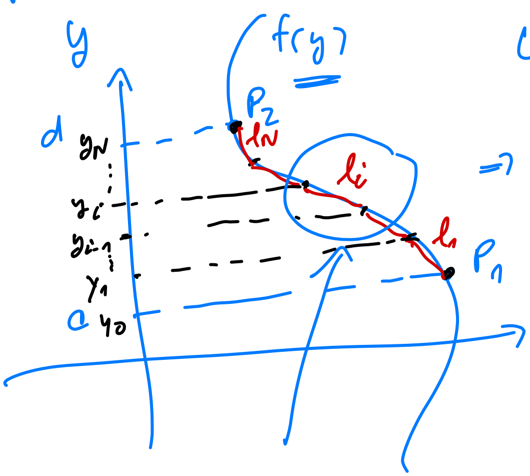
$$l_i = \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sqrt{1+\underbrace{\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}_{\to \frac{df}{dx}}}\ \Delta x_i$$



$$\Rightarrow L_{P_1P_2} = \lim_{\substack{N\to\infty \\ (\Delta x_i \to 0)}} \sum_{i=1}^{N} \sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\ \Delta x_i$$

$$\Rightarrow \boxed{L_{P_1P_2} = \int_{x=a}^{x=b} \sqrt{1+\left(\frac{df}{dx}\right)^2}\, dx}$$

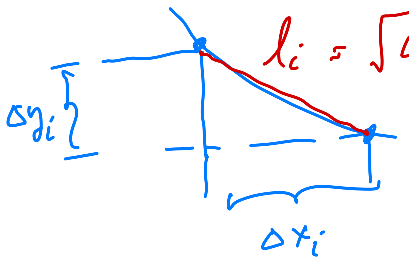
ในทำนองเดียวกันกับข้อ.



$$L_{P_1P_2} \approx \sum_{i=1}^{N} l_i$$

$$\Rightarrow L_{P_1P_2} = \lim_{\substack{N\to\infty \\ (\Delta y \to 0)}} \sum_{i=1}^{N} \sqrt{1+\left(\frac{\Delta x_i}{\Delta y_i}\right)^2}\, \Delta y_i$$

$$\Rightarrow \boxed{L_{P_1P_2} = \int_{y=c}^{y=d} \sqrt{1+\left(\frac{df}{dy}\right)^2}\, dy}$$



$$l_i = \sqrt{\Delta y_i^2 + \Delta x_i^2} = \sqrt{1+\left(\frac{\Delta x_i}{\Delta y_i}\right)^2}\, \Delta y_i$$

Ex: จงหา ความยาวส่วนโค้ง ของ เส้น $y = (4-x^{\frac{2}{3}})^{3/2} = f(x)$

จาก $x = 1$ ถึง $x = 8$

$$\Rightarrow L = \int_{x=1}^{x=8} \sqrt{1+\left(\frac{df}{dx}\right)^2}\, dx$$

$$\frac{df}{dx} = \frac{d}{dx}\left((4-x^{\frac{2}{3}})^{\frac{3}{2}}\right)$$

$$= \cancel{\tfrac{3}{2}}(4-x^{\frac{2}{3}})^{\frac{1}{2}} \cdot \left(-\cancel{\tfrac{2}{3}}x^{-\frac{1}{3}}\right)$$

$$= \frac{-(4-x^{\frac{2}{3}})^{\frac{1}{2}}}{x^{\frac{1}{3}}}$$

$$\Rightarrow L = \int_{x=1}^{x=8} \sqrt{1+\left(\frac{-(4-x^{\frac{2}{3}})^{\frac{1}{2}}}{x^{\frac{1}{3}}}\right)^2}\, dx$$

$$= \int_{x=1}^{x=8} \sqrt{1+\frac{(4-x^{\frac{2}{3}})}{x^{\frac{2}{3}}}}\, dx$$

$$= \int_{x=1}^{x=8} \sqrt{\frac{\cancel{x^{\frac{2}{3}}} + 4 - \cancel{x^{\frac{2}{3}}}}{x^{\frac{2}{3}}}}\, dx = \int_{x=1}^{x=8} 2 \cdot x^{-\frac{1}{3}}\, dx$$

$$= 2\, \frac{x^{-\frac{1}{3}+1}}{\left(-\frac{1}{3}+1\right)} \Bigg|_{x=1}^{x=8} = 3x^{\frac{2}{3}} \Big|_{x=1}^{x=8} = 12 - 3$$

$$= 9 \quad \blacksquare$$

<u>Ex:</u> $x = \underbrace{\frac{y^3}{6} + \frac{1}{2y}}_{f(y)}$ תחום $y = \frac{1}{2}$ עד $y = 2$

$$\Rightarrow L = \int_{y=\frac{1}{2}}^{y=2} \sqrt{1 + \left(\frac{df}{dy}\right)^2}\, dy$$

$$\frac{df}{dy} = \frac{d}{dy}\left(\frac{y^3}{6} + \frac{1}{2y}\right) = \frac{y^2}{2} - \frac{1}{2y^2}$$

$$= \int_{y=\frac{1}{2}}^{y=2} \sqrt{1 + \left(\frac{y^2}{2} - \frac{1}{2y^2}\right)^2}\, dy$$

$$= \int_{y=\frac{1}{2}}^{y=2} \sqrt{\frac{4}{4} + \frac{y^4}{4} - \frac{2 \cdot 1}{4} + \frac{1}{4y^4}} = \frac{1}{2}\int_{y=\frac{1}{2}}^{y=2} \sqrt{y^4 + 2 + \frac{1}{y^4}}$$

$$= \frac{1}{2}\int_{y=\frac{1}{2}}^{y=2} \sqrt{\left(y^2+\frac{1}{y^2}\right)^2}\,dy = \frac{1}{2}\int_{y=1}^{y=2}\left(y^2+\frac{1}{y^2}\right)dy.$$

$$= \frac{1}{2}\left(\frac{y^3}{3}-\frac{1}{y}\right)\Bigg|_{y=\frac{1}{2}}^{y=2} = \frac{1}{2}\left[\left(\frac{8}{3}-\frac{1}{2}\right)-\left(\frac{1}{8\cdot 3}-2\right)\right]$$

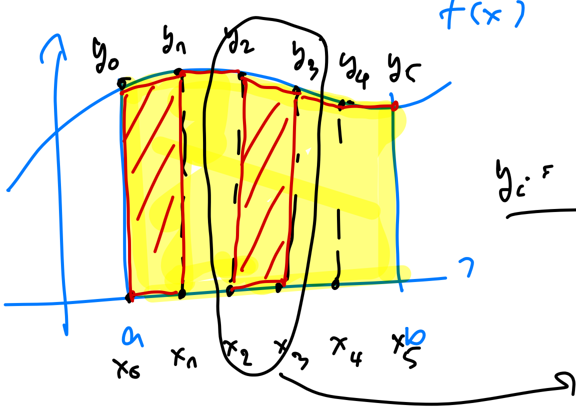
$$= \frac{1}{2}\left(\frac{16-3}{6}-\frac{1-48}{24}\right) = \frac{1}{2}\left(\frac{13\cdot 4}{24}-\frac{(-47)}{24}\right)$$

$$= \frac{1}{2}\left(\frac{52+47}{24}\right) = \frac{99}{48} = \frac{33}{16} \quad \blacksquare$$

⇒ การอินทิเกรตเชิงตัวเลข.

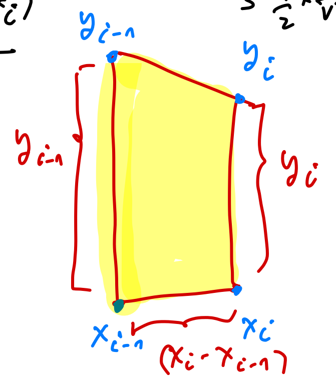
⇒ วิธีสี่เหลี่ยมคางหมู : (Trapezoidal method.)

$f(x)$ ⇒ $\int_a^b f(x)\,dx$



$y_i = f(x_i)$

พท. □ คางหมู $= \frac{1}{2}$ × สูง × ผลบวกความยาวด้านคู่ขนาน



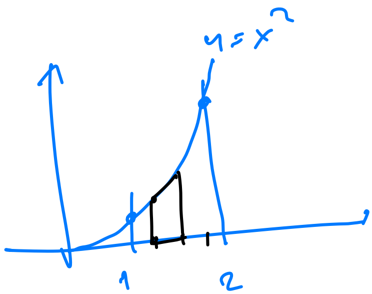
ถ้ากำหนดให้ $x_i - x_{i-1} = h$ ทุก i

เราได้ พท. □ คางหมู ที่ i เป็น $\frac{1}{2}\cdot h\left(y_{i-1}+y_i\right)$

ดังนั้น $A \approx \sum_{i=1}^{N} \square_i = \left(\frac{1}{2}h\left(\underline{y_0}+\underline{y_1}\right) + \frac{1}{2}h\left(\underline{y_1}+\underline{y_2}\right) + \dots + \frac{1}{2}h\left(y_{i-1}+y_i\right) + \dots \frac{1}{2}h\left(y_{N-1}+\underline{y_N}\right)\right)$

$$\boxed{A_{\text{คางหมู}} = \frac{h}{2}\left(y_0+2y_1+2y_2+\dots+2y_i+\dots+2y_{N-1}+y_N\right)}$$

Ex: หาประมาณ $\int_1^2 x^2\,dx$ โดยใช้กฎสี่เหลี่ยมคางหมู ที่มี $N=4$ ช่วง เท่ากัน.



$h=\frac{2-1}{4}=\frac{1}{4}$

$h=\frac{1}{4}$

$x_i = 1+ih$, $x_0=1$, $x_1=1+\frac{1}{4}=\frac{5}{4}$, $x_2=1+\frac{2}{4}=\frac{6}{4}$

$x_3=1+\frac{3}{4}=\frac{7}{4}$, $x_4=2$

$\Rightarrow$ แต่ $y_i = f(x_i)$ โดยที่ $f(x)=x^2$

$y_0=(1)^2$, $y_1=\left(\frac{5}{4}\right)^2$, $y_2=\left(\frac{6}{4}\right)^2$, $y_3=\left(\frac{7}{4}\right)^2$, $y_4=2^2$

$A_{\text{คางหมู}} = \frac{1}{4\cdot 2}\left(y_0+2y_1+2y_2+2y_3+y_4\right)$

(แทนค่า) $= \frac{1}{8}\left(1+2\cdot\left(\frac{5}{4}\right)^2+2\left(\frac{6}{4}\right)^2+2\cdot\left(\frac{7}{4}\right)^2+2^2\right)$ ☒