การบ้าน: แบบฝึกหัด 3.5.

2.) $\int \sin^5(2x)\cos(2x)\,dx$

4.) $\int \frac{\sec x}{\tan^2 x}\,dx$

6.) $\int \sin^4(2\theta)\,dx$

10.) $\int (\sin x + \cos x)^2\,dx$

2.) $\int \sin^5(2x)\cos(2x)\,dx$

แบบ②: ให้ $u = \sin(2x)$

$du = 2\cos(2x)\,dx$

$dx = \frac{du}{2\cos(2x)}$

แทนค่า $= \int \sin^5(2x)\,\cancel{\cos(2x)}\,\frac{du}{2\cancel{\cos(2x)}}$

$(u = \sin(2x))$ $= \frac{1}{2}\int u^5\,du = \frac{1}{2}\frac{u^6}{6} + C$

แทน u กลับ $= \frac{1}{2}\frac{\sin^6(2x)}{6} + C$

แบบ①: ให้ $u = \cos(2x) \Rightarrow du = -\sin(2x)\,2\,dx$

$dx = \frac{du}{-2\sin(2x)}$

จะได้ $\int \sin^5(2x)\cdot\cos(2x)\cdot dx = \int \overbrace{\sin^4(2x)}^{(\sin^2(2x))^2}\cdot\cancel{\sin(2x)}\cdot\cos(2x)\,\frac{du}{-2\cancel{\sin(2x)}}$

$(\sin^2(2x) = 1 - \cos^2(2x))$ $= \frac{1}{-2}\int \left(1 - \underbrace{\cos^2(2x)}_{u^2}\right)^2 \underbrace{\cos(2x)}_{u}\,du$

$(u = \cos(2x))$ $= \frac{-1}{2}\int (1 - 2u^2 + u^4)\cdot u\,du$

$$= -\frac{1}{2}\int u - 2u^3 + u^5\,du$$

$$= -\frac{1}{2}\left[\frac{u^2}{2} - \frac{\cancel{2}u^4}{\cancel{4}\,2} + \frac{u^6}{6}\right] + C$$ แทนค่ากลับ $$= -\frac{1}{2}\left[\frac{\cos^2(2x)}{2} - \frac{\cos^4(2x)}{2} + \frac{\cos^6(2x)}{6}\right] + C$$ ∎

4.) $\int \frac{\sec x}{\tan^2 x}\,dx$ จัดรูป $= \int \frac{1}{\cancel{\cos x}}\cdot\frac{\cos^{\cancel{2}} x}{\sin^2 x}\,dx = \frac{1}{\sin}\,\frac{\cos}{\sin x} = \csc x \cot x$

$$= \int \frac{\cos x}{\sin^2 x}\,dx \quad , \quad u = \sin x \to du = \cos x\,dx$$

$$dx = \frac{du}{\cos x}$$

แทนค่า u $$= \int \frac{\cancel{\cos x}}{u^2}\cdot\frac{du}{\cancel{\cos x}} = -\frac{1}{u} + C$$

แทนค่ากลับ
$$= -\frac{1}{\sin x} + C$$ ∎

6.) $\int \sin^4(2\theta)\,d\theta$

[แบบ 3: ทั้ง n เป็นเลขคู่ (n≥0)]

ทบ. $\sin^2 x = \frac{1-\cos(2x)}{2}$ → ใช้

$$= \int\left(\frac{1-\cos(4\theta)}{2}\right)^2 d\theta = \frac{1}{2}\int 1 - 2\cos(4\theta) + \cos^2(4\theta)\,d\theta$$

ใช้: $\cos^2 x = \frac{1+\cos(2x)}{2}$

$$= \frac{1}{2}\int 1 - 2\cos(4\theta) + \frac{1}{2}\left[1 + \cos(8\theta)\right] d\theta$$

$$= \frac{1}{2}\int \frac{3}{2} - 2\cos(4\theta) + \frac{\cos(8\theta)}{2}\, d\theta$$

เปลี่ยนตัวแปร

$u = 4\theta$
$du = 4d\theta$

$$\int \cos(4\theta)\, d\theta$$

$$= \int \cos(u) \cdot \frac{du}{4}$$

$$= \frac{\sin(u)}{4}$$

เปลี่ยนตัวแปร

$$= \frac{1}{2}\left[\frac{3\theta}{2} - 2\,\frac{\sin(4\theta)}{4} + \frac{\sin(8\theta)}{2\cdot 8}\right] + C$$

10.) $\int (\sin x + \cos x)^2\, dx$

วิธีทำ.

$$= \int \underbrace{\sin^2 x}_{\text{แบบ } 3} + \underbrace{2\sin x\cos x}_{\text{แบบ } 1 \text{ หรือ } 2} + \underbrace{\cos^2 x}_{\text{แบบ } 3}\, dx$$

$$= \int \underbrace{\sin^2 x + \cos^2 x}_{=1}\, dx + \int 2\sin x\, \underbrace{\cos x}_{u}\, dx$$

$u = \sin x$
$du = \cos x\, dx$
$dx = \frac{du}{\cos x}$

$$= \int 1\, dx + \int 2\overbrace{\sin x}^{u}\, \cancel{\cos x}\, \frac{du}{\cancel{\cos x}}$$

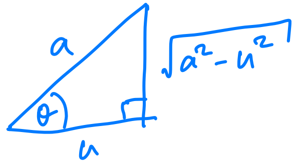
$$= x + \frac{\cancel{2}u^2}{\cancel{2}} + C \quad = \quad x + \sin^2 x + C$$

แบบหลาย.

→ การอินทิเกรตโดยการแทนด้วยฟังก์ชันตรีโกณมิติ.

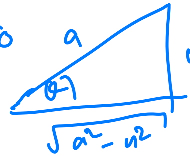
⇒ รูปแบบ:

• $(a^2-u^2)^{\frac{m}{2}}$



ให้ $u = a\cos\theta$.

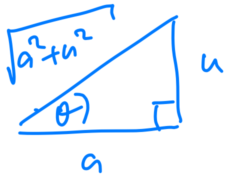
หรือ



$\frac{u}{a} = \sin\theta$

ให้ $\boxed{u = a\sin\theta.}$

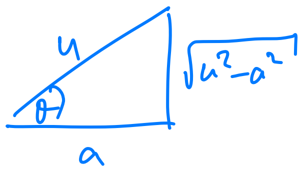
• $(a^2+u^2)^{\frac{m}{2}}$



ให้ $\frac{u}{a} = \tan\theta$

→ $\boxed{u = a\tan\theta.}$

• $(u^2-a^2)^{\frac{m}{2}}$



⇒ ให้ $\frac{a}{u} = \cos\theta$

⇒ $\boxed{u = a\sec\theta}$

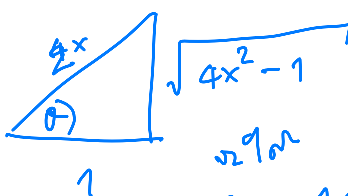
Ex: หาค่า. $\int \frac{1}{(\underbrace{4x^2}_{=(2x)^2}-1)^{3/2}}\,dx$

① check form:

o อยู่ในรูปแบบ $(u^2-a^2)^{\frac{m}{2}}$ ได้.

โดยที่ $u = 2x$, $a = 1$

② สร้างความสัมพันธ์ บน △



จะได้

$2x = 1\sec\theta$

จะได้ $d(2x) = \sec\theta\tan\theta\,d\theta$

⇒ $2dx = \sec\theta\tan\theta\,d\theta$

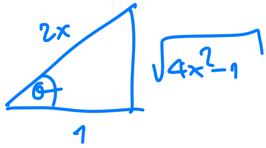
⇒ $dx = \frac{\sec\theta\tan\theta}{2}d\theta$

แทน. ในรูป θ จะได้

$$\int \frac{1}{(\sqrt{4x^2-1})^3}\cdot\frac{\sec\theta\tan\theta}{2}\,d\theta$$

③ เปลี่ยนตัวแปร

(จาก Δ



$$\Rightarrow \frac{\sqrt{4x^2-1}}{1} = \tan\theta$$

$$\Rightarrow \sqrt{4x^2-1} = 1\tan\theta \quad)$$

$$= \frac{1}{2}\int \frac{1}{(\tan\theta)^{\cancel{3}2}} \sec\theta \cancel{\tan\theta}\, d\theta$$

$$= \frac{1}{2}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta = \frac{1}{2}\int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{2}\int \frac{1}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta}\, d\theta = \frac{1}{2}\int \operatorname{cosec}\theta \cdot \cot\theta\, d\theta$$

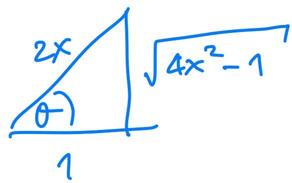
④ อินทิเกรตโดยสูตรตรีโกณฯ.

$$= -\frac{\operatorname{cosec}\theta}{2} + C$$

(ต้องเปลี่ยนกลับในรูป x) ใช้ความสัมพันธ์ในรูป Δ.

แทนกลับ

$$= \frac{1}{2}\left(-\frac{2x}{\sqrt{4x^2-1}}\right) + C \quad \blacksquare$$



⑤ เปลี่ยนตัวแปร θ เป็น x โดยใช้ Δ.

$$\operatorname{cosec}\theta = \frac{2x}{\sqrt{4x^2-1}}$$

(check:

$$\frac{d}{dx}\frac{1}{2}\left(\frac{-2x}{\sqrt{4x^2-1}} + C\right) = \frac{1}{2} \cdot \frac{\sqrt{4x^2-1}\,(-2) - (-2x)\frac{1}{2}(4x^2-1)^{-\frac{1}{2}} \cdot (8x)}{(4x^2-1)^{2/2}}$$

$$= \frac{1}{2}\, \frac{-2\sqrt{4x^2-1} + x \cdot (8x)/(4x^2-1)^{\frac{1}{2}}}{(4x^2-1)^{2/2}} = \frac{1}{2} \cdot \frac{-2(4x^2-1) + x \cdot (8x)}{(4x^2-1)^{3/2}}$$

$$= \frac{1}{2} \cdot \frac{\cancel{-8x^2} + 2 + \cancel{8x^2}}{(4x^2-1)^{3/2}} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}\ 1}{(4x^2-1)^{3/2}}$$

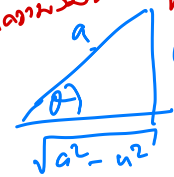
Ex: จงหา. $\int e^x \sqrt{4 - e^{2x}}\, dx$

ให้ $u = e^x \Rightarrow du = e^x dx \Rightarrow dx = \dfrac{du}{e^x}$

แทน u

$$= \int \cancel{e^x} \sqrt{4 - \underbrace{(e^x)}_{u}{}^2} \cdot \frac{du}{\cancel{e^x}}$$

① หาความสัมพันธ์ในรูป △



(จัดรูป)

$$= \int \sqrt{2^2 - u^2}\, du$$

ดูรูป. $(a^2-u^2)^{\frac{m}{2}}$, $a = 2$

ให้. $u = a \sin\theta$

$\boxed{du = a \cos\theta\, d\theta}$

② เปลี่ยนในรูป θ.

แทน θ

$$= \int 2\cos\theta \cdot 2\cos\theta\, d\theta$$

และ $\dfrac{\sqrt{a^2-u^2}}{a} = \cos\theta$

$$= 4\int \cos^2\theta\, d\theta$$

. จะได้ $\boxed{\sqrt{a^2-u^2} = a\cos\theta}$

ถ้าจำได้!

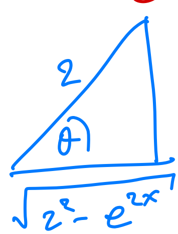
$$= \cancel{4}^{\,2} \cdot \int \frac{1+\cos(2\theta)}{\cancel{2}}\, d\theta$$

③ อินทิเกรต.

$$= 2\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

$$= 2\left(\theta + \frac{\cancel{2}\sin(\theta)\cos(\theta)}{\cancel{2}}\right) + C$$

④ จากสามเหลี่ยม θ กลับ x ที่ใช้ $u = e^x$ แทนค่ากลับ.



แทนค่า $(\theta \to x)$ $= 2\arcsin\left(\frac{e^x}{2}\right) + 2\left(\frac{e^x}{2}\right)\cdot\left(\frac{\sqrt{4-e^{2x}}}{2}\right) + C$

$\sin\theta = \frac{e^x}{2}$

$\cos\theta = \frac{\sqrt{4-e^{2x}}}{2}$

$\theta = \arcsin\left(\frac{e^x}{2}\right)$

Ex: จงหา $\int \frac{1}{x\sqrt{16+9x^2}}\,dx$

แทน $u = 3x \Rightarrow du = 3dx \Rightarrow dx = \frac{du}{3}$

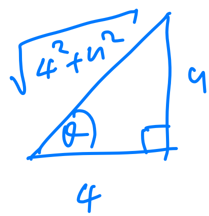
(แทน u) $= \int \frac{1}{x\sqrt{4^2+u^2}}\,\frac{du}{3}$

จาก $u = 3x$ จะได้ $x = \frac{u}{3}$

$= \int \frac{\cancel{3}}{u}\cdot\frac{1}{\sqrt{4^2+u^2}}\,\frac{du}{\cancel{3}}$

$= \int \frac{1}{u\sqrt{4^2+u^2}}\,du$

เขียนสามเหลี่ยมมุมฉากได้ดังนี้



$u = 4\tan\theta$

$\Rightarrow du = 4\sec^2\theta\,d\theta$

และ $\sqrt{4^2+u^2} = 4\sec\theta$

แทนค่าทั้งหมด $= \int \frac{1}{u\cdot 4\sec\theta}\cdot 4\sec^2\theta\,d\theta$

$(u = 4\tan\theta)$ $= \int \frac{1}{4\tan\theta\cdot\cancel{4\sec\theta}}\cdot\cancel{4}\sec^{\cancel{2}}\theta\,d\theta$

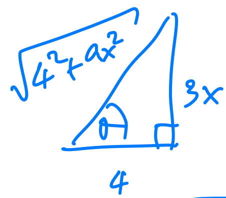
$$= \frac{1}{4}\int \frac{\sec\theta}{\tan\theta}\, d\theta = \frac{1}{4}\int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cancel{\cos\theta}}{\sin\theta}\, d\theta$$

$$= \frac{1}{4}\int \operatorname{cosec}\theta \, d\theta = -\frac{1}{4}\ln\left|\operatorname{cosec}\theta + \cot\theta\right| + C$$

(สูตร: $\int \operatorname{cosec}\theta\, d\theta = -\ln\left|\operatorname{cosec}\theta + \cot\theta\right| + C$)

(เปลี่ยน $\theta \to x$)

$$= -\ln\left| \frac{\sqrt{16+9x^2}}{4} + \frac{9}{3x} \right| + C$$



- $\operatorname{cosec}\theta = \dfrac{\sqrt{16+9x^2}}{4}$
- $\cot\theta = \dfrac{4}{3x}$

การบ้าน: แบบฝึกหัด 3.6

1.) $\displaystyle\int \frac{1}{x^2\sqrt{9-x^2}}\, dx$

7.) $\displaystyle\int \frac{e^t}{(4-e^{2t})^{3/2}}\, dt$

3.) $\displaystyle\int \frac{\sqrt{x^2-49}}{x}\, dx$

8.) $\displaystyle\int \frac{1}{x\sqrt{2+(\ln x)^2}}\, dx$