การบ้าน: แบบฝึกหัด 2.3 (9 + 24 + 15)

ทบทวน: สูตร,

- $f(x) = c \Rightarrow \frac{d}{dx}(c) = 0$
- $f(x) = x^n \Rightarrow \frac{d}{dx}(x^n) = nx^{n-1}, \; n \in \mathbb{R}$ (จำนวนจริง.)

สมบัติ:

1.) $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$

2.) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$

3.) $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + g(x)f'(x)$ (ผลคูณ)

4.) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$ (ผลหาร.)

$(g(x) \neq 0).$

การบ้าน: แบบฝึกหัด. 2.3

9.) จงหาอนุพันธ์ของ. $y = \left(x + \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)$

①. กระจาย. $\rightarrow y = (x + x^{-1})(x^2 - x^{-2}) = x^3 - x^{-1} + x^1 - x^{-3}$

$$\Rightarrow \frac{d}{dx}y = \frac{d}{dx}(x^3 - x^{-1} + x^1 - x^{-3})$$

$$= \frac{d}{dx}(x^3) - \frac{d}{dx}(x^{-1}) + \frac{d}{dx}(x) - \frac{d}{dx}(x^{-3})$$

$$= 3x^2 - (-1)x^{-2} + 1x^0 - (-3)x^{-4}$$

$$= 3x^2 + 1 + x^{-2} + 3x^{-4} \quad \checkmark$$

2. ตัวอย่าง : $y = (x + x^{-1})(x^2 - x^{-2})$

1 2

$$\frac{dy}{dx} = (x+x^{-1})\frac{d}{dx}(x^2 - x^{-2}) + (x^2 - x^{-2})\frac{d}{dx}(x + x^{-1})$$

$$= (x+x^{-1})(2x+2x^{-3}) + (x^2-x^{-2})(1-x^{-2})$$

(คูณ กระจาย)

$$= \left(2x^2 + 2x^{-2} + 2 + 2x^{-4}\right) + \left(x^2 - 1 - x^{-2} + x^{-4}\right)$$

$$= 3x^2 + 1 + x^{-2} + 3x^{-4}. \quad \boxed{\#} \quad \checkmark$$

10.) $y = \dfrac{(x+1)(x-2)}{x^2+1} = \dfrac{x^2 - 2x + x - 2}{x^2+1}$

$$y = \frac{x^2 - x - 2}{x^2+1}$$

1 ผลหาร

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2-x-2}{x^2+1}\right) = \frac{(x^2+1)\frac{d}{dx}(x^2-x-2) - (x^2-x-2)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

2

$$= \frac{(x^2+1)(2x-1) - (x^2-x-2)(2x)}{(x^2+1)^2}$$

คิดบ.

$$= \frac{(\cancel{2x^3} - x^2 + 2x + 1) - (\cancel{2x^3} - 2x^2 - 4x)}{(x^2+1)^2}$$

$$= \frac{x^2 + 6x + 1}{(x^2+1)^2} \quad \boxed{\#}$$

16.) $s = \dfrac{t}{t^3+7}$

$$\Rightarrow \frac{ds}{dt} = \frac{d}{dt}\left(\frac{t}{t^3+7}\right) = \frac{(t^3+7)\frac{d}{dt}(t) - t\frac{d}{dx}(t^3+7)}{(t^3+7)^2}$$

$$= \frac{t^3+7-3t^3}{(t^3+7)^2} = \frac{-2t^3+7}{(t^3+7)^2}$$

อนุพันธ์อันดับสูง ; ⇒ อนุพันธ์มากกว่า 1 ครั้ง

อันดับ 1: $f(x) \xRightarrow{\text{ดิฟ.}} f'(x) = \dfrac{df(x)}{dx}$

อันดับ 2: $f'(x)$ (อันดับ 1) $\Rightarrow (f'(x))' = \dfrac{d}{dx}\left(\dfrac{d}{dx}f\right)$

$$\Downarrow$$

$$f''(x) = \frac{d^2}{dx^2}f(x) \quad \text{— อันดับ 2.}$$

อันดับ 3: $f''(x)$ อันดับ 2 $\xRightarrow{\text{ดิฟ.}} (f''(x))' = \dfrac{d}{dx}\left(\dfrac{d^2}{dx^2}f(x)\right)$

$$\Downarrow$$

$$f'''(x) = \frac{d^3}{dx^3}f(x). \quad \text{— อันดับ 3.}$$

อันดับ ≥4:
อันดับ n:

$$f^{(n)}(x) = \frac{d^n}{dx^n}f(x) \quad \text{— อันดับ } n:$$

$\boxed{Ex}$ จงหา $f^{(4)}(x)$ ของ $f(x) = \dfrac{x^4+2}{x} = \dfrac{x^4}{x} + \dfrac{2}{x}$

วิธีทำ.

$\Rightarrow \quad f(x) = x^3 + 2x^{-1}$

$$f'(x) = \frac{d}{dx}(f(x)) = \frac{d}{dx}(x^3+2x^{-1}) = 3x^2 - 2x^{-2}$$

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}(3x^2-2x^{-2}) = 6x + 4x^{-3}$$

$$f'''(x) = \frac{d}{dx}(f''(x)) = \frac{d}{dx}(6x+4x^{-3}) = 6 - 12x^{-4}$$

$$f^{(4)}(x) = \frac{d}{dx}(f'''(x)) = \frac{d}{dx}(6-12x^{-4}) = 0 + 48x^{-5}$$

$\boxed{Ex}$ 2.2.) จงหา. $\left.\dfrac{d^2y}{dx^2}\right|_{x=1}$ เมื่อ $y = 6x^5 - 4x^2$

$\Rightarrow$ จะ. $\dfrac{dy}{dx} = \dfrac{d}{dx}(6x^5-4x^2) = 30x^4 - 8x$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}(30x^4 - 8x) = 120x^3 - 8$$

ดังนั้น $\left.\dfrac{d^2y}{dx^2}\right|_{x=1} = 120(1) - 8 = 112$

$\Rightarrow$ ກົດລູກໂສ້ (ອະນຸພັນຂອງ ຄອມໂພຊິດຟັງຊັນ.)

Ex: $f(x) = x^{27}$, $g(x) = 2x^2 + 1$

$(f\circ g)(x) = f(g(x)) = (g(x))^{27} = (2x^2+1)^{27}$

ຖາມ: ຫາ ອະນຸພັນ. $\frac{d}{dx}(f\circ g(x)) = \frac{d}{dx}\left(f(g(x))\right)$

ກົດລູກໂສ້: $\boxed{\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\, g'(x)}$

Ex: $f(x) = x^2$, $g(x) = 2x^2 + 1$.

ຫາ $\frac{d}{dx}\left(f(g(x))\right)$

① ແບບເດີມ: $f(g(x)) = (g(x))^2 = (2x^2+1)^2 = 4x^4 + 4x^2 + 1$

$\Rightarrow \frac{d}{dx}\left(f(g(x))\right) = \frac{d}{dx}(4x^4 + 4x^2 + 1) = 16x^3 + 8x$

② ໃຊ້ກົດລູກໂສ້: $\frac{d}{dx} f(g(x)) = \underbrace{f'(g(x))}_{①}\, \underbrace{g'(x)}_{②}$

ໂດຍທີ່. $f(x) = x^2 \Rightarrow f'(x) = 2x$

$g(x) = 2x^2 + 1 \Rightarrow g'(x) = 4x$

ສະນັ້ນ. ①. $f'(g(x)) = 2(2x^2+1) = 4x^2 + 2$

② $g'(x) = 4x$

$$\Rightarrow \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) = (4x^2+2)\cdot(4x) = 16x^3 + 8x \quad \checkmark$$

$\boxed{\text{Ex:}}$ $f(x) = x^{27}$, $g(x) = 2x^2+1$

$\Rightarrow$ วิธี① กระจายเทอมลำบาก ✗.

$\Rightarrow$ วิธี②: กฎลูกโซ่. $\boxed{\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)}$

$f(x) = x^{27} \Rightarrow f'(x) = 27x^{26}$

$g(x) = 2x^2+1 \Rightarrow g'(x) = 4x$

ดังนั้น $\frac{d}{dx}(f(g(x))) \underset{\text{กฎลูกโซ่}}{=} f'(g(x))\, g'(x)$

$$= 27(2x^2+1)^{26}\cdot(4x) = 108x(2x^2+1)^{26}$$

Ex: หา $\frac{df}{dx}$ ของ. $f(x) = (\underbrace{x^3+2x}_{u(x)})^{37}$

$\Rightarrow f(u) = u^{37}$, $u(x) = x^3+2x$

กฎลูกโซ่ $\Rightarrow$ $\boxed{\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}}$

$$\begin{tikzcd}
f \arrow[d, bend left, "\frac{df}{du}"] \arrow[dd, bend left=60, "\frac{df}{dx}"] \\
u \arrow[d, bend left, "\frac{du}{dx}"] \\
x
\end{tikzcd}$$

$$\Rightarrow \frac{df}{dx} = \frac{d(u^{37})}{du} \cdot \frac{d(x^3+2x)}{dx}$$

$$= 37u^{36} \cdot (3x^2+2)$$

$$(u = x^3+2x) \quad = 37(x^3+2x)^{36} \cdot (3x^2+2)$$ ✓

อีกวิธี: $f(x) = (x^3+2x)^{37}$

$$\frac{df}{dx} = \frac{d(x^3+2x)^{37}}{d(x^3+2x)} \cdot \frac{d(x^3+2x)}{dx}$$

$$= 37(x^3+2x)^{36} \cdot (3x^2+2)$$ ✓

$\boxed{Ex}$ 1.7.) $y = \dfrac{4}{(3x^2-2x+1)^3}$ หา $\dfrac{dy}{dx}$

จัดรูป $\Rightarrow$ $y = 4 \cdot \underbrace{(3x^2-2x+1)}_{u(x)}^{-3}$

$$\begin{tikzcd} y \arrow[d, "\frac{dy}{du}"] \arrow[dd, bend left, "\frac{dy}{dx}"] \\ u \arrow[d, "\frac{du}{dx}"] \\ x \end{tikzcd}$$

$\Rightarrow y(u) = 4u^{-3}, \quad u(x) = 3x^2-2x+1.$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d(4u^{-3})}{du} \cdot \frac{d(3x^2-2x+1)}{dx} =$$

$$= -12u^{-4}\cdot(6x-2)$$

$(u=3x^2-2x+1)$ $\Rightarrow$ $= -12(3x^2-2x+1)^{-4}\cdot(6x-2)$ ∎

<u>การบ้าน</u>: แบบฝึกหัด 2.5 (1.8 + 2.2)

$\Rightarrow$ อนุพันธ์ของฟังก์ชันตรีโกณมิติ: $(\sin x, \cos x)$

ทฤษฎี: $f(x)=\sin x \Rightarrow \boxed{\dfrac{d}{dx}\sin(x) = \cos x}$ **

ทฤษฎี: $\boxed{\dfrac{d}{dx}\cos(x) = -\sin(x)}$ ✓

($\cos(x) = \sin\left(\frac{\pi}{2}+x\right)$
$\cos(a+b) = \cos a\cos b - \sin a\sin b$.)

<u>พิสูจน์</u>:

$$\frac{d}{dx}(\cos(x)) = \frac{d}{dx}\left(\sin\left(\frac{\pi}{2}+x\right)\right)$$

(chain rule) $$= \cos\left(\frac{\pi}{2}+x\right)\frac{d}{dx}\left(\frac{\pi}{2}+x\right) = \cos\left(\frac{\pi}{2}+x\right)\cdot 1$$

$$= \cos\left(\frac{\pi}{2}\right)\cos(x) - \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}\cdot\sin(x)$$

①

$$\Rightarrow \frac{d}{dx}\cos x = -\sin(x)$$ ∎

<u>สรุปสูตร</u>:

① $\dfrac{d}{dx}(\sin(x)) = \cos x$

② $\dfrac{d}{dx}(\cos(x)) = -\sin x$

**

ทฤษฎีบท : ③ $$\boxed{\frac{d}{dx}(\tan(x)) = \sec^2(x)}$$

$$\Rightarrow \frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

(ดิฟผลหาร.) $$= \frac{\cos(x)\frac{d}{dx}(\sin x) - \sin(x)\frac{d}{dx}(\cos x)}{\cos^2 x}$$

เอกลักษณ์:

$(\sin^2 x + \cos^2 x = 1)$ $$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x) \quad \blacksquare$$

④: ใช้หลักการเดียวกัน $$\boxed{\frac{d}{dx}(\cot(x)) = -\csc^2(x)}$$

⑤. $$\boxed{\frac{d}{dx}\sec x = \sec(x)\tan x}$$

พิสูจน์.

$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\cos(x)\right)^{-1}$$

(chain rule) $$= (-1)\cos^{-2}(x)\frac{d}{dx}(\cos(x))$$

$$= +\cos^{-2}(x)\sin x = \left(\frac{1}{\cos x}\right)\cdot\left(\frac{\sin x}{\cos x}\right)$$

$$= \sec(x)\cdot\tan(x)$$

6. ทำนองเดียวกัน.

$$\boxed{\frac{d}{dx}\operatorname{cosec}(x) = -\operatorname{cosec}(x)\cot x}$$

Ex: หาอนุพันธ์ของ.

1: $y = t\cdot\cot(t)$

ดิฟผลคูณ

$$\Rightarrow \frac{dy}{dt} = \frac{d}{dt}(t\cdot\cot(t)) = t\frac{d}{dt}\cot(t) + \cot(t)\frac{d}{dt}(t)$$

$$= t(-\operatorname{cosec}^2(t)) + \cot(t)\cdot 1$$

2: $r = \dfrac{\tan\theta}{1+\tan\theta}$

$$\Rightarrow \frac{dr}{d\theta} = \frac{d}{d\theta}\left(\frac{\tan\theta}{1+\tan\theta}\right)$$

ดิฟผลหาร

$$= \frac{(1+\tan\theta)\frac{d}{dx}(\tan\theta) - \tan\theta\frac{d}{dx}(1+\tan\theta)}{(1+\tan\theta)^2}$$

$$= \frac{(1+\cancel{\tan\theta})\sec^2\theta - \cancel{\tan\theta\cdot(\sec^2\theta)}}{(1+\tan\theta)^2}$$

$$= \frac{\sec^2\theta}{(1+\tan\theta)^2}$$

การบ้าน: แบบฝึกหัด 2.6

2.8 + 2.11 + 3.3.