# การผันกลับได้ของทรงหลายหน้า

# (Reversible properties of polyhedra)

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Department of Mathematics, Faculty of Science

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(Asst. Prof. Dr. Supanut Chaidee)

........................................................................... Member

(Asst. Prof. Dr. Chalermpon Bunpog)

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Ponpailin Homsombut

**หัวข้อ (ภาษาไทย)** การผันกลับได้ของทรงหลายหน้า

**(ภาษาอังกฤษ)** Reversible properties of polyhedra

**ชื่อผู้ทำการค้นคว้าอิสระ** Ponpailin Homsombut รหัสนักศึกษา 630510488

**ชื่ออาจารย์ที่ปรึกษา** Asst. Prof. Dr. Supanut Chaidee

**ชื่อกรรมการสอบ** Asst. Prof. Dr. Chalermpon Bunpog

## บทคัดย่อ

การค้นคว้าอิสระนี้ได้แรงบันดาลใจจากการศึกษาบทความของอะคิยาม่า จิน เรื่อง การผันกลับได้ของทรงหลายหน้าด้านขนาน สำหรับการศึกษานี้ตรวจสอบการผันกลับได้ของทรงหลายหน้าที่ไม่สามารถเติมเต็มพื้นที่ โดยการผันกลับแบบการติดกับบานพับ โดยใช้วิธีเพลทผันกลับได้สองแผ่น จากการศึกษาของอะคิยาม่า จิน ทำให้ได้บทแทรกสำหรับการตรวจสอบว่าทรงหลายหน้าที่ไม่สามารถเติมเต็มพื้นที่สามารถใช้วิธีเพลทผันกลับได้สองแผ่นได้หรือไม่ ซึ่งจากสมบัติการไม่แปรผันของเดห์นพบว่า ลูกบาศก์ปลายตัด, รอมบิคคิวบอกทาฮีดรอน และ คิวบอกทาฮีดรอนปลายตัด เป็นตัวเลือกที่เป็นไปได้ จากการศึกษา เมื่อพิจารณาถึงความเท่ากันของปริมาตร พบว่าคู่ของ รอมบิคคิวบอกทาฮีดรอนและคิวบอกทาฮีดรอนปลายตัด และคู่ของลูกบาศก์ปลายตัดและคิวบอกทาฮีดรอนปลายตัด ไม่สามารถใช้วิธีเพลทผันกลับได้สองแผ่นในการผันกลับระหว่างทรงตันได้

## Abstract

In this independent study, we have a motivation from the study of Professor Akiyama Jin on the topic “On Reversibility among Parallelohedra”. This study examines the reversibility of non-space-filling polyhedra by hinge dissection using the double reversal plate method. Using insights from Professor Akiyama Jin’s study, we derive a lemma for peer-checking whether space filla can be achieved using this method. Our analysis identifies the truncated cube, rhombicuboctahedron, and cuboctahedron as potential candidates because they are non-space filla with the same Dehn invariant, suggesting potential reversibility between them. Through further investigation, considering volume equality, we found that the pair of rhombicuboctahedron and cuboctahedron and the pair of truncated cube and cuboctahedron cannot be reversed using the double reversal plate method.

# Contents

# Chapter 1

# Introduction

Reversible polyhedra, characterized by their ability to transform into distinct shapes and then revert to their original forms, represent a fascinating area of study within the realm of geometry.

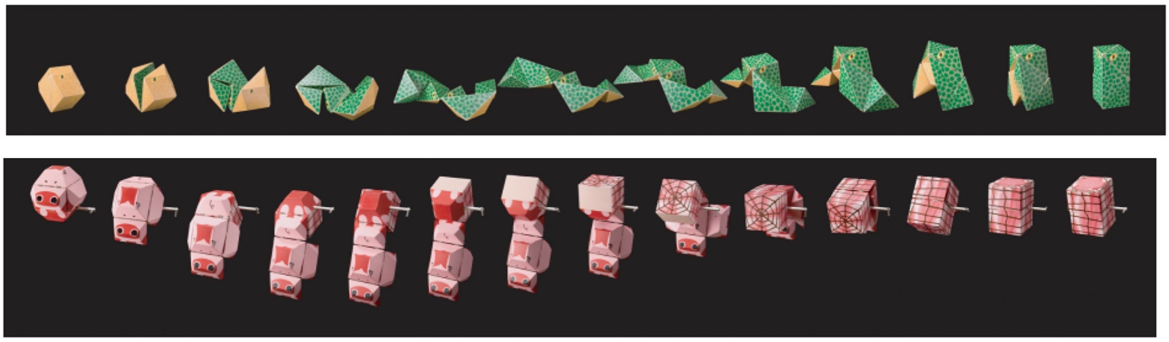

Figure 1.1: Reversible Solids from Professor Akiyama Jin [The figures are captured from Jin Akiyama A Friend and His Mathematics]

This project investigates the concept of reversibility, building upon previous research conducted by Professor Akiyama Jin utilizing the hinged dissection transformation. While existing studies have primarily focused on reversible pairs within space-filling parallelohedra, this study seeks to extend the inquiry to non-space-filling scenarios. Specifically, attention is directed towards the applicability of the Double-Reversal-Plates Method.

In this study, we investigate to determine whether this method can be extended to the Double-Reversal-Plates Method.

By examining the potential challenges and modifications required for such adaptation, this study aims to contribute to a broader understanding of reversible transformations.

## Scope of the study

This study focuses on the investigation of reversibility through the application of hinged dissection transformations in Archimedean solids. The primary objective is to analyze the feasibility of the Double-Reversal-Plates Method.

**Objectives of the study**

1. To identify conditions suitable for the Double Reversible Plate Method.
2. To ascertain conditions conducive to reversal in non-space-filling scenarios.
3. To evaluate the feasibility of applying the Double Reversible Plate Method in non-space-filling.

# Chapter 2

## Preliminaries

**Definition 2.1** (Polyhedron) A polyhedron is a three-dimensional geometric solid bounded by polygonal faces, edges, and vertices.

There are special polyhedra that satisfy some special properties. Platonic solids or regular polyhedra are solids in which all faces are the same regular polygons, the polygon whose edges are equal, and internal angles are equal. It is proved that there exist five Platonic solids including cube, tetrahedron, octahedron, dodecahedron, and icosahedron.

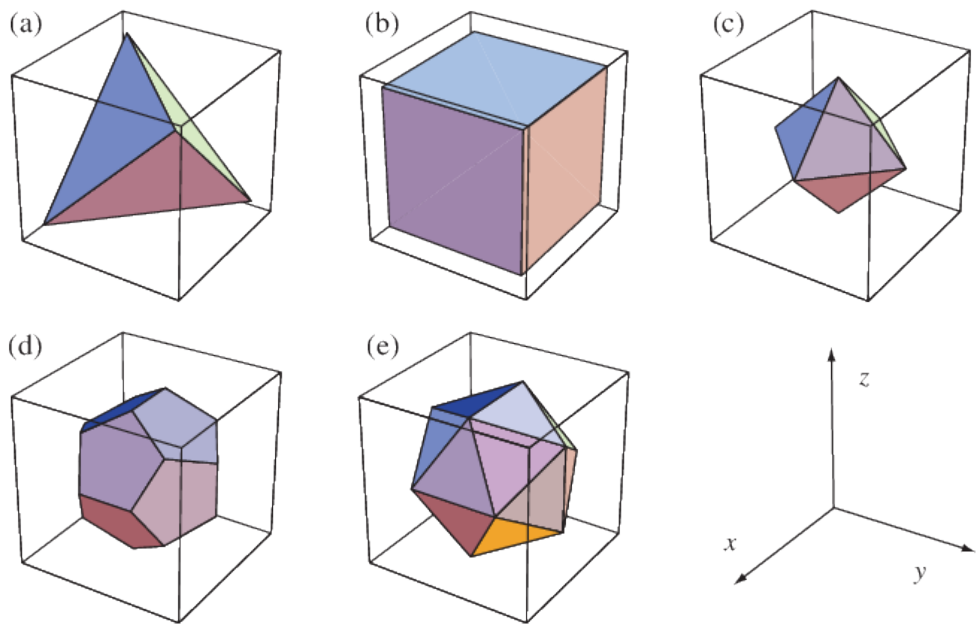


Figure 2.1: The five regular polyhedra known as Platonic solids [The figures are captured from Simple equations giving shapes of various convex polyhedra: The regular polyhedra and polyhedra composed of crystallographically low-index planes].

In the case that a polyhedron is composed of different kinds of regular polygons, it is called Archimedean solids, including 13 polyhedra as follows: truncated tetrahedron, cuboctahedron, truncated cube, truncated octahedron, rhombicuboctahedron, truncated cuboctahedron, Snub cube, Icosidodecahedron, Truncated dodecahedron, Truncated icosahedron, rhombicosidodecahedron, Truncated icosidodecahedron and Snub dodecahedron.

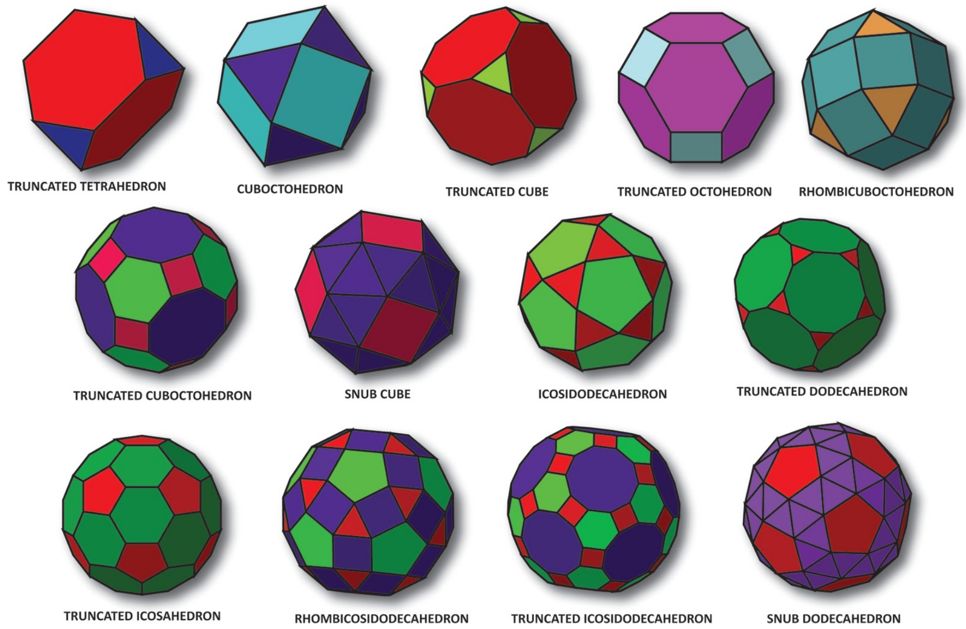


Figure 2.2: The 13 Archimedean solids [The figures are captured from https://xploreandxpress.blogspot.com/2011/04/fun-with-mathematics-archimedian-solids.html]

**Definition 2.2** (Polyhedral net) A polyhedral net is a geometric pattern that can be manipulated by cutting and folding to construct a three-dimensional model representing a solid shape.

Specially, a net of a polyhedron $P$ which is cut through the edges of the polyhedron $P$ is said to e-net. Figure 2.3 shows an example of an e-net of a cube.

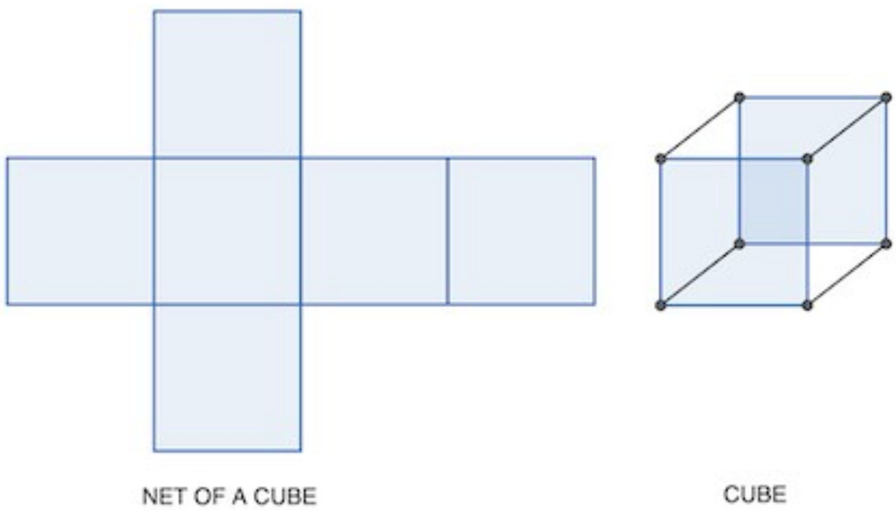


Figure 2.3: An e-net of a cube

| Name | Exact dihedral angle (radians) | Volume with the length of $a$ |
|---|---|---|
| Tetrahedron | $\arccos\left(\frac{1}{3}\right)$ | $\frac{a^3}{6\sqrt{2}}$ |
| Cube | $\arccos\left(\frac{\pi}{2}\right) = 0$ | $a^3$ |
| Octahedron | $\arccos\left(-\frac{1}{3}\right)$ | $\frac{1}{2}\sqrt{2}a^3$ |
| Dodecahedron | $\arccos\left(-\frac{1}{3}\right)$ | $\frac{15+7\sqrt{5}}{4}a^3$ |
| Icosahedron | $\arccos\left(-\frac{1}{3}\right)$ | $\frac{15+5\sqrt{5}}{12}a^3$ |

Table 2.1: The table of the dihedral angle and the volume of regular polyhedrons

**Definition 2.3** (The dihedral angle) The dihedral angle $\theta$ at the edge e of a polyhedron shared between two faces $f_1$ and $f_2$ is the angle between two unit normal vectors $n_1$ and $n_2$ to $f_1$ and $f_2$, respectively.

From Definition 2.3, we remark that $n_1 \cdot n_2 = \cos\theta$ By convention, the normal vectors point to the exterior of the polyhedron, and the dihedral angle at $e$ is the interior angle.

For example, the dihedral angle along each edge of a cube is $\frac{\pi}{2}$. Table 2.1 shows the dihedral angles of 5 regular polyhedra, with their volumes.

**Definition 2.4** (Reversible polyhedra) A reversible polyhedron is a polyhedron capable of undergoing a continuous deformation in three-dimensional space, transforming into another shape.

**Example 2.5** (Juno's Spinner-Reversible Dodecahedron Model)

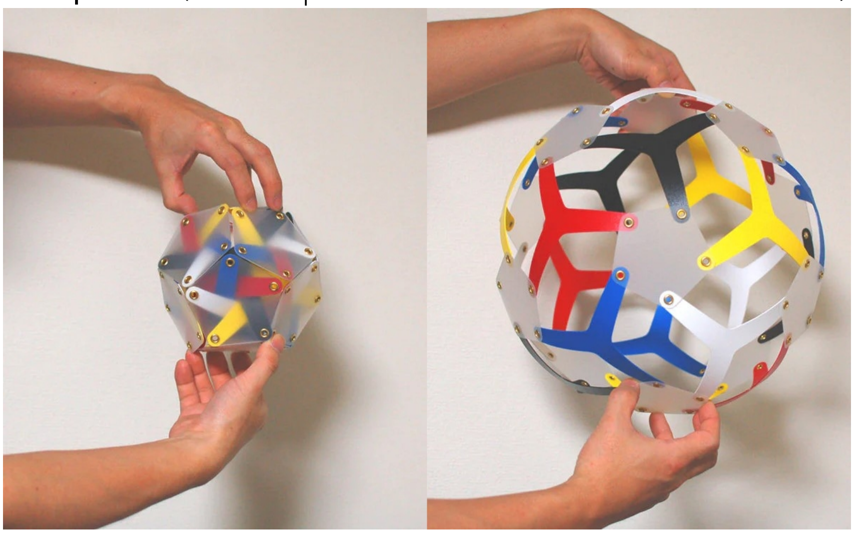

Figure 2.4: Juno's spinner is a unique polyhedral model that links and transforms. With a simple twist, straight edges meld into spherical symmetry. [The figures are captured from https://shop.pluredro.com/products/junos-spinner-dodecahedron-model]

In real world, each face of the polyhedron is connected to an adjacent face by a hinge called **a piano hinge**. Mathematically, the reversible of two polyhedra is considered by the following definition.

**Definition 2.6** (Hinge dissection transformation) A pair of $P, Q$ is said to be **hinge inside-out transformable** (or simply **reversible**) if $P$ and $Q$ satisfy these conditions :

1. The polyhedron $P$ is dissected into several pieces by planes. Such a plane is called a **dissection plane.**
2. The pieces are joined by piano hinges into a tree, and
3. If the pieces of $P$ are resembled inside out, then get a polyhedron $Q$.

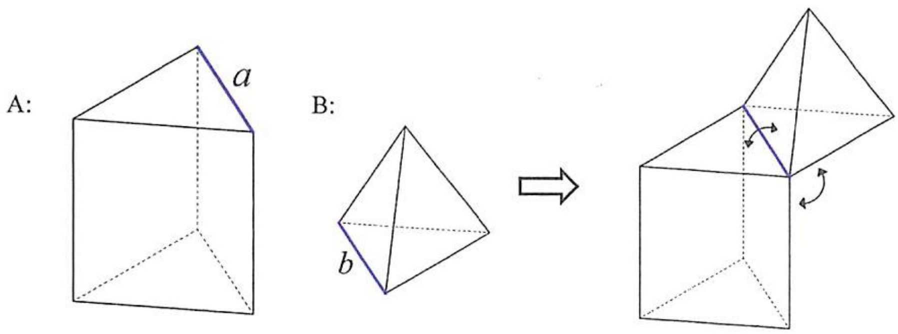


Figure 2.5: An example of a piano hinge [1]

**Definition 2.7** (Minkowski decomposition) The Minkowski sum of two sets of position vectors $P$ and $Q$ in Euclidean space is formed by adding each vector in $P$ to each vector in $Q$:

$$P + Q = \{p + q \mid p \in P, q \in Q\}.$$

**Example 2.8** Let $A = \{(1,0),(0,1),(0,-1)\}$ and $B = \{(0,0),(1,1),(1,-1)\}$ then

$A + B = \{(1,0),(0,1),(0,-1),(2,1),(1,2),(1,0),(2,-1),(1,0),(1,-2)\}$

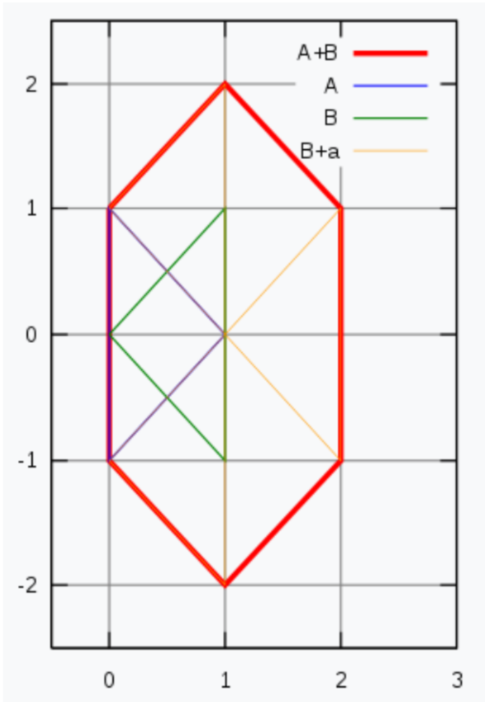


Figure 2.6: The Minkowski sum of $A + B$ [The figures are captured from https://commons.wikimedia.org/wiki/File:Minkowski-sumex4.svg]

**Definition 2.9** (Homothety) A homothety is defined as a transformation within an affine space, characterized by a designated point $S$, referred to as its center, and a non-zero scalar value $k$ denoted as its ratio. This transformation effectively maps $X'$ to $X$ within the space according to the following rule:

$$\overrightarrow{SX'} = k\overrightarrow{SX} \ ; k \neq 0$$

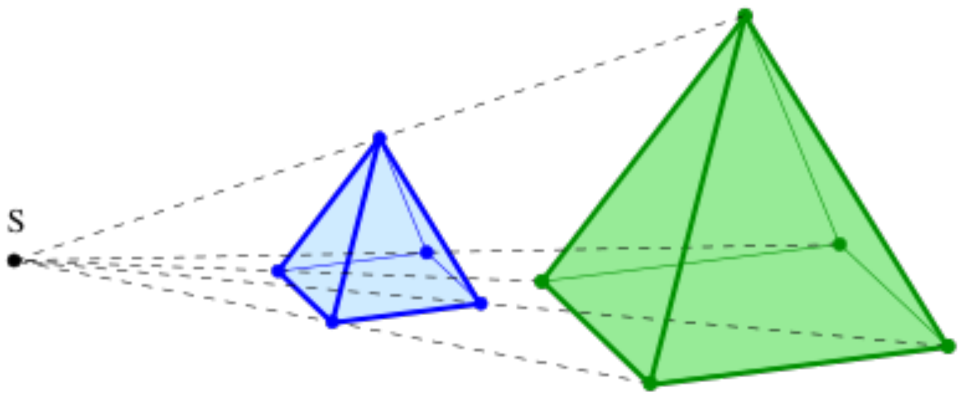


Figure 2.7: The homothety of the pyramid [The figures are captured from https://en.wikipedia.org/wiki/Homothety]

**Definition 2.10** (Decomposable) A polyhedron $P$ is **decomposable** if it is equal to a Minkowski sum $Q+R$ of two polyhedron $Q$ and $R$, which are not homothetic to $P$. [The figures are captured from https://commons.wikimedia.org/wiki/File:Zentr-streck-pyram-e.svg]

**Definition 2.11** (Equidecomposable) Two polyhedra $P$ and $Q$ are **equidecomposable** if $P, Q$ can be decomposed into a finite number of polyhedra $P_1, ..., P_n$ and $Q_1, ..., Q_n$ respectively such that $P = P_1 \cup P_2 \cup .... \cup P_n,\ Q = Q_1 \cup Q_2 \cup .... \cup Q_n$ and $P_i$ is congruent to $Q_i$ for all $i = 1, 2, 3, .., n$

**Definition 2.12** (Equicomplementable) For two polyhedra $P$ and $Q$, polyhedra $\tilde{P}$ and $\tilde{Q}$ **equicomplementable** if polyhedra $\tilde{P}$ and $\tilde{Q}$ that also have decompositions involving $P$ and $Q$ of the form $\tilde{P} = P \cup P_1' \cup .... \cup P_m'$ and $\tilde{Q} = Q \cup Q_1' \cup .... \cup Q_m'$ where $P_k'$ is congruent to $Q_k'$ for all $k$ $(1 \leqslant k < m)$

**Definition 2.13** ($\mathbf{Q}$-linear map ) $f : \mathbb{R} \to \mathbb{Q}$ is called a $\mathbf{Q}$-linear map if a function from the real numbers to the rationals satisfies three properties:

1. $f(v_1 + v_2) = f(v_1) + f(v_2)$ for all $v_1, v_2 \in \mathbb{R}$;
2. $f(qv) = qf(v)$ for all $q \in \mathbb{Q}, v \in \mathbb{R}$;
3. $f(\pi) = 0$.

For instance, for any $\mathbf{Q}$-linear map $f$, we see that $f(\dfrac{3\pi}{2}) = \dfrac{3}{2} \cdot f(\pi) = \dfrac{3}{2} \cdot 0 = 0$.

Let $P$ be a polyhedron that has edges $e_i (i = 1, 2, ..., n)$ with length $l(e_i)$, and dihedral angles $\alpha(e_i)$, which are the angles of the two faces of $P$ that share the edge $e_i$.

Let $D_f(P) = \sum_{e_i \in P} l(e_i) \cdot f(\alpha(e_i))$, where the summation is taken over all edges $e_i$ of $P$. $D_f(P)$ is called the **Dehn invariant** for a polyhedron $P$. The explanation of how the Dehn invariant is found is as follows:

**Example 2.14** Let $C$ be a unit cube with edges $e_i (i = 1, 2, ..., 12)$ in Fig.2.6 Since $\alpha(e_i) = \frac{\pi}{2}$ for any $i (i = 1, 2, ..., 12)$, the Dehn invariant for a cube as follow:

$$D_f(C) = \textstyle\sum_{e_i \in C} l(e_i) \cdot f(\alpha(e_i)) = 12 \cdot f(\frac{\pi}{2}) = 6f(\pi) = 0$$

**Theorem 2.15** (Dehn's Lemma) *Suppose that a polyhedra P is decomposable into a finite number of smaller polyhedra $P_1, P_2, ..., P_n$ and every dihedral angle of $P_i (i = 1, 2, ..., n)$ belongs to $M$. For any $\mathbf{Q}$-linear map, $f(\pi) = 0$ then* ***Dehn invariant of*** *$P$ is $D_f(P) = D_f(P_1) + D_f(P_2) + ... + D_f(P_n)$*

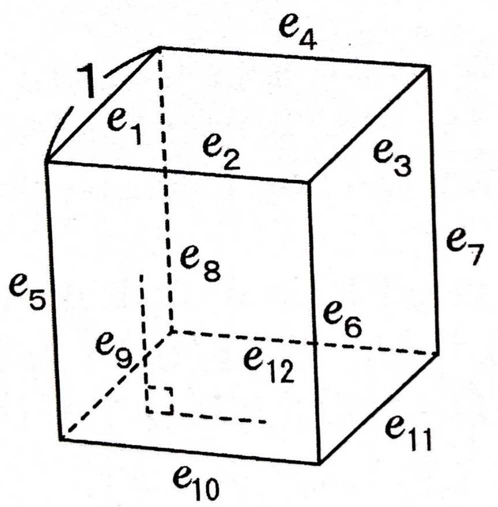


Figure 2.8: A cube $C$ with 12 edges $e_i$ $(i = 1, 2, ..., 12)$ [1]

**Theorem 2.16** For polyhedra $P$ and $Q$ with the same volume. If $D_f(P) \neq D_f(Q)$, then polyhedra $P$ and $Q$ are neither equidecomposable nor equicomplementable.

*Proof.* Let $P$ and $Q$ are equidecomposable or equicomplementable. Then there is a dissection of $P$ into polyhedra $P_1, P_2, ..., P_n$. By the Dehn-invariant theorem,

$$D_f(P) = D_f(P_1) + D_f(P_2) + ... + D_f(P_n) = D_f(Q_1) + D_f(Q_2) + ... + D_f(Q_n) = D_f(Q).$$

Therefore, If $D_f(P) \neq D_f(Q)$, then polyhedra $P$ and $Q$ are neither equidecomposable nor equicomplementable. □

To define a space filla property, we first mention the basic term in a two-dimensional object called **tessellation** as follows:

**Definition 2.17** (A tessellation or tiling) A tessellation or tiling is the covering of a surface, often a plane, using one or more geometric shapes, called tiles, with no overlaps and no gaps.

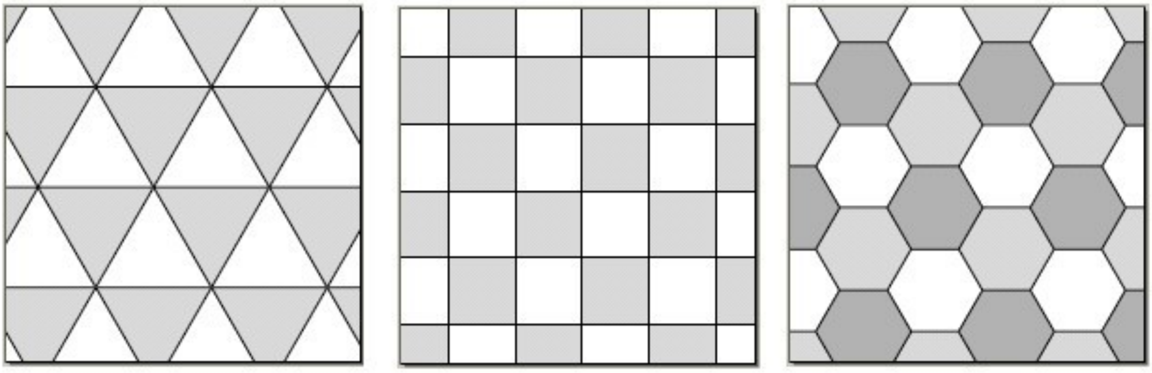

Figure 2.9: Tilling of regular polygons: equilateral triangles, squares, and hexagons. [The figures are captured from https://plus.maths.org/content/trouble-five]

Subsequently, we will elucidate the rationale behind the exclusive utilization of three regular polygons in achieving the tiling of space.

**Theorem 2.18** There are only three regular tessellations, including triangles, squares, and hexagons.

*Proof.* Since the summation of angles at any given point among adjacent polygons equates to $2\pi$ radians and the interior angles in any polygon are $\dfrac{\pi(n-2)}{n}$ where $n$ represents the number of sides in the polygon.

since $\dfrac{\pi(n-2)}{n}$ is a multiple of $2\pi$, then $\dfrac{(n-2)}{n}$ is multiple of 2.
which implies that

$$(n-2) \mid 2n.$$

It means that

$$2n = k(n-2) \Rightarrow k = \frac{2n}{n-2} = \frac{2n-4+4}{n-2} = 2 + \frac{4}{n-2}.$$

Where $k \in \mathbb{N}$, this means that $n-2 \mid 4 \Rightarrow n-2 = 1, 2, 4.$ Therefore, $n = 3, 4, 6,$ which are triangles, squares, and hexagons.

$\square$

**Definition 2.19** (space-filling polyhedron) A space-filling polyhedron is a polyhedron that can be used to generate a tessellation of space.

Examples of space-filling polyhedra are truncated octahedron and rhombicuboctahedron as shown in Figure 2.10.

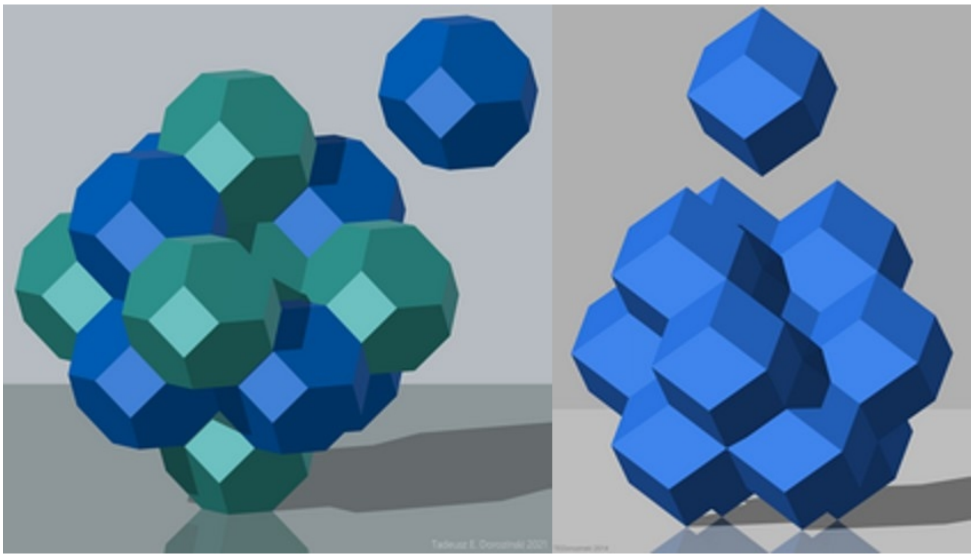

Figure 2.10: Truncated octahedron and rhombicuboctahedron filling space [The figures are adapted from https://polytope.miraheze.org/wiki/File:HC-A4.png]

We remark that Dehn invariant $D_f(P) = 0$ is a necessary but not sufficient condition for a polyhedron to be space-filling.

**Definition 2.20** (parallelohedra) A parallelohedron is a polyhedron that, within 3-dimensional Euclidean space, can be translated without rotations to fill the space in such a manner as to form a honeycomb.

There are 5 parallelohedra as follows: the cube, hexagonal prism, rhombic dodecahedron, elongated dodecahedron, and truncated octahedron.

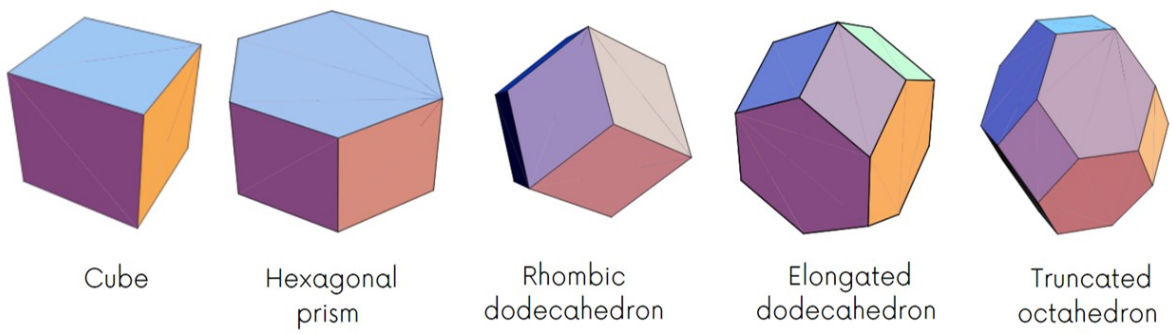


Figure 2.11: The five types of parallelohedron

Moreover, the concept of the Dehn invariant can be extended beyond the confines of a cube to encompass a parallelepiped. In this context, a parallelepiped exhibits a Dehn invariant of 0, as demonstrated by the following elucidation.

**Theorem 2.21** The Dehn invariant of a parallelepiped is equal to 0.

*Proof.* Let $P$ be a parallelepiped edge $e_i(i = 1, 2, ..., 12)$. Since $P$ is a parallelepiped, there exist solely two distinct values for the dihedral angle.

Without loss of generality, let $\alpha(e_i) = \alpha < \frac{\pi}{2}$ where $(i = 1, 2, ..., 6)$ and $\alpha(e_j) = \gamma > \frac{\pi}{2}$ where $(i = 7, 8, ..., 12)$, which is $\alpha(e_i) + \alpha(e_j) = \alpha + \gamma = \pi$.

Where $n < m$ the Dehn invariant for a parallelepiped is :

$$\begin{aligned}
D_f(P) &= \sum_{e_{i \in P}} l(e_i) \cdot f(\alpha(e_i)) \\
&= 4n \cdot f(\alpha(e_i)) + 4n \cdot f(\alpha(e_j)) + 2m \cdot f(\alpha(e_i)) + 2m \cdot f(\alpha(e_j)) \\
&= 4n \cdot (f(\alpha(e_i) + f(\alpha(e_j)) + 2m \cdot (f(\alpha(e_i) + f(\alpha(e_j)) \\
&= 4n \cdot f(\pi) + 2m \cdot f(\pi) \\
&= 0.
\end{aligned}$$

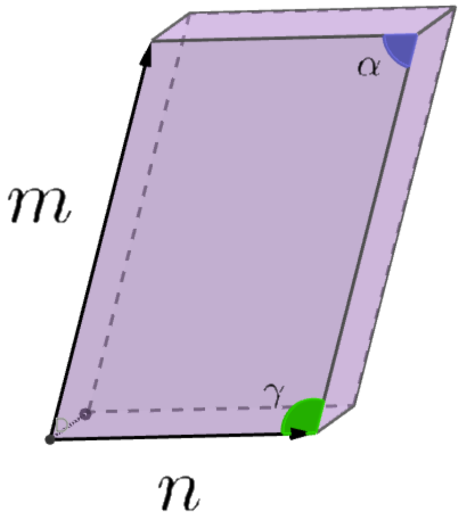


Figure 2.12: A Parallelepiped with the length $m$,$n$, and the dihedral angle $\alpha$, $\gamma$

$\square$

In this study, we also consider finding roots of cubic equations. It is well known that finding the cube root is complicated however, Cardano's method enables us to find all roots as follows. We state it without proof.

Consider a cubic equation with the unknown $x$ and fixed complex coefficients $a, b, c,$ (where $a \neq 0$):

$$ax^3 + bx^2 + cx + d = 0$$

To solve this cubic equation to find $x$, it is convenient to divide both sides by $a$ and complete the first two terms to a full cube $(x + \frac{b}{3a})^3$ In other words, setting

$$y = x + \frac{b}{3a}.$$

we replace $x$ with the simpler equation

$$y^3 + py + q = 0$$

with the unknown $y$ (and some constant coefficients $p$, $q$). However, as any pair of numbers $u$, $v$ satisfies the binomial formula $(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$ , i.e.,

$$(u + v)^3 - 3uv(u + v) - (u^3 + v^3).$$

we will find a solution $y$ in the form

$$y = u + v$$

provided that we have managed to choose (complex) numbers $u, v$ in such a way that

$$p = -3uv$$

and

$$q = (u^3 + v^3).$$

The numbers $u, v$ will also satisfy

$$-\frac{p^3}{27} = u^3v^3.$$

So their cubes $u^3, v^3$ will be the two roots of the quadratic equation

$$t^2 + qt - \frac{p^3}{27} = 0$$

with the unknown $t$; in fact, we have the identity

$$(t - u^3)(t - v^3) = t^2 - (u^3 + v^3)t + u^3v^3$$

Now we obtain the following expressions for all solutions, known as **Cardano's formula:**

$$y_1 = u + v \tag{2.1}$$

$$y_2 = (-\frac{1}{2} + i\frac{\sqrt{3}}{2})u + (-\frac{1}{2} - i\frac{\sqrt{3}}{2})v \tag{2.2}$$

$$y_3 = (-\frac{1}{2} - i\frac{\sqrt{3}}{2})u + (-\frac{1}{2} + i\frac{\sqrt{3}}{2})v \tag{2.3}$$

**Example 2.22** To solve $x^3+6x^2+9x+3=0$.

Let $y=x+\frac{6}{3}=x+2$ rewrite as the simpler equation

$$y^3-3y+1=0$$

we need to find $u, v$ with

$$uv=1,\ u+v=-1$$

and $u^3, v^3$ are the roots of the equation

$$t^2+t+1=0$$

We obtain

$$u^3=\frac{-1+i\sqrt{3}}{2}=e^{\frac{2\pi}{3}},\ v^3=\frac{-1-i\sqrt{3}}{2}=e^{-\frac{2\pi}{3}}$$

Thus

$$u=e^{\frac{2\pi}{9}}$$

and

$$v=e^{-\frac{2\pi}{3}}$$

Hence,

$$y_1=2\cos\frac{2\pi}{9}, y_2=2\cos\frac{8\pi}{9}, y_3=2\cos\frac{4\pi}{9}$$

.

# Chapter 3

# Methodology

To the research conducted by Jin Akiyama, as outlined in reference [1], the proposal entails the utilization of the Double-Reversal-Plates method to effectuate the reversal of each pair of parallelohedra.

**Definition 3.1** (plate) The plate $P$ of a polyhedron $P$ is a dissection of $P$ in such a way that the cut of $P$ corresponds to a net $N$ of $P$

## 3.1 The Double-Reversal-Plates Method for Parallelohedra

In the Double-Reversal-Plates Method, the dissection piece of two polyhedrons is placed on both sides (Head and Tail) of the same net (plate). In this case, the net (plate) for parallelohedra is a parallel pipe (cube).

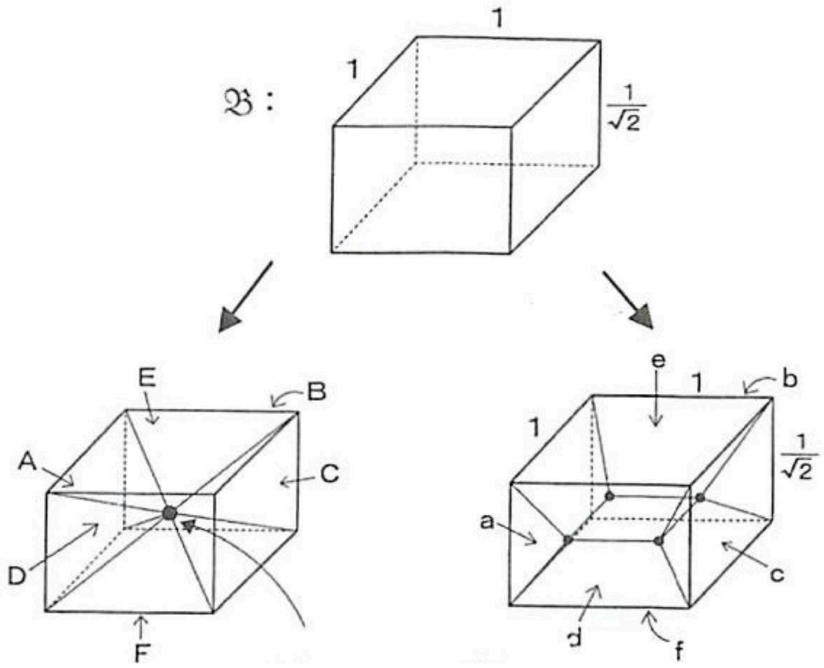


Figure 3.1: For each of the two plate sides, there are two dissections of the box. [1]

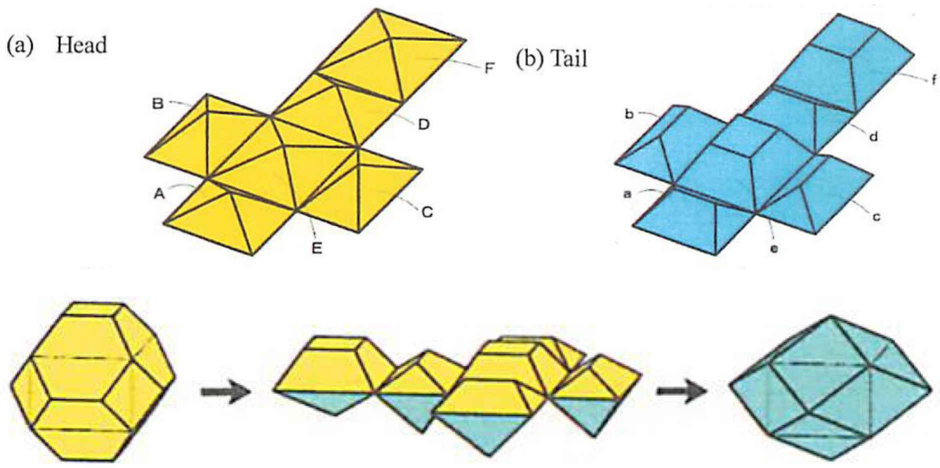


Figure 3.2: Example of the Double-Reversal-Plates Method [1]

According to the concept proposed by Professor Akiyama Jin, the $(B, O)$-chimera superimposition refers to the resultant dissection obtained upon overlaying $B$ (box) and $O$ (octahedron).

Nest, the methodology for dissecting reversible parallelohedra pairs will be demonstrated.

### 3.1.1 A Ham (Box) to A Pig (Truncated Octahedron)

Initially, the volumes of a pig and a ham are the same; that is, a truncated octahedron with side length $\frac{\sqrt{2}}{2}$ and a box of size $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$ have volume = $2\sqrt{2}$. By an $(B, O)$-chimera superimposition, then each copy of a pig (truncated octahedron) and each copy of $B$ must be dissected into $P_1, P_2, P_3, ..., P_6$ (Fig. 3.3). The nets of a truncated octahedron and a box are considered by chimera superimposition, culminating in the implementation of a hinged dissection technique.

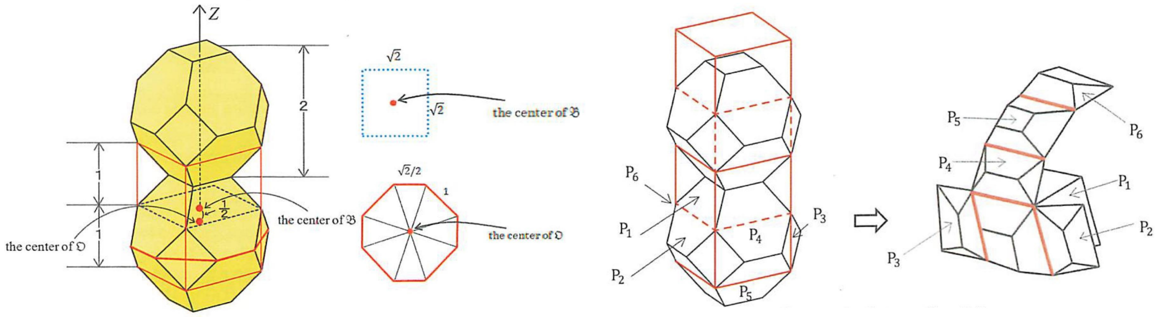


Figure 3.3: An $(B, O)$-chimera superimposition gives the dissections. [1]

Then we got the dissection of a pig (a truncated octahedron) into the box (Fig. 3.4).

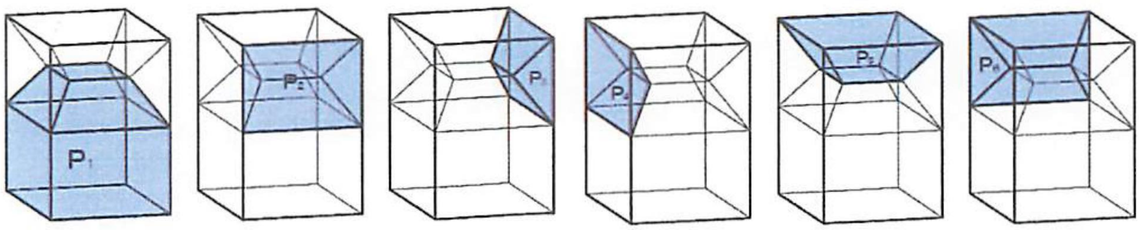

Figure 3.4: The dissected pieces of a pig into a ham [1].

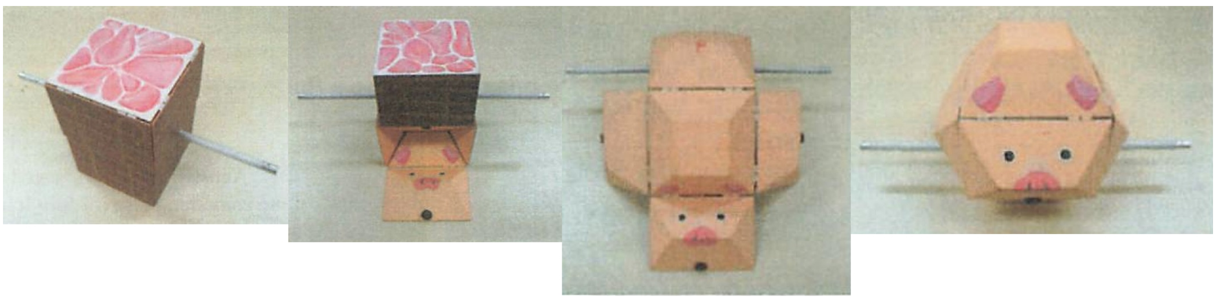

Figure 3.5: A ham to a pig [1].

Thus, the reversible pair of a pig (truncated octahedron) and a ham (box) can be obtained as Figure 3.5.

### 3.1.2 A Ham (Box) to A Fox (Rhombic Dodecahedron)

Let us examine a reversible pair of a fox and a ham, a Rhombic Dodecahedron $R$ with a side length $\frac{\sqrt{3}}{2}$ and the box $B$ of $1 \times 1 \times 2$ have volume = 2. So, in an $(B, O)$-chimera superimposition, each $R$ and $B$ are dissected into six pieces.

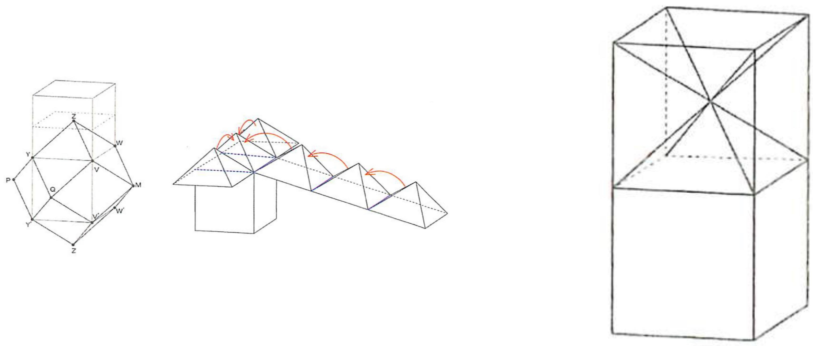

Figure 3.6: The dissected pieces of a $R$ into $B$ [1]

### 3.1.3 A Ham (Box) to A Honeycomb (Hexagonal Prism)

The volume of the hexagonal prism and the box is the same as in the preceding instance. Then, superimposed to create a superimposition of (B,O) chimeras. A $(B, O)$-chimera superimposition guides on dissection.

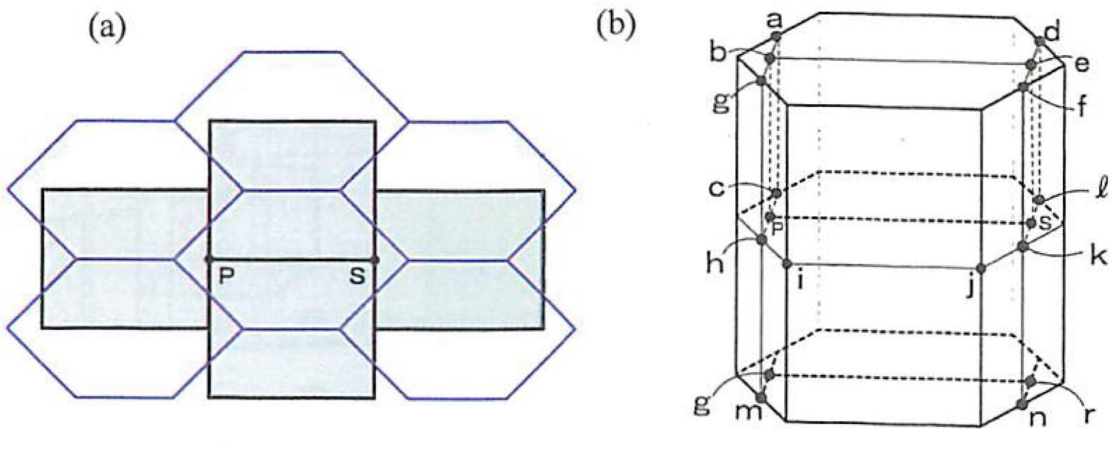


Figure 3.7: A $(B, O)$-chimera superimposition gives 8 pieces of dissection. [1]

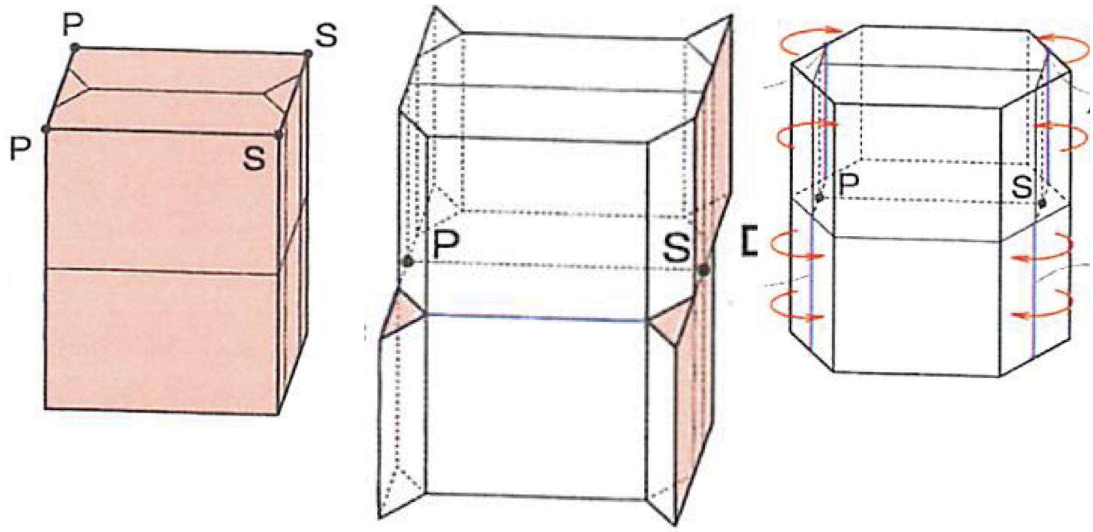


Figure 3.8: A ham to a honeycomb [1]

## 3.2 Properties of Reversibility using the Double-Reversal-Plates Method

We know that the box and the considered solid have **the same volume** from the earlier examples. There are two parts of the box when we superimposed it over the considered solid: the part that is inscribed in the solid (blue part in Fig. 3.9) and the part that is not engraved in the solid under consideration (green part in Fig. 3.9).

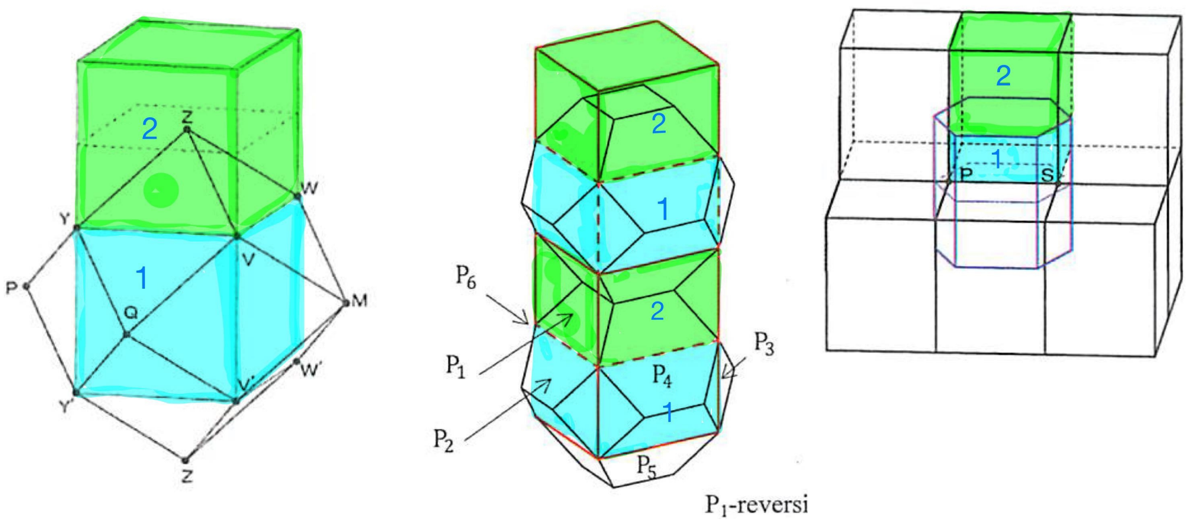


Figure 3.9: Two parts of the box when superimpose with the considered solid [1]

We will refer to the blue portion as **the inscribed box** $I$ going forward. Based on observation, the volume of the two parts is equal.

We denote the notation $V_{\text{inscribed}}$ as the volume of the inscribed polyhedron, and $V_{\text{considered}}$ as the volume of the considered solid. The following lemma is implied as follows:

**Lemma 3.2** If the volume $V_{\text{inscribed}}$ is not equal to the half of $V_{\text{considered}}$, then the polyhedron can not be applied to the Double-Reversal-Plates Method.

We can determine whether the considered solid is capable of using the Double-Reversal Plates Method based on **Lemma 3.2.** We will provide some example cases after that.

### 3.2.1 Octahedron

Without loss of generality, assume that an octahedron is oriented on the first octant, i.e. it is placed on the positive side of $X, Y, Z$ axis in three-dimensional space. Therefore, we assume that each edge has a length equal to $\sqrt{2}$, and each vertex has coordinates as follows: $A(0,0,0), B(0,\sqrt{2},0), C(\sqrt{2},0,0), D(\sqrt{2},\sqrt{2},0), E(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},1)$.

Let us now proceed to ascertain the maximum volume achievable for the inscribed box within the octahedron.

**Theorem 3.3** There is no inscribed box of an octahedron satisfying the condition of Lemma 3.2.

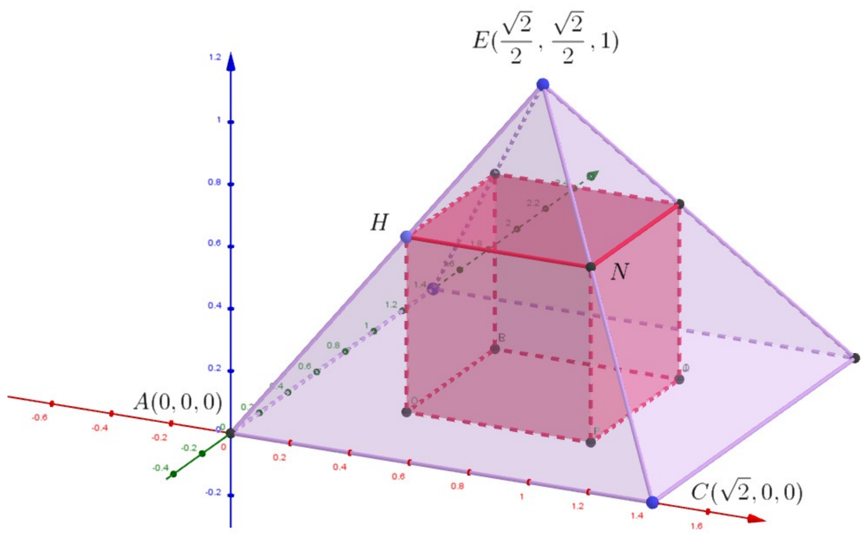


Figure 3.10: Stimulation of Octahedron with side length $\sqrt{2}$ inscribed box

*Proof.* Consider an octahedron with length $\sqrt{2}$. Since the octahedron exhibits symmetry concerning the $xy$ plane, without loss of generality, we will focus our consideration on the maximum rectangular box contained within one-half of the octahedron.

Firstly, select a single point $(H)$ on $\overline{AE} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\rangle$ ; $0 < t < 1$

Since $\triangle AEC \sim \triangle EHC$ and $AE = AC$, we can obtain that $EH = HN$.

Hence, $HN$=$\sqrt{2} - t\sqrt{(\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2 + 1^2} = \sqrt{2} - t\sqrt{2}$.

Since the base is square and the height is $t$, the volume of an inscribed box is $V_i = t(\sqrt{2} - t\sqrt{2})^2$.

We would find the maximum volume of an inscribed box of the upper octahedron. Therefore, we find the derivative of $V_i$ concerning $t$; i.e., we find t such that $\frac{dV}{dt} = 0$.

Therefore,

$$\begin{aligned}
\frac{dV_i}{dt} = 2(3t^2 - 4t + 1) &= 0 \\
t &= \frac{4 \pm \sqrt{16 - 12}}{6} \\
t &= \frac{4 \pm 2}{6} = 1, \frac{1}{3}
\end{aligned}$$

Given that $t < 1$, $t = \frac{1}{3}$. The maximum volume is $V_i = \frac{1}{3}(\sqrt{2} - \frac{\sqrt{2}}{3})^2$ and the volume of one half of the octahedron with the edge length $\sqrt{2}$ is $V_O = (\sqrt{2})^3\frac{\sqrt{2}}{6}$

$$\frac{V_i}{V_O} = \frac{\frac{1}{3}(\sqrt{2} - \frac{\sqrt{2}}{3})^2}{(\sqrt{2})^3\frac{\sqrt{2}}{6}} = \frac{4}{9} < \frac{1}{2}$$

Therefore, the maximum volume of a box inscribed within an octahedron is strictly less than half of the volume of the octahedron.

$\square$

In this part, we conclude that the octahedron is unable to use the Double-Reversal Plates method because the maximum volume of the inscribed box is less than the haft of the octahedron's volume.

## 3.3 Properties of Reversibility for Polyhedra

By Lemma 3.2 we can also conclude that an octahedron is not reversible to a rectangular box since the octahedron in the previous example has an unequal **Dehn invariant**, it cannot be reversed to be a parallel pipe (box). In the study *"On Angles Whose Squared Trigonometric Functions are Rational"* [2], the Dehn Invariant of Archimedean Solid is shown in Figure 3.10.

| | |
|---|---|
| Tetrahedron | $-12\langle 3\rangle_2$ |
| Truncated tetrahedron | $12\langle 3\rangle_2$ |
| Cube | $0$ |
| Truncated cube | $-24\langle 3\rangle_2$ |
| Octahedron | $24\langle 3\rangle_2$ |
| Truncated octahedron | $0$ |
| Rhombicuboctahedron | $-24\langle 3\rangle_2$ |
| Cuboctahedron | $-24\langle 3\rangle_2$ |
| Truncated cuboctahedron | $0$ |
| Icosahedron | $60\langle 3\rangle_5$ |
| Truncated icosahedron | $30\langle 5\rangle_1$ |
| Dodecahedron | $-30\langle 5\rangle_1$ |
| Truncated dodecahedron | $-60\langle 3\rangle_5$ |
| Rhombicosidodecahedron | $60\langle 3\rangle_5 - 30\langle 5\rangle_1$ |
| Icosidodecahedron | $-60\langle 3\rangle_5 + 30\langle 5\rangle_1$ |
| Truncated icosidodecahedron | $0$ |

Figure 3.11: The Dehn invariant for the non-snub unit edge Archimedean polyhedra [2]

Beside the space filler group, which is the class of Dehn Invariant equal to $0$, as you can see in Fig. 3.11., the Dehn Invariant of the **truncated cube, rhombicuboctahedron, and cuboctahedron** are the same. The purpose of this study is to apply the double reversible method to other polyhedra where the Dehn invariant is not equal to 0.

It is established that the Dehn invariant of a parallelepiped is equal to zero. The subsequent inquiry pertains to whether the Dehn invariant of a prism also equals zero. This prompts further investigation into the Dehn invariant of prisms.

### 3.3.1 Dehn invariant of prisms

A point worth mentioning is the concept of **tilling properties**, These are the three regular tilings, which are each made up of identical copies of a regular polygon: **equilateral triangles, squares, and hexagons**, as mentioned in Theorem 2.18.

It can now be established that parallelohedra and prisms are unsuitable candidates as plates for non-space-filling purposes due to their Dehn invariant being equal to zero.

**Lemma 3.4** A prism $P_m$ with polygon $P$ has a Dehn invariant equal to 0.

*Proof.* Let $P_m$ denote a unit prism comprising regular polygonal faces and tilling space, where $e_i(i = 1, 2, ..., n)$ represents the edge of the non-square polygonal face $P$, and

$e_j(i = 1, 2, ..., m)$ denotes as the edge of the square polygonal face. Since $\alpha(e_i) = \frac{\pi}{2}$ for any $i(i = 1, 2, ..., n)$, then we can obtain that

$$\begin{aligned} D_f(P_m) &= \sum_{e_i \in P} l(e_i) \cdot f(\alpha(e_i)) + \sum_{e_j \in P} l(e_i) \cdot f(\alpha(e_j)) \\ &= n \cdot f(\frac{\pi}{2}) + D_f(P) \\ &= 0 + D_f(P) \\ &= D_f(P). \end{aligned}$$

Consider $D_f(P) = n \cdot f(\frac{\pi(n-2)}{n}) = f((n-2)\pi) = (n-2) \cdot f(\pi) = 0.$
Hence, a prism $P_m$ with polygon $P$ has a Dehn invariant equal to 0.

$\square$

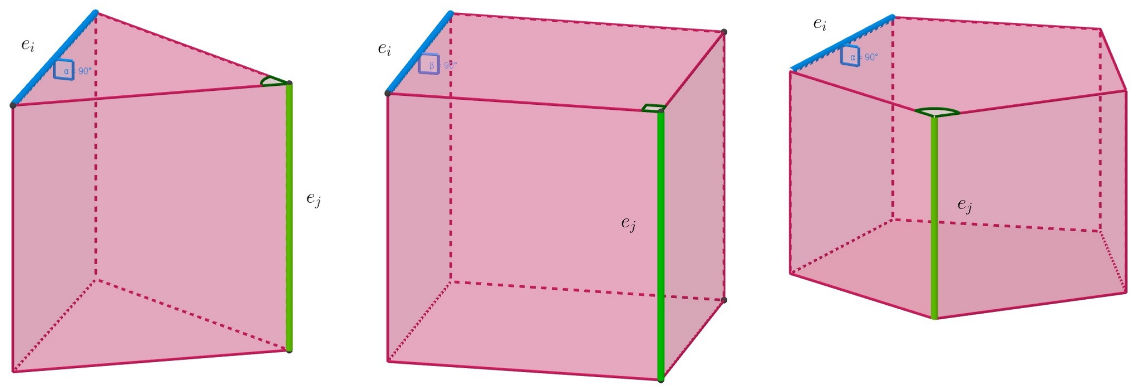


Figure 3.12: $e_i$ is represented by the blue line, and $e_j$ by the green line.

### 3.3.2 Equality of volume

The study by Akiyama [1] explored the idea of reversibility among parallelohedra, especially those are space-filling. A key question raised was whether non-space-filling can also be reversible. Based on previous knowledge suggesting that polyhedra with similar Dehn invariants can be reversible, this study aims to find another condition for reversibility. Akiyama's study [1] showed that the relationships between the lengths of edges in polyhedra go beyond just having the same volume. Therefore, this investigation will look into how these length relationships play out, using examples like the rhombic dodecahedron and parallelepiped.

**The volume relation between a rhombic dodecahedron and a cube**

Rhombic dodecahedron has a side length equal to $\frac{\sqrt{3}}{2}$ and a box of $1 \times 1 \times 2$ which is 2 unit cubes. Since Volume of rhombic dodecahedron : $V_R = \frac{16\sqrt{3}a^3}{9}$ and $V_{cube} = b^3$; where $a, b$ are edge sides of rhombic dodecahedron and cube respectively, we will consider $V_R = 2V_{cube}$ (from the lemma 3.2). Therefore, $a = k\sqrt{3}$ where $k \in \mathbb{Q}$. Then $V_R = \frac{16\sqrt{3}a^3}{9} = 16k^3 = 2b^3$. Thus $8k^3 = b^3$ or $2k = b$. Without loss of generality, let $b = 1$, then we can obtain that $a = \frac{\sqrt{3}}{2}$ when $b = 1$.

This example implies that the situation in which $V_R = 2V_C$ occurs when $a = \frac{\sqrt{3}}{2}$ if $b = 1$.

Next, we will ascertain the relationships between the edges of the candidate polyhedra.

As we previously stated in Figure 3.11, the Dehn Invariant of the truncated cube, rhombicuboctahedron, and cuboctahedron are the same. Therefore, we can consider $P$ and $Q$ to be reversible if they have the same volume, We will consider the equality of the volume of the pairs of these three polyhedra.

$$V_{Cuboctahedron}; V_C = \frac{5\sqrt{2}a^3}{3} \tag{3.1}$$

$$V_{Rhombicuboctahedron}; V_R = \frac{2}{3}b^3(6 + 5\sqrt{2}) \tag{3.2}$$

$$V_{TruncatedCube}; V_T = (7 + \frac{14\sqrt{2}}{3})c^3 \tag{3.3}$$

A number of $a$ is said to be a pure irrational number if $a$ is written as a multiple of a root of a number without any addition of rational numbers.

We denote the notation $\mathbb{Q}'_*$ as a set of impure irrational numbers. For example, $a_1 = b\sqrt{2}$ is said to be an impure irrational number whereas $a_2 = c + d\sqrt{2}$ such that $c, d \in \mathbb{Q}$ is an impure irrational number but not a pure irrational number.

**Case 1.** The pair of cuboctahedron and rhombicuboctahedron

**Lemma 3.5** For a volume $V_C$ as defined in (3.1) and $V_R$ as defined in (3.2) there is no $b \in \mathbb{Q}$ and $b \in \mathbb{Q}'_*$ whereas $a \in \mathbb{Q}$ such that $V_C = kV_R$ or $V_R = kV_C$ for some integer $k$.

*Proof.* Assume that there exists $b \in \mathbb{Q}$ and $b \in \mathbb{Q}'_*$ satisfies the condition $V_C = kV_R$ or $V_R = kV_C$. Since $a \in \mathbb{Q}$, $V_C = \dfrac{5\sqrt{2}a^3}{3}$ is a pure irrational number, which means that $kV_R$ is an irrational number for all integer $k$ satisfying the condition. Without loss of generality, assume that $k = 1$. Consider $V_R = \dfrac{2}{3}b^3(6 + 5\sqrt{2}) = \dfrac{5\sqrt{2}a^3}{3} = V_C$

In the case that $b \in \mathbb{Q}$, then $V_R \in \mathbb{Q}'_*$, **a contradict to** $V_R$ the fact that is an irrational number.

Suppose that $b \in \mathbb{Q}'_*$ then $b = r + q$ and $r \in \mathbb{Q}, q \in \mathbb{Q}'$. The fact that $V_R = \frac{2}{3}b^3(6 + 5\sqrt{2}) = (r^3 + 3r^2q + 3rq^2 + q^3)(6 + 5\sqrt{2})\frac{2}{3}$. From the equation, at least $r^3 \in \mathbb{Q}$ and $3r^2q \in \mathbb{Q}'$, then $b^3 \in \mathbb{Q}'_*$, That means $V_R \in \mathbb{Q}'_*$, **a contradict to** the fact that $V_R$ is an pure irrational number.

Hence, there is no $b \in \mathbb{Q}$ and $b \in \mathbb{Q}'_*$ satisfies the condition $V_C = kV_R$ or $V_R = kV_C$ for some integer $k$ if $a \in \mathbb{Q}$.

$\square$

**Lemma 3.6** For a volume $V_C$ as defined in (3.1) and $V_R$ as defined in (3.2) there is no $b \in \mathbb{Q}$ and $b \in \mathbb{Q}'_*$ whereas $a \in \mathbb{Q}'$ such that $V_C = kV_R$ or $V_R = kV_C$ for some integer $k$.

*Proof.* Assume that there exists $b \in \mathbb{Q}$ and $b \in \mathbb{Q}'_*$ satisfies the condition $V_C = kV_R$ or $V_R = kV_C$. Without loss of generality, assume that $k = 1$.

Since $a \in \mathbb{Q}'$, in the case of $a = \sqrt{2}$ then $V_C \in \mathbb{Q}$. Thus $V_R = \dfrac{2}{3}b^3(6+5\sqrt{2})$. Obviously, there is no $b \in \mathbb{Q}$ and $b \in \mathbb{Q}'_*$ that $V_R \in \mathbb{Q}$

In the case of $a \neq \sqrt{2}$, then $V_C \in \mathbb{Q}'$ so $V_R \in \mathbb{Q}'$.

Suppose that $b \in \mathbb{Q}$ then $V_R \in \mathbb{Q}'_*$, **a contradict to** the fact that $V_R \in \mathbb{Q}'$.

Assume that $b \in \mathbb{Q}'_*$ then $b = r + q \ : r \in \mathbb{Q}, q \in \mathbb{Q}'$ Therefore, $V_R = \frac{2}{3}b^3(6 + 5\sqrt{2}) = (r^3 + 3r^2q + 3rq^2 + q^3)(6 + 5\sqrt{2})\frac{2}{3}$. From the equation, at least $r^3 \in \mathbb{Q}$ and $3r^2q \in \mathbb{Q}'$, then $b^3 \in \mathbb{Q}'_*$, which means $V_R \in \mathbb{Q}'_*$, **a contradict to** the fact that $V_R \in \mathbb{Q}'$.

Thus, there is no $b \in \mathbb{Q}$ and $b \in \mathbb{Q}'_*$ satisfies the condition $V_C = kV_R$ or $V_R = kV_C$ for some integer $k$ if $a \in \mathbb{Q}'$.

$\square$

Initially, we establish that certain values of $a \notin \mathbb{Q}$ and $a \notin \mathbb{Q}'$. Then we can obtain that $a \in \mathbb{Q}'_*$ Subsequently, we propose an assumption regarding the permissible range of values for $a = p + q\sqrt{2}$ based on $V_R$ has the term of $\sqrt{2}$. Next, we will formally explore the relationship between the volumes of $V_C$ and $V_R$ and a specific parameter $a = p + q\sqrt{2}$.

**Lemma 3.7** For a volume $V_C$ as defined in (3.1) and $V_R$ as defined in (3.2) there is no $a \in \mathbb{Q}'_* : a = p + q\sqrt{2}$, where $p, q \in \mathbb{Q}$ and $p, q \neq 0$ such that $V_C = kV_R$ or $V_R = kV_C$ for some integer $k$.

*Proof.* Assume that there exists $a = p + q\sqrt{2}\ ;\ p, q \neq 0$, then $V_C = \dfrac{5\sqrt{2}(p + q\sqrt{2})^3}{3}$. Remark that $V_C = \dfrac{5\sqrt{2}(p^3 + 3p^2q\sqrt{2} + 6pq^2 + 2q^3\sqrt{2})}{3} = \dfrac{5\sqrt{2}}{3}[(p^3 + 6pq^2) + (3p^2q\sqrt{2} +$

$2q^3\sqrt{2})]$. Without loss of generality, assume that $k = 1$ then $V_C = \frac{5\sqrt{2}(p+q\sqrt{2})^3}{3} = \frac{2k(6b^3+5b^3\sqrt{2})}{3} = V_R$. Equating the corresponding coefficients now results in this:

1. In the case of the term of $\sqrt{2}$ :

$$\frac{5\sqrt{2}(p^3+6pq^2)}{3} = \frac{5\sqrt{2}(2kb^3)}{3} \tag{3.4}$$

2. In the case of the term without $\sqrt{2}$ :

$$\frac{30p^2q+20q^3}{3} = \frac{12kb^3}{3} \tag{3.5}$$

Solving it results in:

$$p^3+6pq^2 = 2kb^3 \tag{3.6}$$

$$15p^2q+10q^3 = 6kb^3 \tag{3.7}$$

From (3.6) and (3.7), we can obtain that $3p^3+18pq^2 = 15p^2q+10q^3$ which means that $3p^3-15p^2q+18pq^2-10q^3 = 0$

From Cardano's method, Let $p = x+\frac{5q}{3}$ then we can obtain the reduced form $3x^3 - 7q^2x - \frac{70q^3}{9} = 0$, Let $x = u+v$ rewrite above equation as follow:

$$\begin{aligned}
3(u+v)^3-7q^2(u+v)-\frac{70q^3}{9} &= 0 && (3.8)\\
3u^3+3v^3+9u^2v+9uv^2-7q^2(u+v)-\frac{70q^3}{9} &= 0 && (3.9)\\
3u^3+3v^3+(u+v)(3uv-7q^2)-\frac{70q^3}{9} &= 0 && (3.10)
\end{aligned}$$

Setting $3uv-7q^2 = 0$, the above equation become $3u^3+3v^3-\frac{70q^3}{9} = 0$ in this way, we can obtain the system below

$$u^3+v^3 = \frac{70q^3}{27} \tag{3.11}$$

$$u^3v^3 = \frac{343q^6}{27} \tag{3.12}$$

It allows us to identify a quadratic equation with $u^3$ and $v^3$ as its roots. This equation is :

$$t^2-\frac{70q^3t}{27}+\frac{343q^6}{27} = 0 \tag{3.13}$$

Where $u^3 = q^3(\frac{70}{27}+\sqrt{\frac{4900}{729}-\frac{1372}{27}})$ and $v^3 = q^3(\frac{70}{27}-\sqrt{\frac{4900}{729}-\frac{1372}{27}})$

Thus $x = q(\sqrt[3]{\frac{70}{27} - \sqrt{\frac{4900}{729} - \frac{1372}{27}}} + \sqrt[3]{\frac{70}{27} + \sqrt{\frac{4900}{729} - \frac{1372}{27}}})$

Therefore $p = q(\sqrt[3]{\frac{70}{27} - \sqrt{\frac{4900}{729} - \frac{1372}{27}}} + \sqrt[3]{\frac{70}{27} + \sqrt{\frac{4900}{729} - \frac{1372}{27}}} + \frac{5}{3})$

since $\sqrt{\frac{4900}{729} - \frac{37044}{729}}$ is a complex number, **a contradict to** $p \in \mathbb{Q}$.

Hence, there is no $a \in \mathbb{Q}'_* : a = p + q\sqrt{2}$, where $p, q \neq 0$ such that $V_C = kV_R$ or $V_R = kV_C$ for some integer $k$. For a volume $V_C$ as defined in (3.1) and $V_R$ as defined in (3.2). □

Based on our comprehensive analysis, it has been established that there is no discernible correlation between the lengths of cuboctahedron and rhombicuboctahedron. Consequently, we can infer that employing the Double-Reversal-Plates Method is not feasible for a pair of cuboctahedron and rhombicuboctahedron.

**Case 2.** The pair of cuboctahedron and truncated cube

**Lemma 3.8** For a volume $V_C$ as defined in (3.1) and $V_T$ as defined in (3.3) there is no $b \in \mathbb{Q}$ and $b \in \mathbb{Q}'_*$ whereas $a \in \mathbb{Q}$ such that $V_C = kV_T$ or $V_T = kV_C$ for some integer $k$.

*Proof.* Assume that there exists $c \in \mathbb{Q}$ and $c \in \mathbb{Q}'$ such that satisfies these conditions : $V_C = kV_T$ or $V_T = kV_C$ Without loss of generality, assume that $k = 1$.

Since $a \in \mathbb{Q}$, $V_C = \dfrac{5\sqrt{2}a^3}{3} \in \mathbb{Q}'$ which means that $V_T \in \mathbb{Q}'$. Therefore, $V_T = (7 + \dfrac{14\sqrt{2}}{3})c^3 = \dfrac{5\sqrt{2}a^3}{3} \in \mathbb{Q}'$ .

In the case that $c \in \mathbb{Q}$, then $V_T \in \mathbb{Q}'_*$, **a contradict to** the fact that $V_T \in \mathbb{Q}'$.

Suppose that $c \in \mathbb{Q}'_*$ then $c = r + q\ : r \in \mathbb{Q}, q \in \mathbb{Q}'$ .

Thus,

$$V_T = (7 + \frac{14\sqrt{2}}{3})c^3 = (7 + \frac{14\sqrt{2}}{3})(r^3 + 3r^2q + 3rq^2 + q^3)$$

. From the equation, at least $r^3 \in \mathbb{Z}$ and $3r^2q \in \mathbb{Q}'$, then $b^3 \in \mathbb{Q}'_*$, which means that $V_T \in \mathbb{Q}'_*$, **a contradict to** the fact that $V_T \in \mathbb{Q}'$.

Therefore, there is no $b \in \mathbb{Q}$ and $b \in \mathbb{Q}'_*$ whereas $a \in \mathbb{Q}$ satisfies these conditions : $V_C = kV_R$ or $V_R = kV_C$ for some integer $k$.

□

**Lemma 3.9** For a volume $V_C$ as defined in (3.1) and $V_T$ as defined in (3.3) there is no $b \in \mathbb{Q}$ and $b \in \mathbb{Q}'_*$ whereas $a \in \mathbb{Q}'$ such that $V_C = kV_T$ or $V_T = kV_C$ for some integer $k$.

*Proof.* Assume that there exists $c \in \mathbb{Q}$ and $c \in \mathbb{Q}'$ such that satisfies these conditions : $V_C = kV_T$ or $V_T = kV_C$ Without loss of generality, assume that $k = 1$.

In the case that $a = \sqrt{2}$, then $V_C \in \mathbb{Q}$. Thus, $V_T = (7 + \frac{14\sqrt{2}}{3})c^3$. Obviously, there is no $c \in \mathbb{Q}$ and $c \in \mathbb{Q}'$ that $V_R \in \mathbb{Q}$

In the case that $a \neq \sqrt{2}$, then $V_C \in \mathbb{Q}'$ so $V_T \in \mathbb{Q}'$ .

Suppose that $c \in \mathbb{Q}$ then $V_T \in \mathbb{Q}'_*$, **a contradict to** the fact that $V_T \in \mathbb{Q}'$.

Assume that $c \in \mathbb{Q}'_*$ then $c = r + q \ : r \in \mathbb{Q}, q \in \mathbb{Q}'$. Thus,

$$V_T = (7 + \frac{14\sqrt{2}}{3})c^3 = (7 + \frac{14\sqrt{2}}{3})(r^3 + 3r^2q + 3rq^2 + q^3)$$

. From the equation, at least $r^3 \in \mathbb{Z}$ and $3r^2q \in \mathbb{Q}'$, then $b^3 \in \mathbb{Q}'_*$, which means that $V_T \in \mathbb{Q}'_*$, **a contradict to** the fact that $V_T \in \mathbb{Q}'$.

Therefore, there is no $b \in \mathbb{Q}$ and $b \in \mathbb{Q}'_*$ whereas $a \in \mathbb{Q}'$ such that $V_C = kV_T$ or $V_T = kV_C$ for some integer $k$. □

Following a similar line of reasoning as in Case 1, we deduce that $a = p + q\sqrt{2}$ based on $V_T$ having the term of $\sqrt{2}$. Next, we will formally explore the relationship between the volumes of $V_C$ and $V_T$ with $a = p + q\sqrt{2}$.

**Lemma 3.10** For a volume $V_C$ as defined in (3.1) and $V_T$ as defined in (3.3) there is no $a \in \mathbb{Q}'_* : a = p + q\sqrt{2}$, where $p, q \in \mathbb{Q}$ and $p, q \neq 0$ such that $V_C = kV_T$ or $V_T = kV_C$ for some integer $k$.

*Proof.* Assume that there exists $a = p+q\sqrt{2}\,; p, q \neq 0$ then $V_C = \dfrac{5\sqrt{2}(p + q\sqrt{2})^3}{3}$. Without loss of generality, assume that $k = 1$ then $V_C = V_T$. Thus,

$$V_C = \frac{5\sqrt{2}(p + q\sqrt{2})^3}{3} = (7 + \frac{14\sqrt{2}}{3})kc^3 = V_T$$

. Equating the corresponding coefficients now results in this:

1. In the case of the term $\sqrt{2}$ :

$$\frac{5\sqrt{2}(p^3 + 6pq^2)}{3} = \frac{14\sqrt{2}(kc^3)}{3} \tag{3.14}$$

2. In the case of the term without $\sqrt{2}$ :

$$\frac{30p^2q + 20q^3}{3} = 7kc^3 \tag{3.15}$$

Solving it results in:

$$5p^3 + 30pq^2 = 14kc^3 \tag{3.16}$$

$$30p^2q + 20q^3 = 21kc^3 \tag{3.17}$$

From the results, we can obtain that $\frac{3}{2}(5p^3 + 30pq^2) = 30p^2q + 20q^3$ which means that

$$3p^3 - 12p^2q + 18pq^2 - 8q^3 = 0 \tag{3.18}$$

From Cardano's method, Let $p = x + \frac{4q}{3}$ then we can obtain the reduced form $3x^3 + 2q^2x + \frac{16q^3}{9} = 0$, Letting $x = u + v$ rewrite above equation as follows:

$$3(u+v)^3 + 2q^2(u+v) + \frac{16q^3}{9} = 0 \tag{3.19}$$

$$3u^3 + 3v^3 + 9u^2v + 9uv^2 + 2q^2(u+v) + \frac{16q^3}{9} = 0 \tag{3.20}$$

$$3u^3 + 3v^3 + (u+v)(9uv + 2q^2) + \frac{16q^3}{9} = 0 \tag{3.21}$$

Setting $9uv + 2q^2 = 0$, the above equation become $3u^3 + 3v^3 + \frac{16q^3}{9} = 0$ in this way, we can obtain the system below:

$$u^3 + v^3 = -\frac{16q^3}{27} \tag{3.22}$$

$$u^3v^3 = -\frac{8q^6}{729} \tag{3.23}$$

It allows us to identify a quadratic equation with $u^3$ and $v^3$ as its roots. This equation is :

$$t^2 + \frac{16q^3t}{27} - \frac{8q^6}{729} = 0 \tag{3.24}$$

Where $u^3 = q^3(-\frac{16}{27} + \sqrt{\frac{256}{729} + \frac{32}{729}})$ and $v^3 = q^3(-\frac{16}{27} - \sqrt{\frac{256}{729} + \frac{32}{729}})$

Thus $x = q(\sqrt[3]{-\frac{16}{27} - \frac{\sqrt{288}}{27}} + \sqrt[3]{-\frac{16}{27} + \frac{\sqrt{288}}{27}}$

Therefore $p = \frac{q}{3}(\sqrt[3]{-16 - \sqrt{288}} + \sqrt[3]{-16 + \sqrt{288}})$, Thus $p$ is an irrational number, **a contradicts to** $p \in \mathbb{Q}$.

Hence, there is no $a \in \mathbb{Q}'_* : a = p + q\sqrt{2}$, where $p, q \in \mathbb{Q}$ and $p, q \neq 0$ such that $V_C = kV_T$ or $V_T = kV_C$ for some integer $k$.

$\square$

Based on the evidence presented in the proof above, we conclude that both cuboctahedron and truncated cube are not suitable candidates for the application of the Double-Reversal-Plates Method.

**Remarks on the case of rhombicuboctahedron and truncated cube**

When we employ a similar strategy to the previous cases, we found that $c^3 = \dfrac{2b^3}{7}(-2 + 3\sqrt{2})$ as follows:

*Proof.* Consider $V_T = (7 + \frac{14\sqrt{2}}{3})c^3$ and $V_R = \frac{2}{3}b^3(6 + 5\sqrt{2})$

Assume that $V_T = V_R$ then

$$
\begin{aligned}
(7 + \frac{14\sqrt{2}}{3})c^3 &= \frac{2}{3}b^3(6 + 5\sqrt{2}) \\
\frac{(7 + \frac{14\sqrt{2}}{3})c^3}{\frac{2}{3}b^3(6 + 5\sqrt{2})} &= 1 \\
\frac{c^3}{b^3} &= \frac{\frac{2}{3}(6 + 5\sqrt{2})}{(7 + \frac{14\sqrt{2}}{3})} \\
\frac{c^3}{b^3} &= \frac{(12 + 10\sqrt{2})}{(21 + 14\sqrt{2})} \\
\frac{c^3}{b^3} &= \frac{2}{7}\frac{(6 + 5\sqrt{2})}{(3 + 2\sqrt{2})} \\
\frac{c^3}{b^3} &= \frac{2}{7}(6 + 5\sqrt{2})(3 - 2\sqrt{2}) \\
\frac{c^3}{b^3} &= \frac{2}{7}(-2 + 3\sqrt{2})
\end{aligned}
$$

$\square$

Due to $V_R$ and $V_T$ being components of the same term, no definitive conclusions can be drawn. However, based on the orientation analysis between truncated cube and rhombic cuboctahedron, we have a conjecture that they are unsuitable candidates for the proposed method.

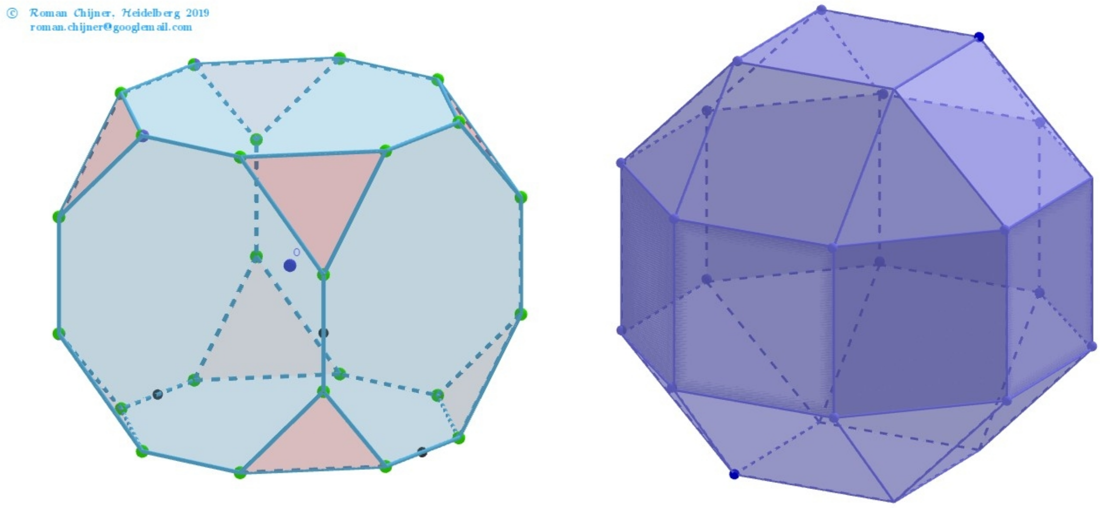


Figure 3.13: The picture of truncated cube and rhombic cuboctahedron [The figures are adapted from https://www.geogebra.org/m/rgqyn3vt]

# Chapter 4

# Conclusion

Firstly, our project thoroughly examined the Double-Reversal-Plates Method to see if it could work for different polyhedra. We developed a criterion called Lemma 3.2 to help us decide if a polyhedron could be dissected effectively using this method.

Secondly, we explored the idea of reversibility among non-space-filling polyhedra, using Dehn invariants as our guide. We found that some polyhedra, like the Cuboctahedron, Truncated Cube, and Rhombicuboctahedron, showed potential for being reversible.

In summary, we found that some polyhedra like the Cuboctahedron, Truncated Cube, and Rhombicuboctahedron showed promisisng for being reversible. However, Cuboctahedron and Truncated Cube, and Cuboctahedron and Rhombicuboctahedron, do not have a clear volume relationship needed for the Double-Reversal-Plates Method to work.

In conclusion, our project represents a significant contribution to the field of geometric analysis and properties of reversibility, offering both theoretical insights and practical implications for the application of the Double-Reversal-Plates Method. Moreover, it underscores the ongoing need for further research and refinement in understanding the properties of reversibility among polyhedral forms, with a particular emphasis on enhancing methodologies for identifying suitable dissections.

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# REVERSIBLE PROPERTIES OF POLYHEDRA


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In this independent study, we have a motivation from the study of Professor Akiyama Jin on the topic “On Reversibility among Parallelohedra”. This study examines the reversibility of non-space filling polyhedra by hinge dissection using the double reversal plate method. Using insights from Professor Akiyama Jin’s study, we derive a lemma for peer-checking whether space filla can be achieved using this method. Our analysis identifies the trun-cated cube, rhombicuboctahedron, and cuboctahedron as potential candidates because they are non-space filla with the same Dehn invariant, suggesting potential reversibility between them. Through further investigation, considering volume equality, we found that the pair of rhombicuboctahedron and cuboctahedron and the pair of truncated cube and cuboctahedron cannot be reversed using the double reversal plate method.


## 1 Introduction

In this study, we investigate to determine whether this method can be extended to the Double-Reversal-Plates Method.
By examining the potential challenges and modifications required for such adaptation, this study aims to contribute to a broader understanding of reversible transformations

## 2 Objective

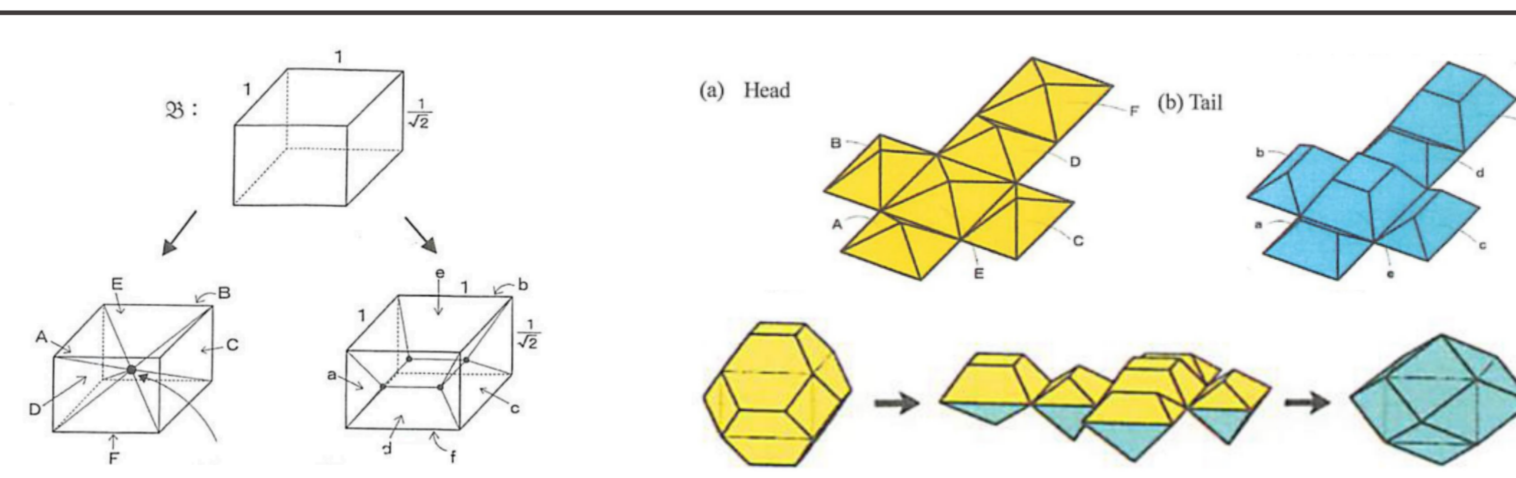
- To identify conditions suitable for the Double Reversible Plate Method.
- To ascertain conditions conducive to reversal in non-space-filling scenarios.
- To evaluate the feasibility of applying the Double Reversible Plate Method in non-space-filling.

## 3 Scope of the study

This study focuses on the investigation of reversibility through the application of hinged dissection transformations in Archimedean solids. The primary objective is to analyze the feasibility of the Double-Reversal-Plates Method.

## 4 Methodology

In the Double-Reversal-Plates Method, the dissection piece of two polyhedrons is placed on both sides (Head and Tail) of the same net (plate). In this case, the net (plate) for parallelohedra is a parallel pipe (cube).
We can determine whether the considered solid is capable of using the Double-Reversal Plates Method based on Lemma 3.2 : If the volume V(inscribed) is not equal to the half of V(considered), then the polyhedron can not be applied to the Double-Reversal-Plates Method.




Example of the Double-Reversal-Plates Method

## 5 Dehn invariant

Beside the space filler group, which is the class of Dehn Invariant equal to 0, the Dehn Invariant of the truncated cube, rhombicuboctahedron, and cuboctahedron are the same. The purpose of this study is to apply the double reversible method to other polyhedra where the Dehn invariant is not equal to 0

Cuboctahedron

Rhombi cuboctahedron

Truncated Cube

A point worth mentioning is the concept of tilling properties, These are the three regular tilings, which are each made up of identical copies of a regular polygon: equilateral triangles, squares, and hexagons, as mentioned in Theorem 2.18. It can now be established that parallelohedra and prisms are unsuitable candidates as plates for non-space-filling purposes due to their Dehn invariant being equal to zero.

## 6 Equality of volume

The study by Akiyama [1] explored the idea of reversibility among parallelohedra, espe□cially those are space-filling. A key question raised was whether non-space-filling can also be reversible. Based on previous knowledge suggesting that polyhedra with similar Dehn invariants can be reversible, this study aims to find another condition for reversibility. Akiyama’s study [1] showed that the relationships between the lengths of edges in poly□hedra go beyond just having the same volume.

### Case 1. The pair of cuboctahedron and rhombicuboctahedron

Based on our comprehensive analysis, it has been established that there is no dis□cernible correlation between the lengths of cuboctahedron and rhombicuboctahedron. Consequently, we can infer that employing the Double-Reversal-Plates Method is not feasible for a pair of cuboctahedron and rhombicuboctahedron.

### Case 2. The pair of cuboctahedron and truncated cube

Based on the evidence presented in the proof above, we conclude that both cubocta□hedron and truncated cube are not suitable candidates for the application of the Double□Reversal-Plates Method.

### Remarks on the case of rhombicuboctahedron and truncated cube

Due to volume of rhombicuboctahedron and volume of truncated cube being components of the same term, no definitive conclusions can be drawn. However, based on the orientation analysis between truncated cube and rhombic cuboctahedron, we have a conjecture that they are unsuitable candidates for the proposed method.

## 7 Conclusion

Our project explored the Double-Reversal-Plates Method for dissecting polyhedra, introducing Lemma 3.2 as a criterion for effectiveness. We also investigated reversibility among polyhedra using Dehn invariants, identifying potential reversibility in the Cuboctahedron, Truncated Cube, and Rhombicuboctahedron. However, the volume relationships of some pairs hindered the method's applicability. Our findings contribute to understanding reversibility in polyhedra, emphasizing the need for further research and improved dissection methodologies.


**Related literature**

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