การบ้าน: แบบฝึกหัด 3.7

2.) $\displaystyle\int \frac{x^2+2x+1}{(x^2+1)^2}\,dx$ 3.) $\displaystyle\int \frac{9x^2-3x+1}{x^3-x^2}\,dx$

2.) $\displaystyle\int \frac{x^2+2x+1}{(x^2+1)^2}\,dx$

พิจารณา. ① เขียนในรูปเศษส่วนย่อย **

$$\frac{x^2+2x+1}{(x^2+1)^2} = \frac{A_1x+B_1}{(x^2+1)} + \frac{A_2x+B_2}{(x^2+1)^2} \quad \text{—} (*)$$

② หาค่า unknowns : A_1, B_1, A_2, B_2 มี 4 unknowns.

คูณ (*) ด้วย $(x^2+1)^2$ ทั้งสองข้าง จะได้

$$x^2+2x+1 = \underbrace{(A_1x+B_1)(x^2+1)}_{=(A_1x^3+A_1x+B_1x^2+B_1)} + A_2x+B_2$$

สรุป.

$$0\cdot x^3+1\cdot x^2+2x+1 = A_1x^3+B_1x^2+(A_1+A_2)x+(B_1+B_2)$$

① ② ③ ④

→ เทียบสปส.

①: $A_1 = 0$

②: $B_1 = 1$

③: $A_1 + A_2 = 2 \Rightarrow A_2 = 2 - A_1 \overset{①}{=} 2 - 0 = 2$

④: $B_1 + B_2 = 1 \Rightarrow B_2 = 1 - B_1 \overset{②}{=} 1 - 1 = 0$

ดังนั้น. $A_1 = 0$, $B_1 = 1$, $A_2 = 2$, $B_2 = 0$

ขั้น ③ อินทิเกรต

$$\int \frac{x^2+2x+1}{(x^2+1)^2}\,dx = \int \frac{0\cdot x + 1}{(x^2+1)} + \frac{2\cdot x + 0}{(x^2+1)^2}\,dx$$

$$= \int \frac{1}{x^2+1}\,dx + \int \frac{2x}{(\underbrace{x^2+1}_{u=x^2+1})^2}\,dx \Rightarrow \int \frac{\cancel{2x}}{u^2}\frac{du}{\cancel{2x}}$$

$\Rightarrow dx = \frac{du}{2x}$

$$= \arctan(x) + \frac{-1}{(x^2+1)} + C \quad \boxed{}$$

3.) $$\int \frac{9x^2-3x+1}{x^3-x^2}\,dx \overset{\text{จัดรูป.}}{=} \int \frac{9x^2-3x+1}{x^2(x-1)}\,dx$$

พิจารณา. $\boxed{1}$ เขียนในรูปเศษส่วนย่อย.

$$\frac{9x^2-3x+1}{\underbrace{x^2}_{①}\underbrace{(x-1)}_{②}} = \underbrace{\frac{A_1}{x} + \frac{A_2}{x^2}}_{①} + \underbrace{\frac{B_1}{(x-1)}}_{②} \quad —(*)$$

คูณ (*) ด้วย $x^2(x-1)$

$$9x^2-3x+1 = A_1x(x-1) + A_2(x-1) + B_1x^2$$

$\boxed{2}$ หา unknowns: A_1, A_2, B_1 3 ตัว.

แทนค่า:

$x=0: \Rightarrow 1 = 0 + A_2(0-1) + 0$

$\Rightarrow A_2 = -1$

$x=1: \Rightarrow 9-3+1 = 0+0+B_1 \cdot 1^2$

$\Rightarrow B_1 = 7$

แทน $x=-1$ และ $B_1=7$, $A_2=-1$ จะได้.

$$9\cdot(-1)^2-3(-1)+1 = A_1(-1)(-2) + \underbrace{A_2}_{(-1)}(-2) + \underbrace{B_1}_{7}(-1)^2$$

$$\Rightarrow 13 = 2A_1 + 9 \Rightarrow A_1 = \frac{13-9}{2} = 2$$

จะได้: $A_1 = 2$, $A_2 = -1$, $B_1 = 7$

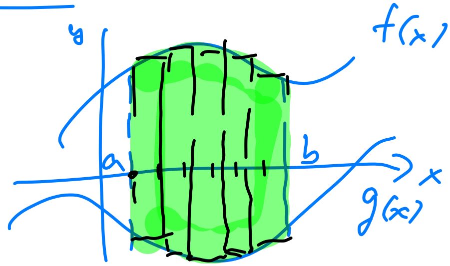
ดังนั้น.

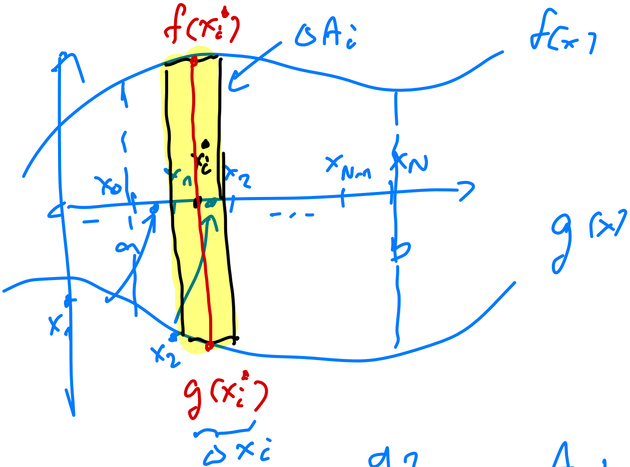
$$\int \frac{9x^2-3x+1}{x^2(x-1)}\,dx = \int \frac{2}{x} + \frac{(-1)}{x^2} + \frac{7}{(x-1)}\,dx$$

$$= 2\ln|x| + \frac{1}{x} + 7\ln|x-1| + \underline{\underline{C}}$$

$\Rightarrow$ การประยุกต์ของอินทิกรัล.

$\Rightarrow$ การหาพื้นที่ ระหว่างเส้นโค้ง





พท. $\Delta A_i = \Delta x_i \cdot (f(x_i^*) - g(x_i^*))$

$$A_{ab} \approx \sum_{i=1}^{N} \Delta A_i$$

$$A_{ab} = \sum_{i=1}^{N} (f(x_i^*) - g(x_i^*))\Delta x_i$$

มาจาก นิยามอินทิกรัล

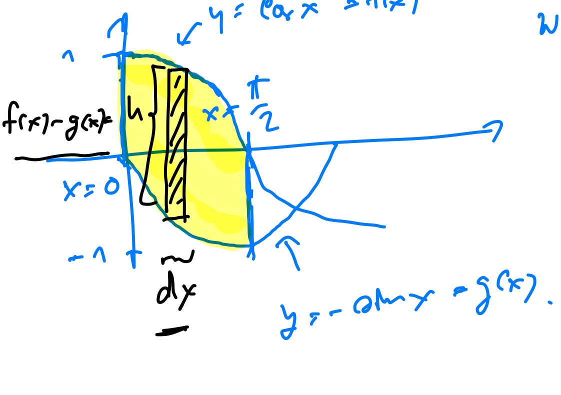
$$A_{ab} = \lim_{\substack{N \to \infty \\ (\Delta x_i \to 0)}} \sum_{i=1}^{N} (f(x_i^*) - g(x_i^*))\Delta x_i$$

สรุป

$$\boxed{A_{ab} = \int_a^b f(x) - g(x)\, dx}$$

Ex: หาพื้นที่ระหว่าง $y = \cos x$ และ $y = -\sin x$

จาก $x = 0$ ถึง $x = \frac{\pi}{2}$



พท. $A_{ab} = \int_{x=0}^{x=\frac{\pi}{2}} (f(x) - g(x))\,dx$

$$= \int_0^{\frac{\pi}{2}} \cos x - (-\sin x)\, dx$$

$$= (\sin x - \cos x)\Big|_{x=0}^{x=\frac{\pi}{2}}$$

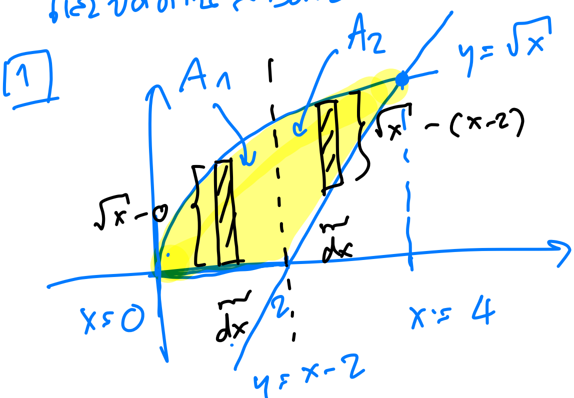
$$\Rightarrow A_{ab} = \left(\sin\left(\tfrac{\pi}{2}\right) - \cancel{\cos\left(\tfrac{\pi}{2}\right)}^{\,0}\right) - \left(\cancel{\sin(0)}^{\,0} - \cos(0)\right)$$

$$= 1 - (-1) = 2 \quad \boxed{}$$

Ex: หาพท. ปิดล้อมด้วยเส้นโค้ง $y = \sqrt{x}$, $y = x - 2$

และปิดด้านล่างด้วยแกน x

$\boxed{1}$



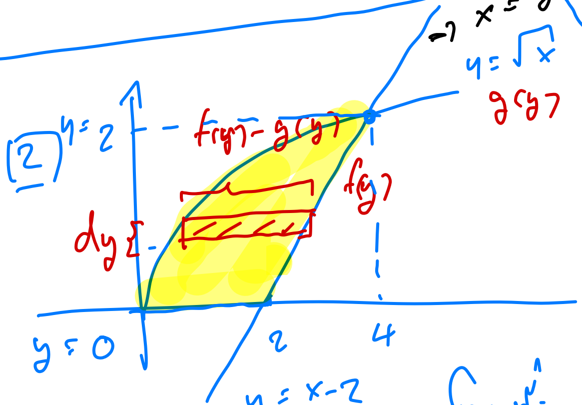
$$A = A_1 + A_2$$

$$= \int_{x=0}^{x=2} (\sqrt{x} - 0)\,dx \qquad \boxed{1}$$

$$+ \int_{x=2}^{x=4} \sqrt{x} - (x-2)\,dx$$

$\Rightarrow x = y^2$

$\boxed{2}$



$$A = \int_{y=0}^{y=2} \left(f(y) - g(y)\right) dy$$

$\Rightarrow x = y + 2$

โดยที่ $f(y) = y + 2$

$g(y) = y^2$

จะได้ $A = \int_{y=0}^{y=2} (y+2) - y^2\,dy$. $\quad$ — $\boxed{2}$

$\boxed{1}$:

$$A = \int_{x=0}^{x=2} (\sqrt{x} - 0)\, dx + \int_{x=2}^{x=4} \sqrt{x} - (x-2)\, dx$$

$$= \left(\frac{2}{3}x^{\frac{3}{2}}\right)\Bigg|_{x=0}^{x=2} + \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2} + 2x\right)\Bigg|_{x=2}^{x=4}$$

$$= \left[\cancel{\frac{2\cdot 2^{\frac{3}{2}}}{3}} - 0\right] + \left[\left(\frac{2\cdot 4^{\frac{3}{2}}}{3} - \frac{4^2}{2} + 2\cdot 4\right) - \left(\cancel{\frac{2\cdot 2^{\frac{3}{2}}}{3}} - \frac{2^2}{2} + 2\cdot 2\right)\right]$$

$$= \left(\frac{2\cdot 8}{3} - \cancel{8} + \cancel{8}\right) - (-2+4) = \frac{16}{3} - 2 = \frac{10}{3} \checkmark$$

$\boxed{2}$

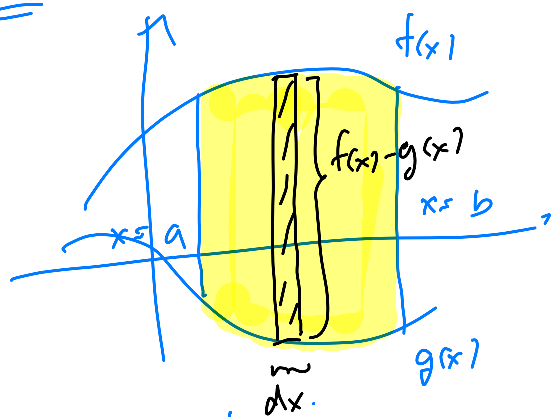
$$A = \int_{y=0}^{y=2} (y+2) - y^2\, dy$$

$$= \left(-\frac{y^3}{3} + \frac{y^2}{2} + 2y\right)\Bigg|_{y=0}^{y=2}$$

$$= \left(-\frac{2^3}{3} + \frac{2^2}{2} + 2\cdot 2\right) - 0 = -\frac{8}{3} + 6 = \frac{10}{3} \checkmark$$

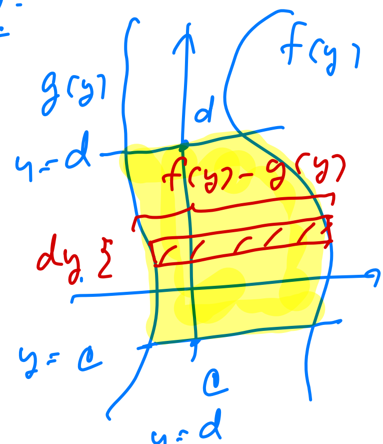
สรุป: หาพท. ระหว่างกราฟ.

dx:



$$A_{ab} = \int_{x=a}^{x=b} [f(x) - g(x)]\,dx$$

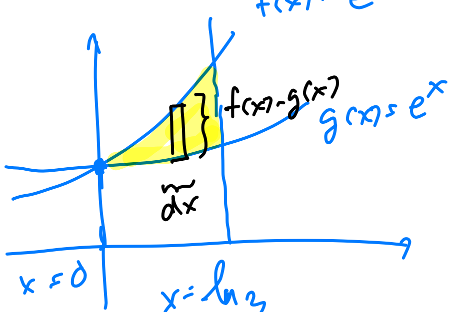
dy:



$$A_{cd} = \int_{y=c}^{y=d} [f(y) - g(y)]\,dy$$

Ex: แบบฝึกหัด 3.8.

1.) จงหาพื้นที่แรเงา.



$$\boxed{A = \int_{x=0}^{x=\ln 3} [e^{2x} - e^{x}]\,dx} \quad (**)$$

$$= \left(\frac{e^{2x}}{2} - e^{x}\right)\Bigg|_{x=0}^{x=\ln 3}$$

$$= \left(\frac{e^{2\ln 3}}{2} - e^{\ln 3}\right) - \left(\frac{e^{2\cdot 0}}{2} - e^{0}\right)$$

$$= \left(\frac{3^2}{2} - 3\right) - \left(\frac{1}{2} - 1\right) = \frac{3}{2} + \frac{1}{2} = 2 \;\blacksquare$$

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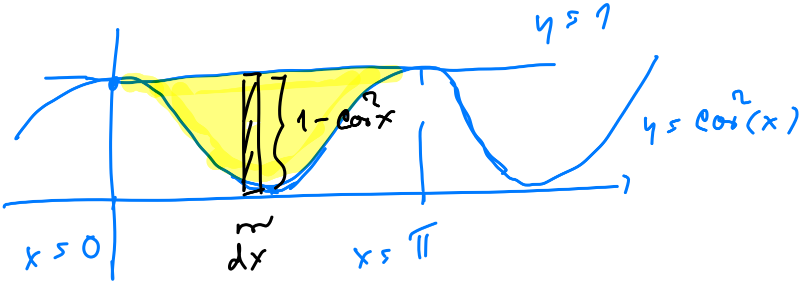
④. หาพท. แรเงา



$$A = \int_{x=0}^{x=\pi} \left[1 - \underbrace{\cos^2 x}_{=\frac{1-\cos(2x)}{2}}\right] dx$$

$$= \int_{x=0}^{x=\pi} \frac{1}{2} + \frac{\cos(2x)}{2}\, dx$$

$$= \left(\frac{1}{2}x + \frac{\sin(2x)}{2 \cdot 2}\right)\Bigg|_{x=0}^{x=\pi}$$

$$= \left(\frac{1}{2}\cdot\pi + \cancelto{0}{\frac{\sin(2\pi)}{4}}\right) - \left(\frac{1}{2}\cdot 0 + \cancelto{0}{\frac{\sin(2\cdot 0)}{4}}\right)$$

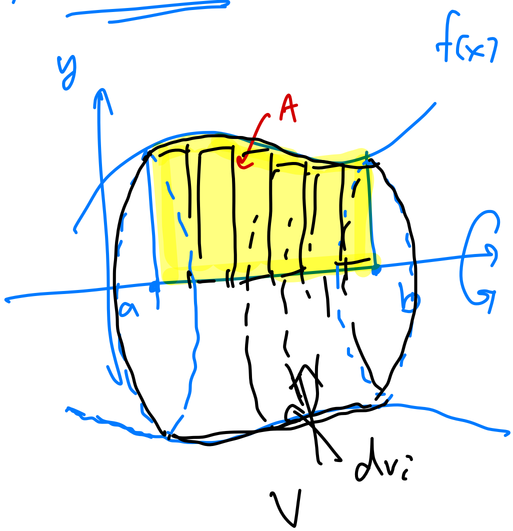
$$= \frac{\pi}{2} \;\blacksquare$$

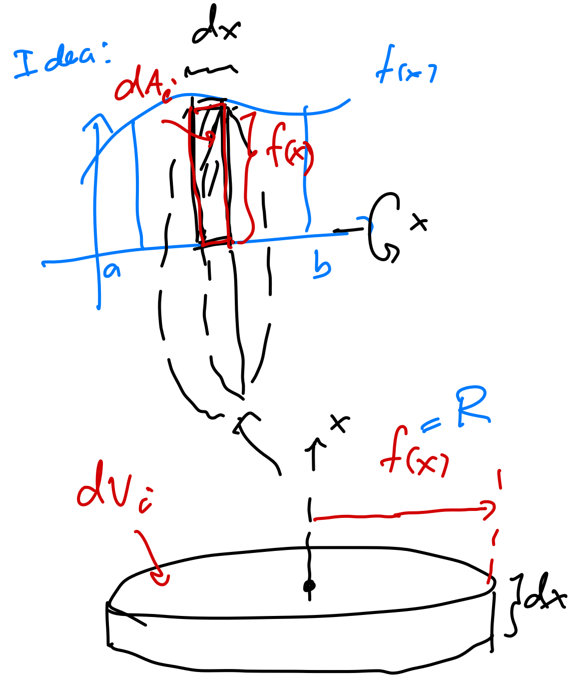
การบ้าน: แบบฝึกหัด 5.8

ข้อ. 5+6

⇒ การหาปริมาตรของทรงตันที่เกิดจากการหมุน:

⇒ แบบ Disk:





$$V \approx \sum_{i=1}^{N} dV_i$$

dV_i = พท.ฐาน × สูง

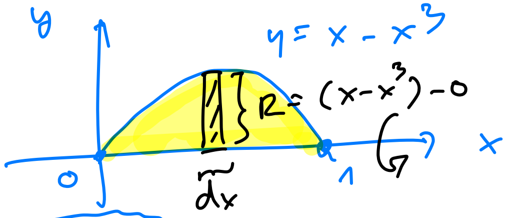
$\pi R^2 \times dx$, $R = f(x)$

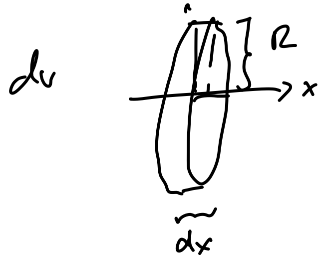
จะได้ $dV_i = \pi (f(x_i^*))^2 \cdot dx$

$$V = \lim_{N \to \infty} \sum_{i=1}^{N} \pi (f(x_i^*))^2 dx_i$$

$$\Rightarrow \boxed{V = \int_{x=a}^{x=b} \pi (f(x))^2 \, dx}$$

Ex: จงหาปริมาตรทรงตันที่เกิดจากการหมุน พท. ดังรูป รอบแกน x





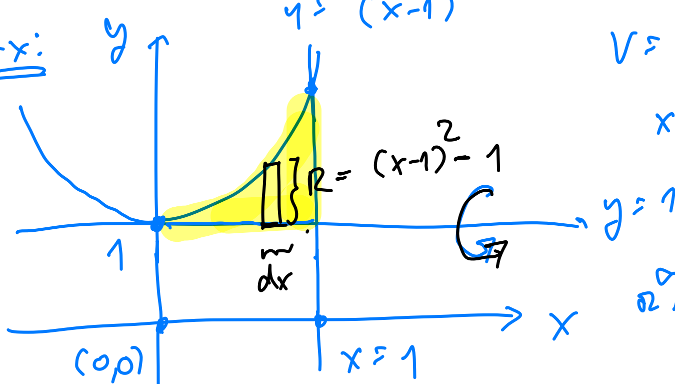
$$\boxed{V = \int_{x=0}^{x=1} \pi R^2 \, dx}, \quad R = (x - x^3) - 0$$

$$= \int_{x=0}^{x=1} \pi \left(x - x^3\right)^2 dx$$

$$= \pi \int_{x=0}^{x=1} x^2 - 2x^4 + x^6 \, dx$$

$$= \pi \left(\frac{x^3}{3} - \frac{2x^5}{5} + \frac{x^7}{7} \right) \Bigg|_{x=0}^{x=1} = \pi \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7} \right)$$

Ex:



$$V = \int_{x=1}^{x=1} \pi R^2 \, dx$$

$$\boxed{V = \int_{x=0}^{x=1} \pi \Big(\underbrace{(x-1)^2 - 1}_{R} \Big)^2 dx}$$