Performance of a compact structure-preserving finite difference scheme for a model of nonlinear dispersive wave equations

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AN INDEPENDENT STUDY SUBMITTED TO CHIANG MAI UNIVERSITY IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS

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Title Performance of a compact structure-preserving finite difference scheme for a model of nonlinear dispersive wave equations

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Abstract

In this study, a high-order algorithm based on the finite difference method is developed to solve the nonlinear shallow water equation modeled by the BBM-KdV equation. The proposed scheme is implemented using compact difference operators and precisely conserves the mass-preserving property on any time region. Numerical experiments are conducted to illustrate the performance and accuracy of the proposed method in comparison with other schemes in various benchmark problems.

Keywords: Finite difference method, Compact difference operator, BBM-KdV equation, Mass-preserving property

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Chapter 1

Introduction

The study of nonlinear dispersive wave equations is an active area of research in applied mathematics and engineering due to its widespread applications in various physical phenomena, such as water waves, optics, and plasma physics. In particular, the BBM-KdV equation, which combines the characteristics of the Benjamin-Bona-Mahony (BBM) equation and the Korteweg-de Vries (KdV) equation, provides a versatile framework for analyzing the dynamics of nonlinear waves in different contexts. The BBM-KdV equation that we focus in this study can be expressed in the following form :

$$u_t + u_x - u_{xxt} + u_{xxx} + uu_x = 0. \tag{1.1}$$

Here, u represents the wave profile as a function of space x and time t. Furthermore, we consider the BBM-KdV equation with the following initial and boundary conditions

$$u(x, 0) = u_0(x), \quad x \in [x_L, x_R], \tag{1.2}$$

$$u(x_L, t) = u(x_R, t) = 0, \quad u_x(x_L, t) = u_x(x_R, t) = 0, \quad t \in (0, T]. \tag{1.3}$$

In the study of the BBM-KdV equation, various numerical methods have been employed to obtain solutions and analyze the wave dynamics. One widely utilized numerical technique is the finite difference method (FDM), which normally approximates the derivative terms in the BBM-KdV equation by expressing the wave profile u at discrete points in space and time.

The FDM has proven to be a tool for studying the BBM-KdV equation in several research studies. In [1], Rouatbi et al. have presented a high-order conservative difference scheme for solving the BBM-KdV equation. Using the Brouwer fixed-point theorem to establish solution existence and demonstrate unconditional stability and uniqueness of their scheme. Later, Suebcharoen et al. [2] have focused on the bifurcation analysis and numerical investigation of the nonlinear BBM-KdV equation and also implemented a two-level linear implicit finite difference algorithm for numerical simulations.

To enhance the accuracy and efficiency of the FDM, compact finite difference operators are often employed. Compact operators are difference operators that provide high-order accuracy while requiring fewer computational resources compared to traditional finite difference approximations. These operators can effectively capture the intricate behavior of the BBM-KdV equation while minimizing numerical artifacts. In this study, the authors focus on evaluating the performance of a specific compact structure-preserving finite difference scheme for the BBM-KdV equation. In particular, the study introduces a novel compact operator for the third-order derivative term in the equation. The developed compact operator for the third-order derivative is designed to achieve high-order accuracy, allowing for the precise representation of the wave profile u and its derivatives. By incorporating this operator within the structure-preserving finite difference scheme, we aim to capture the nonlinear interactions and dispersive effects in the BBM-KdV equation with improved accuracy compared to traditional finite difference approximations. Furthermore, the propose scheme is linearized. This linearization process simplifies the numerical solution procedure and facilitates the application of the compact finite difference operators. The linearized scheme preserves important physical properties of the BBM-KdV equation while providing a more efficient framework for numerical simulations.

Through a comprehensive series of numerical experiments and comparisons with existing methods, we assess the performance of the developed compact structure-preserving finite difference scheme with the new compact operator. The evaluation includes examining the accuracy, stability, and efficiency of the scheme in capturing the complex dynamics of the BBM-KdV equation, such as soliton formation, propagation.

Chapter 2

Preliminary

2.1 Notations

In this section, we provide the notations which are used through out this study

x_L	is the initial point in one-dimentional,
x_R	is the terminal point in one-dimentional,
T	represents the final time,
M	represents the number of grid points in space,
N	represents the number of grid points in time,
x	is space variable,
t	is time variable,
h	is space step size defined by $h = \dfrac{x_R - x_L}{M}$,
τ	is time step size defined by $\tau = \dfrac{T}{N}$,
U_i^n	denotes the numerical approximation to $U(x_i, t^n)$,
$\Vert U^n\Vert_\infty$	denotes the maximum-norms : $\Vert U^n\Vert_\infty = \max \vert U_i^n\vert$ for $i = 0, 1, 2, \ldots, M$,
$\Vert U^n\Vert$	denotes the L_2-norms : $\Vert U^n\Vert = \left(\sum_i \vert U_i^n\vert^2\right)^{\frac{1}{2}}$ for $i = 0, 1, 2, \ldots, M$.

2.2 Partial Differential Equation

Partial Differential Equations (PDEs) are fundamental mathematical equations used to describe various physical and natural phenomena in science and engineering. Unlike ordinary differential equations (ODEs), which involve derivatives with respect to a single independent variable, PDEs involve derivatives with respect to multiple independent variables. They are powerful tools for modeling and understanding complex systems that involve multiple interacting variables.

PDEs find applications in diverse fields such as physics, engineering, biology, economics, etc. They provide a mathematical framework to describe and analyze phenomena that exhibit spatial and temporal variations. By formulating a PDE, one can study the behavior of a system under various conditions and make predictions about its future state.

2.2.1 Classification of PDEs

PDEs can be classified into several types based on their properties and characteristics. Some common classifications include elliptic, parabolic, and hyperbolic equations.

Example 1.

1-D Wave equation :	$u_{tt} = c^2 u_{xx}.$
2-D Wave equation :	$u_{tt} = c^2(u_{xx} + u_{yy}).$
1-D Heat or diffusion equation :	$u_t = cu_x.$
2-D Laplace equation :	$u_{xx} + u_{yy} = 0.$

PDEs can be further classified into linear and nonlinear equations based on the linearity of the equations with respect to the dependent variables and their derivatives.

1. A linear PDE. is an equation where the dependent variable and its derivatives appear linearly. This means that the dependent variable and its derivatives have a power of one and do not appear in products, powers, or other nonlinear combinations. Linear PDEs have the property of superposition, meaning that if u_1 and u_2 are solutions, then any linear combination of them (such as $\alpha u_1 + \beta u_2$, where α and β are constants) is also a solution.
   Example of linear PDEs : The equation mentioned earlier in **Example 1**. are all linear PDEs.

2. A nonlinear PDE. is an equation where the dependent variable or its derivatives appear nonlinearly. This means that the dependent variable and its derivatives can appear in products, powers, or other nonlinear combinations. Nonlinear PDEs do not exhibit the property of superposition, making their analysis and solution more challenging compared to linear PDEs.
   Example of nonlinear PDEs :

Benjamin–Bona–Mahony equation (BBM equation) :	$u_t + u_x + uu_x - u_{xxt} = 0.$
Korteweg–de Vries equation (KdV equation) :	$u_t + u_{xxx} - 6uu_x = 0.$

   Here, respectively, the term uu_x and $6uu_x$ represent a nonlinear term.

It is important to note that some PDEs can exhibit both linear and nonlinear behavior, depending on the specific form of the equation and the properties of the dependent variables involved. In such cases, the behavior of the equation is typically analyzed by linearizing it around certain points or using perturbation methods.

2.3 Finite difference method

Finite difference method (FDM) is a numerical technique used to approximate solutions of differential equations. It is based on the Taylor's series expansion of a function, which allows us to approximate the value of the function at a point by considering its values and derivatives at neighboring points. Taylor's series is an expansion of a function as an infinite sum of terms involving its derivatives evaluated at a point. The formula is:

$$f\left(x+h\right)=\sum_{n=0}^{\infty}\frac{f^{(n)}\left(x\right)}{n!}h^{n},$$

where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$ evaluated at x. Taylor's formula allows us to approximate the value of a function at a nearby point by truncating the series after a finite number of terms.

FDM uses a discrete grid of points to approximate derivatives of the function at the grid points. By applying the Taylor's formula to neighboring points on the grid, we can obtain different forms of finite difference formulas. The three most commonly used forms for first-order derivative are:

• Forward Difference Formula: This formula approximates the first derivative of a function at a point using the values of the function at that point and a neighboring point ahead. The formula is :

$$f'\left(x_i\right)\approx\frac{x_{i+1}-f\left(x_i\right)}{h}.$$

• Backward Difference Formula : This formula approximates the first derivative of a function at a point using the values of the function at that point and a neighboring point behind. The formula is :

$$f'\left(x_i\right)\approx\frac{f\left(x_i\right)-f\left(x_{i-1}\right)}{h}.$$

• Central Difference Formula : This formula approximates the first derivative of a function at a point using the values of the function at two neighboring points, one ahead and one behind. The formula is :

$$f'\left(x_i\right)\approx\frac{f\left(x_{i+1}\right)-f\left(x_{i-1}\right)}{2h}.$$

Using these finite difference formulas, we can discretize the domain of a differential equation and approximate its solution. FDM is a powerful tool for solving a wide range of differential equations, including those that cannot be solved analytically. However, the accuracy of FDM depends on the choice of grid size and the order of the difference formula used, which requires careful consideration to achieve optimal results.

Example 2. We want to approximate the first derivative of the function $f(x) = \sin(x)$ at a specific point $x_0 = 1$ with the exact solution is $f'(x) = \cos(1) \approx 0.540302$. So, we get the results that show differences in accuracy from sample grid sizes and order of the formulas as follows :

Difference formula	Grid size (h)	Approximate $f'(1)$	Error
Forward difference	0.1	0.4973	4.2938×10^{-2}
	0.01	0.5360	4.2163×10^{-3}
	0.001	0.5398	4.2082×10^{-4}
Backward difference	0.1	0.5814	4.1138×10^{-2}
	0.01	0.5445	4.1983×10^{-3}
	0.001	0.5407	4.2064×10^{-4}
Central difference	0.1	0.5394	9.0005×10^{-4}
	0.01	0.5402	9.0049×10^{-6}
	0.001	0.5403	9.0050×10^{-8}

Chapter 3

Main Result

3.1 A novel finite difference scheme

To solve the initial boundary value problem (IBVP) (1.1)-(1.3) numerically, we will use a finite difference method. First, we need to define the solution domain and number of grid points. Let the computational domain be $[x_L, x_R]$, which is divided into M equally spaced grid points $\{x_i\}_{i=0}^{M}$, where $x_i = x_L + ih$ and $h = \dfrac{x_R - x_L}{M}$ is the uniform step size for a fixed positive integer M. The time domain is discretized uniformly by $t_n = n\tau$, where τ is the time step size. We let the numerical approximation of a function u at the grid point (x_i, t^n) as U_i^n. By physical boundary conditions

$$\partial_x^n u \to 0 \quad \text{as} \quad |x| \to +\infty, \ \ n \in \mathbb{N} \cup \{0\},$$

we logically assume $U_{-1} = U_0 = U_1 = 0$ and $U_{M-1} = U_M = U_{M+1} = 0$. Therefore, we defined

$$Z_{h,0} = \{U = (U_i) | U_{-1} = U_0 = U_1 = U_{M-1} = U_M = U_{M+1} = 0\}.$$

For simplicity, we define the notations of the different operators as shown below:

$$\bar{U}_i^n = \frac{U_i^{n+1} + U_i^{n-1}}{2}, \qquad \delta_{\hat{t}}(U_i^n) = \frac{U_i^{n+1} - U_i^{n-1}}{2\tau},$$

$$\delta_x(U_i^n) = \frac{U_{i+1}^n - U_i^n}{h}, \qquad \delta_{\bar{x}}(U_i^n) = \frac{U_i^n - U_{i-1}^n}{h},$$

$$\delta_{\hat{x}}(U_i^n) = \frac{U_{i+1}^n - U_{i-1}^n}{2h}, \qquad \delta_{x\bar{x}}(U_i^n) = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2},$$

$$\delta_{x\bar{x}\hat{x}}(U_i^n) = \frac{U_{i+2}^n - 2U_{i+1}^n + 2U_{i-1}^n - U_{i-2}^n}{2h^3}.$$

Here, this study will use only standard compact finite difference operators in space, so for simplicity, we introduced the compact operators as follows

$$\mathcal{A}_x(U_i^n) = (1 + \frac{h^2}{6}\delta_{x\bar{x}})U_i^n, \tag{3.1}$$

$$\mathcal{B}_x(U_i^n) = (1 + \frac{h^2}{12}\delta_{x\bar{x}})U_i^n, \tag{3.2}$$

and corresponding matrices representations

$$\mathbf{H}_1 = \frac{1}{6}\begin{bmatrix} 4 & 1 & 0 & \dots & 0 \\ 1 & 4 & 1 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & 1 & 4 & 1 \\ 0 & \dots & 0 & 1 & 4 \end{bmatrix}_{(M-1)\times(M-1)},$$

$$\mathbf{H}_2 = \frac{1}{12}\begin{bmatrix} 10 & 1 & 0 & \dots & 0 \\ 1 & 10 & 1 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & 1 & 10 & 1 \\ 0 & \dots & 0 & 1 & 10 \end{bmatrix}_{(M-1)\times(M-1)},$$

where the operators $\mathcal{A}_x$ and $\mathcal{B}_x$ are obtained by [3]. With the same idea from [3], we will derive operator for third order derivative term. By Taylor series, we have

$$\begin{aligned} U_{i+2} &= U_i + 2h(U_i') + \frac{(2h)^2}{2!}U_i'' + \frac{(2h)^3}{3!}U_i''' + \frac{(2h)^4}{4!}U_i^{(4)} + \frac{(2h)^5}{5!}U_i^{(5)} + \dots, \\ U_{i+1} &= U_i + h(U_i') + \frac{h^2}{2!}U_i'' + \frac{h^3}{3!}U_i''' + \frac{h^4}{4!}U_i^{(4)} + \frac{h^5}{5!}U_i^{(5)} + \dots, \\ U_{i-1} &= U_i - h(U_i') + \frac{h^2}{2!}U_i'' - \frac{h^3}{3!}U_i''' + \frac{h^4}{4!}U_i^{(4)} - \frac{h^5}{5!}U_i^{(5)} + \dots, \\ \text{and} \quad U_{i-2} &= U_i - 2h(U_i') + \frac{(2h)^2}{2!}U_i'' - \frac{(2h)^3}{3!}U_i''' + \frac{(2h)^4}{4!}U_i^{(4)} - \frac{(2h)^5}{5!}U_i^{(5)} + \dots \quad . \end{aligned}$$

Then, consider

$$U_{i+2} - 2U_{i+1} + 2U_{i-1} + U_{i-2} = 2h^3U_i''' + \frac{h^5}{2}U_i^{(5)} + \mathcal{O}(h^7).$$

That is

$$\begin{aligned} U_i''' &= \frac{1}{2h^3}\left(U_{i+2} - 2U_{i+1} + 2U_{i-1} + U_{i-2}\right) - \frac{h^2}{4}U_i^{(5)} + \mathcal{O}(h^4) \\ &= \delta_{x\bar{x}\hat{x}}(U_i) - \frac{h^2}{4}U_i^{(5)} + \mathcal{O}(h^4). \end{aligned}$$

Let $f = U_i'''$, so

$$\begin{aligned} f &= \delta_{x\bar{x}\hat{x}}(U_i) - \frac{h^2}{4}f'' + \mathcal{O}(h^4) \\ &= \delta_{x\bar{x}\hat{x}}(U_i) - \frac{h^2}{4}\delta_{x\bar{x}}f + \mathcal{O}(h^4). \end{aligned}$$

That is

$$\left(1+\frac{h^2}{4}\delta_{x\bar{x}}\right)f=\delta_{x\bar{x}\hat{x}}(U_i)+\mathcal{O}(h^4).$$

So, we define

$$\mathcal{C}_x(U_i^n)=(1+\frac{h^2}{4}\delta_{x\bar{x}})U_i^n \tag{3.3}$$

Next, we will find the corresponding metric representation, for $i=1,2,\ldots,M$ consider

$$\begin{aligned}\mathcal{C}_x f_i &= \left(1+\frac{h^2}{4}\delta_{x\bar{x}}\right)f_i\\ &= f_i+\frac{h^2}{4}\left(\frac{f_{i+1}-2f_i+f_{i-1}}{h^2}\right)\\ &= \frac{f_{i+1}+2f_i+f_{i-1}}{4},\end{aligned}$$

and we get the corresponding metric representation as follows

$$\mathbf{H}_3=\frac{1}{4}\begin{bmatrix}2 & 1 & 0 & \ldots & 0\\ 1 & 2 & 1 & \ldots & 0\\ & \ddots & \ddots & \ddots & \\ 0 & \ldots & 1 & 2 & 1\\ 0 & \ldots & 0 & 1 & 2\end{bmatrix}_{(M-1)\times(M-1)}.$$

Here, we provide some modification of the results related to the matrices $\mathbf{H}_1,\mathbf{H}_2$ and $\mathbf{H}_3$ based on the techniqe used in [4, 5, 6]. The results can be verified straightforwardly; therefore, we leave the proof.

Lemma 3.1 ([7, 8, 9, 5]). *For matrices H_1, H_2 and H_3 of order $(M-1)\times(M-1)$, the following results are obtained:*

1. *The eigenvalues of the matrices $\mathbf{H}_1,\mathbf{H}_2$ and $\mathbf{H}_3$ are*

   $$\lambda_{H_1,i}=\frac{1}{6}(5+\cos\frac{i\pi}{M}),\qquad \lambda_{H_2,i}=\frac{1}{3}(2+\cos\frac{i\pi}{M}),\qquad \lambda_{H_3,i}=\frac{1}{2}(1+\cos\frac{i\pi}{M}),\quad i=1,2,\ldots,M-1$$

2. *The eigenvectors corresponding to the above eigenvalues of the matrices $\mathbf{H}_1,\mathbf{H}_2$ and $\mathbf{H}_3$ are the same, represented as v, and exists*

   $$v_k=\left(\sin\left(\frac{k\pi}{M}\right),\sin\left(\frac{2k\pi}{M}\right),\ldots,\sin\left(\frac{(M-1)k\pi}{M}\right)\right)^T,\quad k=1,2,\ldots,M-1$$

3. *The symmetric matrices $\mathbf{H}_1,\mathbf{H}_2$ and $\mathbf{H}_3$ are positive definite matrices, and the inverse matrices are also symmetric and positive definite matrices.*

From Lemma 3.1, we know that the eigenvalues of the three corresponding matrices are all positive, this implies the inverse of these matrices are exist.

Hence, resulting in heightened accuracy in the numerical computation, the operators are used to approximate the derivative terms as

$$\begin{aligned} u_x|_{(x_i,t^n)} &= \mathcal{A}_x^{-1}\delta_{\hat{x}}(U_i^n) + \mathcal{O}(h^4), \\ u_{xx}|_{(x_i,t^n)} &= \mathcal{B}_x^{-1}\delta_{x\bar{x}}(U_i^n) + \mathcal{O}(h^4), \\ u_{xxx}|_{(x_i,t^n)} &= \mathcal{C}_x^{-1}\delta_{x\bar{x}\hat{x}}(U_i^n) + \mathcal{O}(h^4). \end{aligned}$$

Here, we are ready to design a higher-order FDM for solving the IBVP (1.1)-(1.3). Combine with Crank-Nicolson/Adam-Bashforth method $u|_{(x,t^n)} = \bar{u}_i^n + \mathcal{O}(\tau^2)$. After discretizing the equation (1.1) in time and space, we obtain a description of a FDS and an algorithm for the formulation of the problem (1.1)-(1.3) as

$$\delta_{\hat{t}}(U_i^n) - \delta_{\hat{t}}\mathcal{B}_x^{-1}\delta_{x\bar{x}}(U_i^n) + \mathcal{A}_x^{-1}\delta_{\hat{x}}(\bar{U}_i^n) + \mathcal{C}_x^{-1}\delta_{x\bar{x}\hat{x}}(\bar{U}_i^n) + \frac{1}{2}\mathcal{A}_x^{-1}\delta_{\hat{x}}\left[(U_i^n)^2\right] = 0, \quad 0 \le i \le M, \tag{3.4}$$

with initial condition

$$U_i^0 = U_0(x_i), \quad 0 \le i \le M, \tag{3.5}$$

and boundary conditions

$$U_0^n = U_M^n = 0, \qquad \delta_{\hat{x}}(U_0^n) = \delta_{\hat{x}}(U_M^n) = 0, \qquad 1 \le n \le N. \tag{3.6}$$

It is worth to note that, by using the Taylor expansion, it can be seen from the above reformulations of the scheme (3.4) achieves the truncation error of order $\mathcal{O}(\tau^2 + h^4)$. Noting that the vector form of the scheme (3.4) is

$$\delta_{\hat{t}}(U^n) - \mathbf{H}_2^{-1}\mathbf{D}_2\delta_{\hat{t}}(U^n) + \mathbf{H}_1^{-1}\mathbf{D}_1(\bar{U}^n) + \mathbf{H}_3^{-1}\mathbf{D}_3(\bar{U}^n) + \frac{1}{2}\mathbf{H}_1^{-1}\delta_x\left[(\mathrm{diag}U^n)U^n\right] = 0, \tag{3.7}$$

where the matrices $\mathbf{D}_1, \mathbf{D}_2$ and $\mathbf{D}_3$ are defined by

$$\mathbf{D}_1 = \frac{1}{2h}\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & -1 & 0 & 1 \\ 0 & \dots & 0 & -1 & 0 \end{bmatrix}_{(M-1)\times(M-1)},$$

$$\mathbf{D}_2 = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 \end{bmatrix}_{(M-1)\times(M-1)},$$

$$\mathbf{D}_3 = \frac{1}{2h^3}\begin{bmatrix} 0 & -2 & 1 & 0 & \dots & 0 \\ 2 & 0 & -2 & 1 & \dots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & \dots & -1 & 2 & 0 & -2 \\ 0 & \dots & 0 & -1 & 2 & 0 \end{bmatrix}_{(M-1)\times(M-1)}.$$

Consequently, the vector form of the scheme (3.7) is equivalent to

$$\mathbf{A}U^{n+1} = \mathbf{b}^n \tag{3.8}$$

where

$$\begin{aligned} \mathbf{A} &= \frac{1}{2\tau}\mathbf{I} - \frac{1}{2\tau}\mathbf{H}_2^{-1}\mathbf{D}_2 + \frac{1}{2}\mathbf{H}_1^{-1}\mathbf{D}_1 + \frac{1}{2}\mathbf{H}_3^{-1}\mathbf{D}_3 \\ \mathbf{b}^n &= \left(\frac{1}{2\tau}\mathbf{I} - \frac{1}{2\tau}\mathbf{H}_2^{-1}\mathbf{D}_2 - \frac{1}{2}\mathbf{H}_1^{-1}\mathbf{D}_1 - \frac{1}{2}\mathbf{H}_3^{-1}\mathbf{D}_3\right)U^{n-1} - \frac{1}{2}\mathbf{H}_1^{-1}\mathbf{D}_1\left[(\mathrm{diag}(U^n))U^n\right]. \end{aligned}$$

Notice that the proposed scheme is to form a linear system with a constant coefficient matrix at each time step which significantly reduced computational cost.

3.2 Structural preserving property

In this section, we discuss the convergence and stability of our numerical scheme. From now on, we will denote C as a generic constant independent of step sizes h and τ. Next, we denote the matrix $\mathbf{R}_j$ by

$$\mathbf{H}_j^{-1} = \mathbf{R}_j^T\mathbf{R}_j, \quad \text{for } j = 1, 2$$

where $\mathbf{R}_j$ is obtained by Cholesky decomposition of $\mathbf{H}_j^{-1}$. According to Lemma 3.1 and the spectral radius of $\mathbf{R}_j$, we obtain the following estimations which are essential for existence and uniqueness, convergence, and stability of our numerical solution.

Lemma 3.2 ([4]). *For any real symmetric positive definite matrices $\mathbf{H}_j$ for $j = 1, 2$ and for*

$U, V \in Z_{h,0}$, *we have*

$$\langle \mathbf{H}_j \delta_{x\bar{x}} U, V \rangle = -\langle \mathbf{H}_j \delta_x U, \delta_x V \rangle = -\langle \mathbf{R}_j \delta_x U, \mathbf{R}_j \delta_x V \rangle,$$
$$\langle \mathbf{H}_j \delta_{\hat{x}} U, V \rangle = -\langle \mathbf{H}_j U, \delta_{\hat{x}} V \rangle = -\langle U, \mathbf{H}_j \delta_{\hat{x}} V \rangle.$$

Lemma 3.3 ([3, 10]). *For $U \in Z_{h,0}$, we have*

1. $\|U\|^2 \le \langle \mathbf{H}_1^{-1} U, U \rangle = \|\mathbf{R}_1 U\|^2 \le \frac{3}{2}\|U\|^2$,
2. $\|U\|^2 \le \langle \mathbf{H}_2^{-1} U, U \rangle = \|\mathbf{R}_2 U\|^2 \le 3\|U\|^2$.

Proof. It follows from Lemma 3.1 that the eigenvalues of $\mathbf{H}_1$ and $\mathbf{H}_2$ satisfy

$$\frac{2}{3} \le \lambda_{\mathbf{H}_1,k} \le 1, \quad k = 1, 2, 3, \ldots, M-1,$$
$$\frac{1}{3} \le \lambda_{\mathbf{H}_2,k} \le 1, \quad k = 1, 2, 3, \ldots, M-1,$$

which yields the spectral radius

$$\frac{2}{3} \le \|\mathbf{H}_1\| = \rho(\mathbf{H}_1) \le 1$$
$$\frac{1}{3} \le \|\mathbf{H}_2\| = \rho(\mathbf{H}_2) \le 1.$$

Consequently , we obtain the inequality (i) and (ii) by applying the identities

$$\langle \mathbf{H}_j^{-1} U, U \rangle = \langle \mathbf{R}_j^T \mathbf{R}_j U, U \rangle = \langle \mathbf{R}_j U, \mathbf{R}_j U \rangle = \|\mathbf{R}_j U\|^2,$$

when $j = 1, 2$. This completes the proof. □

Lemma 3.4. *For any grid functions $U \in Z_{h,0}$, the defined operator $\mathcal{A}_x, \mathcal{B}_x$ and $\mathcal{C}_x$ are three linear transformations defined by Eqs.(3.1)-(3.2), respectively, so their inverse transformations $\mathcal{A}_x^{-1}, \mathcal{B}_x^{-1}$and $\mathcal{C}_x^{-1}$ are also linear transformations and there are the following relationships:*

$$\sum_{i=1}^{M-1} \mathcal{A}_x^{-1} U_i = \sum_{i=1}^{M-1} U_i, \tag{3.3.1}$$
$$\sum_{i=1}^{M-1} \mathcal{B}_x^{-1} U_i = \sum_{i=1}^{M-1} U_i, \tag{3.3.2}$$
$$\sum_{i=1}^{M-1} \mathcal{C}_x^{-1} U_i = \sum_{i=1}^{M-1} U_i. \tag{3.3.3}$$

Proof. The ralations (3.3.1) and (3.3.2) can be obtained from [10]. With the same idea, we can get the relation (3.3.3) as follows

$$\sum_{i=1}^{M-1} U_i = \sum_{i=1}^{M-1} \mathcal{C}_x^{-1} \mathcal{C}_x U_i = \frac{1}{4} \sum_{i=1}^{M-1} \mathcal{C}_x^{-1} U_i + \frac{1}{2} \sum_{i=1}^{M-1} \mathcal{C}_x^{-1} U_i + \frac{1}{4} \sum_{i=1}^{M-1} \mathcal{C}_x^{-1} U_i = \sum_{i=1}^{M-1} \mathcal{C}_x^{-1} U_i.$$

□

Lemma 3.5 (Discrete Sobolev's inequality [11]). *There exist two constants C_1 and C_2 such that*

$$\|U^n\|_\infty \leq C_1\|U^n\| + C_2\|\delta_x U^n\|.$$

Lemma 3.6 (Discrete Gronwall inequality [11]). *Suppose that $\omega(k)$ and $\rho(k)$ are nonnegative functions and $\rho(k)$ is a nondecreasing function. If $C > 0$ and*

$$\omega(k) \leq \rho(k) + C\tau \sum_{l=0}^{k-1} \omega(l), \qquad \forall k,$$

then

$$\omega(k) \leq \rho(k) e^{C\tau k}, \qquad \forall k.$$

Lemma 3.7. *For any grid functions $U, V \in Z_{h,0}$,*

1. $\langle \mathbf{H}_1^{-1}\delta_{\hat{x}}U, U\rangle = 0$,
2. $\langle \mathbf{H}_3^{-1}\delta_{x\bar{x}\hat{x}}U, U\rangle = 0$.

Proof. By Lemma 3.2, we have

$$\langle \mathbf{H}_1^{-1}\delta_{\hat{x}}U, U\rangle = -\langle \mathbf{H}_1^{-1}U, \delta_{\hat{x}}U\rangle = -\langle U, \mathbf{H}_1^{-1}\delta_{\hat{x}}U\rangle,$$

and

$$\begin{aligned}
\langle \mathbf{H}_3^{-1}\delta_{x\bar{x}\hat{x}}U, U\rangle &= -\langle \mathbf{H}_3^{-1}\delta_{x\bar{x}}U, \delta_{\hat{x}}U\rangle \\
&= \langle \mathbf{H}_3^{-1}\delta_x U, \delta_x\delta_{\hat{x}}U\rangle \\
&= -\langle \mathbf{H}_3^{-1}U, \delta_{x\bar{x}\hat{x}}U\rangle \\
&= -\langle U, \mathbf{H}_3^{-1}\delta_{x\bar{x}\hat{x}}U\rangle,
\end{aligned}$$

Therefore, $\langle \mathbf{H}_1^{-1}\delta_{\hat{x}}U, U\rangle = 0$ and $\langle \mathbf{H}_3^{-1}\delta_{x\bar{x}\hat{x}}U, U\rangle = 0$. □

Theorem 3.8. *Suppose that $U^n \in Z_{h,0}$ for $n = 0, 1, 2, \ldots, N$. Then, the scheme (3.4) with Eqs. (3.5)-(3.6) is mass conservative as appeared in the following:*

$$Q^n = \frac{h}{2}\sum_{i=1}^{M-1}\left(U_i^{n+1} + U_i^n\right) = Q^{n-1} = \ldots = Q^0. \tag{3.9}$$

Proof. Multiply the scheme with h and summing up for i from 1 to $M-1$, we get

$$\begin{aligned}
h\sum_{i=1}^{M-1}\delta_{\hat{t}}(U_i^n) - h\sum_{i=1}^{M-1}\mathcal{B}_x^{-1}\delta_{\hat{t}}\delta_{x\bar{x}}(U_i^n) + h\sum_{i=1}^{M-1}\mathcal{A}_x^{-1}\delta_{\hat{x}}(\bar{U}_i^n) \\
+ h\sum_{i=1}^{M-1}\mathcal{C}_x^{-1}\delta_{x\bar{x}\hat{x}}(\bar{U}_i^n) + \frac{1}{2}h\sum_{i=1}^{M-1}\mathcal{A}_x^{-1}\delta_x[(U_i^n)^2] = 0.
\end{aligned} \tag{3.10}$$

By using Lemma 3.4, we obtain

$$h\sum_{i=1}^{M-1}\delta_{\hat{t}}(U_i^n)-h\sum_{i=1}^{M-1}\delta_{\hat{t}}\delta_{x\bar{x}}(U_i^n)+h\sum_{i=1}^{M-1}\delta_{\hat{x}}(\bar{U}_i^n)+h\sum_{i=1}^{M-1}\delta_{x\bar{x}\hat{x}}(\bar{U}_i^n)+\frac{1}{2}h\sum_{i=1}^{M-1}\delta_{\hat{x}}[(U_i^n)^2]=0. \quad (3.11)$$

Since $U^n \in Z_{h,0}$, consider each term of (3.11)

$$\begin{aligned}
h\sum_{i=1}^{M-1}\delta_{\hat{t}}\delta_{x\bar{x}}(U_i^n) &= h\delta_{\hat{t}}\sum_{i=1}^{M-1}\left(\frac{U_{i+1}^n-2U_i^n+U_{i-1}^n}{h^2}\right)\\
&= \frac{1}{h}\delta_{\hat{t}}\left[U_M^n-U_{M-1}^n-U_1^n+U_0^n\right]=0, && (3.12)\\
h\sum_{i=1}^{M-1}\delta_{\hat{x}}(\bar{U}_i^n) &= h\sum_{i=1}^{M-1}\frac{\bar{U}_{i+1}^n-\bar{U}_{i-1}^n}{2h}\\
&= \frac{1}{2}\left[\bar{U}_M^n+\bar{U}_{M-1}^n-\bar{U}_1^n-\bar{U}_0^n\right]=0, && (3.13)\\
h\sum_{i=1}^{M-1}\delta_{\hat{x}}(U_i^n)^2 &= h\sum_{i=1}^{M-1}\frac{(U_{i+1}^n)^2-(U_{i-1}^n)^2}{2h} && (3.14)\\
&= \frac{1}{2}\left[(U_M^n)^2+(U_{M-1}^n)^2-(U_1^n)^2-(U_0^n)^2\right]=0. && (3.15)
\end{aligned}$$

Similarly, from $U^n \in Z_{h,0}$, we get

$$\begin{aligned}
h\sum_{i=1}^{M-1}\delta_{x\bar{x}\hat{x}}(U_i^n) &= h\sum_{i=1}^{M-1}\left(\frac{\bar{U}_{i+2}^n-2\bar{U}_{i+1}^n+2\bar{U}_{i-1}^n-\bar{U}_{i-2}^n}{2h^3}\right)\\
&= \frac{1}{2h^2}\left[\bar{U}_{M+1}^n-\bar{U}_M^n-\bar{U}_{M-1}^n+\bar{U}_{M-2}^n-\bar{U}_2^n+\bar{U}_1^n+\bar{U}_0^n-\bar{U}_{-1}^n\right]=0, && (3.16)
\end{aligned}$$

Thus, we left

$$h\sum_{i=1}^{M-1}\delta_{\hat{t}}(U_i^n)=0.$$

Which we can arrange to be in the form as follows

$$h\sum_{i=1}^{M-1}\delta_{\hat{t}}(U_i^n)=\frac{h}{2\tau}\sum_{i=1}^{M-1}\left(U_i^{n+1}-U_i^{n-1}\right)=0.$$

Then, this gives Eq.(3.9), which completes the proof. □

Theorem 3.9 (Uniquely solvability)**.** *Suppose that $U^n \in Z_{h,0}$, then the compact difference scheme (3.4) is uniquely solvable.*

Proof. Suppose that $U^0, U^1, \ldots, U^n$ are solved uniquely, and using mathematical induction, let both $U^{n+1}$and V^{n+1} are the solution of the scheme (3.4), that means

$$\mathbf{A}U^{n+1}=\mathbf{b}^n, \quad (3.17)$$

$$\mathbf{A}V^{n+1}=\mathbf{b}^n. \quad (3.18)$$

Let $W^{n+1} = U^{n+1} - V^{n+1}$. Subtracting Eq. (3.17) by Eq. (3.18), we get

$$\left(\frac{1}{2\tau}\mathbf{I} - \frac{1}{2\tau}\mathbf{H}_2^{-1}\mathbf{D}_2 + \frac{1}{2}\mathbf{H}_1^{-1}\mathbf{D}_1 + \frac{1}{2}\mathbf{H}_3^{-1}\mathbf{D}_3\right) W^{n+1} = 0. \tag{3.19}$$

That is

$$\frac{1}{2\tau}W^{n+1} - \frac{1}{2\tau}\mathcal{B}_x^{-1}\delta_{x\bar{x}}W^{n+1} + \frac{1}{2}\mathcal{A}_x^{-1}\delta_{\hat{x}}W^{n+1} + \frac{1}{2}\mathcal{C}_x^{-1}\delta_{x\bar{x}\hat{x}}W^{n+1} = 0. \tag{3.20}$$

Taking an inner product of (3.20) with W^{n+1} respectively, where Lemma 3.4 - 3.2 are used, we obtain each term as follows

$$\begin{aligned}
\langle\frac{1}{2\tau}W^{n+1}, W^{n+1}\rangle &= \frac{1}{2\tau}\|W^{n+1}\|^2,\\
\langle\frac{1}{2\tau}\mathbf{H}_2^{-1}\delta_{x\bar{x}}W^{n+1}, W^{n+1}\rangle &= -\frac{1}{2\tau}\langle\mathbf{H}_2^{-1}\delta_x W^{n+1}, \delta_x W^{n+1}\rangle = -\frac{1}{2\tau}\|\mathbf{R}_2\delta_x\left(W^{n+1}\right)\|^2,\\
\langle\frac{1}{2}\mathbf{H}_1^{-1}\delta_{\hat{x}}W^{n+1}, W^{n+1}\rangle &= 0,\\
\langle\frac{1}{2}\mathbf{H}_3^{-1}\delta_{x\bar{x}\hat{x}}W^{n+1}, W^{n+1}\rangle &= 0.
\end{aligned}$$

Thus, when combined together, we have

$$\frac{1}{2\tau}\|W^{n+1}\|^2 + \frac{1}{2\tau}\|\mathbf{R}_2\delta_x\left(W^{n+1}\right)\|^2 = 0 \tag{3.21}$$

Which each term is positive, it shows that $\|W^{n+1}\| = 0$. That is the scheme (3.4)-(3.6) is uniquely solvable. $\square$

The next theorem shows that our numerical solution is bounded for any given initial functions $u_0(x)$.

Theorem 3.10 (Boundedness)**.** *Suppose $U^n \in Z_{h,0}$ and $u_0 \in C^2[x_L, x_R]$. If τ is sufficiently small, then the solution U^n of the scheme (3.4) with Eqs. (3.5)-(3.6) satisfies*

$$\|U^n\| \le C, \quad \|\delta_x U^n\| \le C,$$

and

$$\|U^n\|_\infty \le C,$$

for $n = 1, 2, \ldots, N$.

Proof. To prove the boundedness, by mathematical induction, we assume

$$\|U^k\| \le C, \quad \|\delta_x U^k\| \le C, \tag{3.22}$$

and

$$\|U^k\|_\infty \le C, \tag{3.23}$$

for $k = 1, 2, \ldots, n$. Taking the inner product in Eqs. (3.4) with $2\bar{U}^n$, respectively, where Lemma

3.3, Lemma 3.2 and Lemma 3.7 are used, we get each term as follows

$$
\begin{aligned}
\langle \delta_{\hat{t}}\left(U^n\right), 2\bar{U}^n \rangle &= \langle \frac{1}{2\tau}\left(U^{n+1} - U^{n-1}\right), U^{n+1} + U^{n-1} \rangle \\
&= \frac{1}{2\tau} \langle \left(U^{n+1} - U^{n-1}\right), U^{n+1} + U^{n-1} \rangle \\
&= \frac{1}{2\tau}\left[\langle U^{n+1}, U^{n+1} \rangle + \langle U^{n+1}, U^{n-1} \rangle + \langle -U^{n-1}, U^{n+1} \rangle + \langle -U^{n-1}, U^{n-1} \rangle\right] \\
&= \frac{1}{2\tau}\left(\|U^{n+1}\|^2 - \|U^{n-1}\|^2\right), \\
\langle \mathbf{H}_2^{-1}\delta_{\hat{t}}\delta_{x\bar{x}}\left(U^n\right), 2\bar{U}^n \rangle &= \frac{1}{2\tau} \langle \mathbf{H}_2^{-1}\delta_{x\bar{x}}\left(U^{n+1}\right) - \mathbf{H}_2^{-1}\delta_{x\bar{x}}\left(U^{n-1}\right), U^{n+1} + U^{n-1} \rangle \\
&= \frac{1}{2\tau}\left[-\langle \mathbf{H}_2^{-1}\delta_x\left(U^{n+1}\right), \delta_x\left(U^{n+1}\right) \rangle + \langle \mathbf{H}_2^{-1}\delta_x\left(U^{n-1}\right), \delta_x\left(U^{n-1}\right) \rangle\right] \\
&= \frac{1}{2\tau}\left[-\|\mathbf{R}_2\delta_x\left(U^{n+1}\right)\|^2 + \|\mathbf{R}_2\delta_x\left(U^{n-1}\right)\|^2\right], \\
\langle \mathbf{H}_1^{-1}\delta_{\hat{x}}\left(\bar{U}^n\right), 2\bar{U}^n \rangle &= 2\langle \mathbf{H}_1^{-1}\delta_{\hat{x}}\left(\bar{U}^n\right), \bar{U}^n \rangle = 0,
\end{aligned}
$$

and $\langle \mathbf{H}_3^{-1}\delta_{\hat{x}}\left(\bar{U}^n\right), 2\bar{U}^n \rangle = 0.$

When taken together, we have

$$
\frac{1}{2\tau}\left(\|U^{n+1}\|^2 - \|U^{n-1}\|^2\right) + \frac{1}{2\tau}\left[\|\mathbf{R}_2\delta_x\left(U^{n+1}\right)\|^2 - \|\mathbf{R}_2\delta_x\left(U^{n-1}\right)\|^2\right] + \langle \mathbf{H}_1^{-1}\delta_{\hat{x}}\left(\mathrm{diag}\left(U^n\right)\right)U^n, \bar{U}^n \rangle = 0. \quad (3.24)
$$

For the nonlinear term, we also apply Lemma 3.3, Lemma 3.2, the Cauchy-Schwarz inequality, and the assumption (3.23) to obtain

$$
\begin{aligned}
\langle \mathbf{H}_1^{-1}\delta_{\hat{x}}\left(\operatorname{diag}\left(U^n\right)\right)U^n, \bar{U}^n\rangle &= -\langle\left(\operatorname{diag}\left(U^n\right)\right)U^n, \mathbf{H}_1^{-1}\delta_{\hat{x}}\left(\bar{U}^n\right)\rangle \\
&= -h\sum_{i=1}^{M-1}\left(U_i^n\right)^2\mathcal{A}_x^{-1}\delta_{\hat{x}}\left(\bar{U}_i^n\right) \\
&\le h\sum_{i=1}^{M-1}|U_i^n|^2\left|\mathcal{A}_x^{-1}\delta_{\hat{x}}\left(\bar{U}_i^n\right)\right| \\
&\le \|U^n\|_\infty h\sum_{i=1}^{M-1}|U_i^n|\left|\mathcal{A}_x^{-1}\delta_{\hat{x}}\left(\bar{U}_i^n\right)\right| \\
&\le \|U^n\|_\infty\left[\left(h\sum_{i=1}^{M-1}|U_i^n|^2\right)^{\frac{1}{2}}\left(h\sum_{i=1}^{M-1}\left|\mathcal{A}_x^{-1}\delta_{\hat{x}}\left(\bar{U}_i^n\right)\right|^2\right)^{\frac{1}{2}}\right] \\
&= \|U^n\|_\infty\left(\|U^n\|\cdot\|\mathbf{H}^{-1}\delta_{\hat{x}}\left(\bar{U}^n\right)\|\right) \\
&\le \|U^n\|_\infty\frac{1}{2}\left(\|U^n\|^2+\|\mathbf{H}^{-1}\delta_{\hat{x}}\left(\bar{U}^n\right)\|^2\right) \\
&\le C\left(\|U^n\|+\frac{1}{2}\|\mathbf{H}_1^{-1}\delta_x U^{n+1}\|^2+\frac{1}{2}\|\mathbf{H}_1^{-1}\delta_x U^{n-1}\|^2\right) \\
&\le C\left(\|U^n\|+\frac{3}{4}\|\mathbf{R}_1\delta_x U^{n+1}\|^2+\frac{3}{4}\|\mathbf{R}_1\delta_x U^{n-1}\|^2\right) \\
&\le C\left(\|U^n\|+\frac{9}{4}\|\mathbf{R}_2\delta_x U^{n+1}\|^2+\frac{9}{4}\|\mathbf{R}_2\delta_x U^{n-1}\|^2\right) \\
&= C\left(\|U^n\|+\|\mathbf{R}_2\delta_x U^{n+1}\|^2+\|\mathbf{R}_2\delta_x U^{n-1}\|^2\right).
\end{aligned}
$$

Here, by substitute the results into (3.24), we have

$$
\begin{aligned}
\frac{1}{2\tau}\left(\|U^{n+1}\|^2-\|U^{n-1}\|^2\right)+\frac{1}{2\tau}\left[\|\mathbf{R}_2\delta_x\left(U^{n+1}\right)\|^2-\|\mathbf{R}_2\delta_x\left(U^{n-1}\right)\|^2\right] \\
\le C\left(\|U^n\|+\|\mathbf{R}_2\delta_x(U^{n+1})\|^2+\|\mathbf{R}_2\delta_x(U^{n-1})\|^2\right).
\end{aligned} \tag{3.25}
$$

Let

$$
B^n \equiv \left(\|U^{n-1}\|^2+\|U^n\|^2\right)+\left(\|\mathbf{R}_2\delta_x U^{n-1}\|^2+\|\mathbf{R}_2\delta_x U^n\|^2\right).
$$

Then, Eq. (3.25) can be rewritten as

$$
B^{n+1}-B^n \le \tau C\left(B^{n+1}+B^n\right),
$$

where Lemma 3.2 is used. Therefore, if τ is sufficiently small, which satisfies $\tau \le \dfrac{k-2}{kC}$ and $k>2$, then

$$
B^{n+1} \le \frac{(1+\tau C)}{(1-\tau C)}B^n \le (1+\tau kC)\,B^n \le \exp(kCT)\,B^0.
$$

Hence,

$$
\|U^{n+1}\|^2 \le C, \quad \|\mathbf{R}_2\delta_x U^{n+1}\|^2 \le C,
$$

which proves Eq. (3.22) by using Lemma 3.3. We complete the proof by using Lemma 3.5 to obtain

$$\|U^{n+1}\|_\infty \leq C$$

□

Lastly, in this section, to prove the convergence and stability of the scheme (3.4) with Eqs. (3.5)-(3.6), we let $u_i^n = u(x_i, t^n)$ be the solution to Eqs. (1.1)-(1.3). By letting $e_i^n = u_i^n - U_i^n$, the truncation error of the scheme (3.4) with Eqs. (3.5)-(3.6) can be obtained from

$$\delta_{\hat{t}}(e_i^n) - \mathcal{B}_x^{-1}\delta_{\hat{t}}\delta_{x\bar{x}}(e_i^n) + \mathcal{A}_x^{-1}\delta_{\hat{x}}(\bar{e}_i^n) + \mathcal{C}_x^{-1}\delta_{x\bar{x}\hat{x}}(\bar{e}_i^n) + \frac{1}{2}\mathcal{A}_x^{-1}\delta_{\hat{x}}\left[(u_i^n)^2 - (U_i^n)^2\right] = r_i^n \tag{3.26}$$

By Taylor expansion, it can be easily demonstrated that $|r_i^n| = \mathcal{O}(\tau^2 + h^4)$, as τ, $h \to 0$. Furthermore, by previous observation, the truncation error in the term of u_i^n becomes $\mathcal{O}(\tau^2 + h^4)$. Now, we present the convergence theorem of the proposed scheme.

Theorem 3.11 (Convergence)**.** *Assume that u_0 are sufficiently smooth and $U^n \in Z_{h,0}$. If τ is sufficiently small, then the solution U^n of scheme (3.4) with Eqs. (3.5)-(3.6) converges to the solution of Eqs. (1.1)-(1.3) in a sense of $\|\cdot\|_\infty-$norm with the convergence rate of order $\mathcal{O}(\tau^2+h^4)$.*

Proof. By taking the inner product of Eqs. (3.26) with $2\bar{e}^n$, respectively, we obtain

$$\begin{aligned}
\langle\delta_{\hat{t}}\left(e^n\right), 2\bar{e}^n\rangle &= \langle\frac{1}{2\tau}\left(e^{n+1} - e^{n-1}\right), e^{n+1} + e^{n-1}\rangle \\
&= \frac{1}{2\tau}\langle\left(e^{n+1} - e^{n-1}\right), e^{n+1} + e^{n-1}\rangle \\
&= \frac{1}{2\tau}\left[\langle e^{n+1}, e^{n+1}\rangle + \langle e^{n+1}, e^{n-1}\rangle + \langle -e^{n-1}, e^{n+1}\rangle + \langle -e^{n-1}, e^{n-1}\rangle\right] \\
&= \frac{1}{2\tau}\left(\|e^{n+1}\|^2 - \|e^{n-1}\|^2\right), \\
\langle\mathbf{H}_2^{-1}\delta_{\hat{t}}\delta_{x\bar{x}}\left(e^n\right), 2\bar{e}^n\rangle &= \frac{1}{2\tau}\langle\mathbf{H}_2^{-1}\delta_{x\bar{x}}\left(e^{n+1}\right) - \mathbf{H}_2^{-1}\delta_{x\bar{x}}\left(e^{n-1}\right), e^{n+1} + e^{n-1}\rangle \\
&= \frac{1}{2\tau}\left[-\langle\mathbf{H}_2^{-1}\delta_x\left(e^{n+1}\right), \delta_x\left(e^{n+1}\right)\rangle + \langle\mathbf{H}_2^{-1}\delta_x\left(e^{n-1}\right), \delta_x\left(e^{n-1}\right)\rangle\right] \\
&= \frac{1}{2\tau}\left[-\|\mathbf{R}_2\delta_x\left(e^{n+1}\right)\|^2 + \|\mathbf{R}_2\delta_x\left(e^{n-1}\right)\|^2\right], \\
\langle\mathbf{H}_1^{-1}\delta_{\hat{x}}\left(\bar{e}^n\right), 2\bar{e}^n\rangle &= 2\langle\mathbf{H}_1^{-1}\delta_{\hat{x}}\left(\bar{e}^n\right), \bar{e}^n\rangle = 0, \\
\langle\mathbf{H}_3^{-1}\delta_{\hat{x}}\left(\bar{e}^n\right), 2\bar{e}^n\rangle &= 0.
\end{aligned}$$

When taken together, we have

$$\begin{aligned}
&\frac{1}{2\tau}\left(\|e^{n+1}\|^2 - \|e^{n-1}\|^2\right) + \frac{1}{2\tau}\left[\|\mathbf{R}_2\delta_x\left(e^{n+1}\right)\|^2 - \|\mathbf{R}_2\delta_x\left(e^{n-1}\right)\|^2\right] \\
&\qquad + \langle\mathbf{H}_1^{-1}\delta_{\hat{x}}\left[\left(\operatorname{diag}\left(u^n\right)\right)u^n - \left(\operatorname{diag}\left(U^n\right)\right)U^n\right], \bar{e}^n\rangle = \langle r^n, 2\bar{e}^n\rangle.
\end{aligned} \tag{3.27}$$

For the nonlinear term, we also apply Lemma 3.4, Lemma 3.1, the Cauchy-Schwarz inequality, and Theorem 3.10 to obtain

$$
\begin{aligned}
&\langle \mathbf{H}_1^{-1}\delta_{\hat{x}}[(\operatorname{diag}(u^n))\,u^n - (\operatorname{diag}(U^n))\,U^n], \bar{e}^n\rangle \\
&\qquad = -\langle [(\operatorname{diag}(u^n))\,u^n - (\operatorname{diag}(U^n))\,U^n]\,, \mathbf{H}_1^{-1}\delta_{\hat{x}}\bar{e}^n\rangle \\
&\qquad = -h\sum_{i=1}^{M-1}\left[(u_i^n)^2 - (U_i^n)^2\right]\mathcal{A}_x^{-1}\delta_{\hat{x}}\left(\bar{e}_i^n\right) \\
&\qquad = h\sum_{i=1}^{M-1}\left(u_i^n + U_i^n\right)e_i^n\mathcal{A}_x^{-1}\delta_{\hat{x}}\left(\bar{e}_i^n\right) \\
&\qquad \le h\sum_{i=1}^{M-1}\left(\|u^n\|_\infty + \|U^n\|_\infty\right)e_i^n\mathcal{A}_x^{-1}\delta_{\hat{x}}\left(\bar{e}_i^n\right) \\
&\qquad \le h\sum_{i=1}^{M-1} Ce_i^n\mathcal{A}_x^{-1}\delta_{\hat{x}}\left(\bar{e}_i^n\right) \\
&\qquad \le C\left[\left(h\sum_{i=1}^{M-1}|e_i^n|^2\right)^{\frac{1}{2}}\left(h\sum_{i=1}^{M-1}\left|\mathcal{A}_x^{-1}\delta_{\hat{x}}\left(\bar{e}_i^n\right)\right|^2\right)^{\frac{1}{2}}\right] \\
&\qquad = C\left(\|e^n\|\cdot\|\mathbf{H}_1^{-1}\delta_{\hat{x}}\left(\bar{e}^n\right)\|\right) \\
&\qquad = C\frac{1}{2}\left(\|e^n\|^2 + \|\mathbf{H}_1^{-1}\delta_{\hat{x}}\left(\bar{e}^n\right)\|^2\right) \\
&\qquad = C\left(\|e^n\|^2 + \frac{1}{2}\|\mathbf{H}_1^{-1}\delta_x e^{n+1}\|^2 + \frac{1}{2}\|\mathbf{H}_1^{-1}\delta_x e^{n-1}\|^2\right) \\
&\qquad \le C\left(\|e^n\|^2 + \frac{3}{2}\|\mathbf{R}_2\delta_x e^{n+1}\|^2 + \frac{3}{2}\|\mathbf{R}_2\delta_x e^{n-1}\|^2\right) \\
&\qquad = C\left(\|e^n\|^2 + \|\mathbf{R}_2\delta_x e^{n+1}\|^2 + \|\mathbf{R}_2\delta_x e^{n-1}\|^2\right),
\end{aligned}
$$

which is

$$
\langle \mathbf{H}_1^{-1}\delta_{\hat{x}}[(\operatorname{diag}(u^n))\,u^n - (\operatorname{diag}(U^n))\,U^n], \bar{e}^n\rangle \le C\left(\|e^n\|^2 + \|\mathbf{R}_2\delta_x e^{n+1}\|^2 + \|\mathbf{R}_2\delta_x e^{n-1}\|^2\right). \tag{3.28}
$$

Furthermore,

$$
\langle r^n, 2\bar{e}^n\rangle \le \|r^n\|^2 + \frac{1}{2}\left(\|e^{n-1}\|^2 + \|e^{n+1}\|^2\right). \tag{3.29}
$$

By combining Eq. (3.27) together with Eqs. (3.28)-(3.29), we have

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\|e^{n+1}\|^2 - \|e^{n-1}\|^2\right) + \frac{1}{2\tau}\left[\|\mathbf{R}_2\delta_x\left(e^{n+1}\right)\|^2 - \|\mathbf{R}_2\delta_x\left(e^{n-1}\right)\|^2\right] \\
&\qquad\qquad \le \|r^n\|^2 + C\left(\|e^n\|^2 + \|\mathbf{R}_2\delta_x e^{n+1}\|^2 + \|\mathbf{R}_2\delta_x e^{n-1}\|^2\right).
\end{aligned} \tag{3.30}
$$

Let

$$
\text{Error}^n = \left(\|e^{n-1}\|^2 + \|e^n\|^2\right) + \left(\|\mathbf{R}_2\delta_x e^{n-1}\|^2 + \|\mathbf{R}_2\delta_x e^n\|^2\right).
$$

Then, Eq. (3.30) can be rewritten as

$$
\text{Error}^{n+1} - \text{Error}^n \le 2\tau\|r^n\|^2 + \tau C\left(\text{Error}^{n+1} + \text{Error}^n\right), \tag{3.31}
$$

or

$$(1-\tau C)\left(\text{Error}^{n+1}-\text{Error}^{n}\right)\leq 2\tau\|r^n\|^2+\tau C\text{Error}^{n}.$$

If τ is sufficiently small, which satisfies $1-C\tau>0$, then

$$\text{Error}^{n+1}-\text{Error}^{n}\leq 2\tau\|r^n\|^2+\tau C\text{Error}^{n}. \tag{3.32}$$

Summing up Eq. (3.32) from 1 to n, we have

$$\begin{aligned}\text{Error}^{n+1}-\text{Error}^{1}&\leq 2\tau\sum_{k=2}^{n}\|r^k\|^2+\tau C\sum_{k=1}^{n}\text{Error}^{k}\\&\leq \mathcal{O}(\tau^2+h^4)^2+\tau C\sum_{k=1}^{n}\text{Error}^{k},\end{aligned} \tag{3.33}$$

where we used

$$2\tau\sum_{k=2}^{n}\|r^k\|^2\leq 2(n-1)\tau\max_{2\leq k\leq n}\|r^k\|^2\leq C(\tau^2+h^4)^2$$

Since we can approximate u^1 using the available method, we then have $\text{Error}^1=C(\tau^2+h^4)^2$. Hence, by applying Lemma 3.3, we obtain

$$\text{Error}^{n+1}\leq C(\tau^2+h^4)^2,$$

that is

$$\|e^{n+1}\|^2\leq C(\tau^2+h^4)^2,\quad \|\delta_x e^{n+1}\|^2\leq C(\tau^2+h^4)^2.$$

Finally, by applying Lemma 3.4 and Lemma 3.5, we get

$$\|e^{n+1}\|_\infty^2\leq C(\tau^2+h^4)^2.$$

This completes the proof. □

Chapter 4

Numerical experiments

In this section, several numerical experiments are computed to affirm the effectiveness and correctness of our theoretical analysis in the previous section using the evolution of the single solitary wave solutions. As mentioned earlier, the proposed scheme is to form a linear system with a constant coefficient matrix at each time step.As a result in significantly reduced the computational time and resources.

Since a three-level approximation in time is used in the algorithm together with the initial conditions u^0, we need u^1 as known condition. Therefore, we will choose an available method with the same accuracy such as Crank–Nicolson method or Adams-Bashforth to solve for u^1. To start the process, the accuracy of the method is measured by a comparison of numerical solutions and the exact solutions. The terms $\|\cdot\|_\infty$-norm is defined by

$$\|\text{Error}_u^n\|_\infty = \max_i |U(x_i, t^n) - u_i^n| \,,$$

and the rate of convergence can be measured as according to the formula

$$\text{Rate} = \log_2 \left(\frac{\text{Error}_h}{\text{Error}_{h/2}} \right),$$

where Error_h and $\text{Error}_{h/2}$ are the norm errors with the grid sizes h and $h/2$, respectively.

4.1 Error and rate of convergence

In this subsection, we will look at the BBM-KdV equation [2]

$$u_t - u_{xxt} + u_x + u_{xxx} + uu_x = 0, \tag{4.1}$$

with the analytical solution to the BBM-KdV equation is as follows:

$$u(x,t) = 3csech^2(m(x - (c+1)t - x_0)),$$

which stands for a single solitary wave having amplitude $3c$ centered at x_0 with velocity $v = 1 + c$ and $m = \sqrt{\frac{c}{4c+8}}$. As a result, the initial condition corresponding to the precise solution is provided for Eq. (4.1) as follows:

$$u(x, 0) = 3csech^2[m(x - x_0)]$$

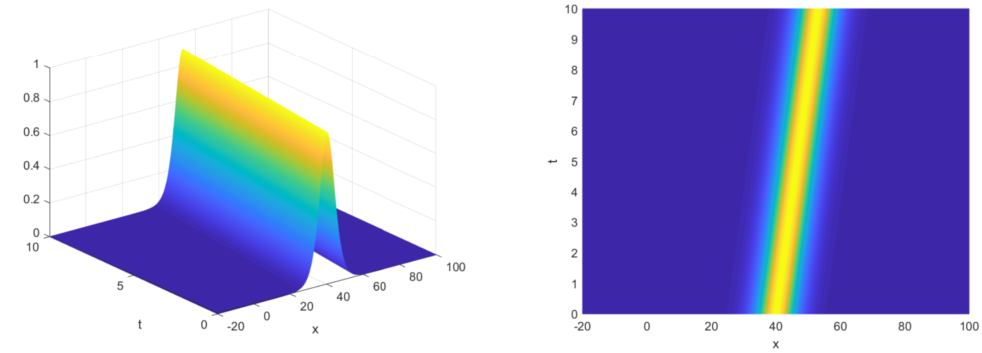
and the boundary conditions (1.3).

The current approach is quantitatively tested in the simulation for the case of $c = 0.3$ with $x_0 = 40$ over the space interval $[-20, 180]$ and the time interval $[0, 10]$. Table 4.1 presents the maximum errors for various step sizes τ and h at the final time $T = 10$. The obtained data confirms the fourth-order rate of convergence, which aligns significantly with the theoretical expectations for each scheme. In order to establish the accuracy of the proposed scheme, a comparative assessment will be conducted on the errors associated with the finite difference schemes described in [13], which consist of a second-order nonlinear scheme (Scheme I) and a linear three-level scheme (Scheme II).

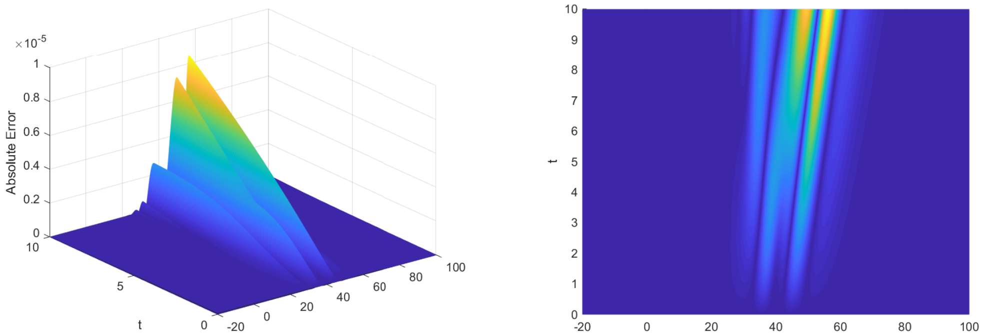
In Figure 4.1, we depict the numerical solutions and their distribution errors generated by the present scheme, using $h = 0.125$ and $\tau = h^2$. The results indicate that the scheme can effectively replicate the exact solution, with the greatest discrepancy observed near the wave amplitude's maximum displacement.

Table 4.1: The error and convergence rate of numerical solutions for the BBM-KdV system at $T = 10$ using $\tau = h^2$.

Sceme	h	$\|\|\text{Error}\|\|_\infty$	Rate	$\|\|\text{Error}\|\|_{L_2}$	Rate
Presented scheme	0.5	2.1503×10^{-3}	-	5.9504×10^{-3}	-
	0.25	1.3593×10^{-4}	3.9795	3.7766×10^{-4}	3.9767
	0.125	8.5223×10^{-6}	3.9826	2.3681×10^{-5}	3.9842
	0.0625	5.3032×10^{-7}	3.9857	1.4790×10^{-6}	4.0012
scheme I	0.5	9.5002×10^{-3}	-	2.3928×10^{-2}	-
	0.25	2.0790×10^{-3}	2.1591	5.2421×10^{-3}	2.1577
	0.125	5.0023×10^{-4}	2.0568	1.2617×10^{-3}	2.0561
	0.0625	1.2385×10^{-4}	2.0084	3.1234×10^{-4}	2.0086
scheme II	0.5	1.3086×10^{-2}	-	3.3496×10^{-2}	-
	0.25	2.3102×10^{-3}	2.5019	5.8545×10^{-3}	2.5163
	0.125	5.1477×10^{-4}	2.1367	1.3001×10^{-3}	2.1491
	0.0625	1.2476×10^{-4}	2.0385	3.1475×10^{-4}	2.0405



(a) Profiles of numerical solution.



(b) Profiles of absolute error.

Figure 4.1: Numerical solutions and its absolute error using $h = 0.125, \tau = h^2$, computational domain $[-20, 100]$ and $t = 0$ to 10.

Table 4.2: The quantities of Q^n and its errors using $h = 0.125$ and $\tau = h^2$.

	Present scheme		Scheme I		Scheme II	
t	Q^n	$\lVert Q^n - Q(0)\rVert$	Q^n	$\lVert Q^n - Q(0)\rVert$	Q^n	$\lVert Q^n - Q(0)\rVert$
0	9.9679486318	-	9.9679486355	-	9.9679486355	-
1	9.9679486352	3.3284×10^{-9}	9.9679486354	6.5263×10^{-12}	9.9679484074	2.2809×10^{-7}
2	9.9679486374	5.5818×10^{-9}	9.9679486354	1.3312×10^{-11}	9.9679481782	4.5728×10^{-7}
3	9.9679486398	7.9470×10^{-9}	9.9679486354	1.9893×10^{-11}	9.9679479606	6.7490×10^{-7}
4	9.9679486423	1.0484×10^{-8}	9.9679486354	2.6636×10^{-11}	9.9679477581	8.7734×10^{-7}
5	9.9679486450	1.3148×10^{-8}	9.9679486354	3.3239×10^{-11}	9.9679475727	1.0627×10^{-6}
6	9.9679486477	1.5873×10^{-8}	9.9679486354	3.9874×10^{-11}	9.9679474049	1.2306×10^{-6}
7	9.9679486504	1.8585×10^{-8}	9.9679486354	4.6422×10^{-11}	9.9679472541	1.3814×10^{-6}
8	9.9679486530	2.1200×10^{-8}	9.9679486354	5.2957×10^{-11}	9.9679471193	1.5161×10^{-6}
9	9.9679486554	2.3591×10^{-8}	9.9679486354	5.9490×10^{-11}	9.9679469994	1.6361×10^{-6}
10	9.9679486574	2.5599×10^{-8}	9.9679486354	6.6104×10^{-11}	9.9679468928	1.7427×10^{-6}

Chapter 5

Conclusion

In this study, we present a new compact operator for the third-derivative term that achieves a fourth-order accuracy. Furthermore, we have developed a specific scheme using finite difference method and compact operators to solve the numerical solution of the BBM-KdV equation. The scheme is a linear explicit system, providing second-order accuracy in time and fourth-order accuracy in space, matching the theoretical predictions. Moreover, the structural preserving properties have been proof.

The linearity of the scheme eliminates the need for implicit treatment of nonlinear terms, simplifying the computational process and conserving computing resources. This advantage, combined with the structural preserving properties, enhances the robustness and efficiency of the numerical framework. The obtained numerical results demonstrate the scheme's ability to accurately capture the behavior of nonlinear waves in different scientific domains.

Overall, the findings of this research support the effectiveness and applicability of the proposed numerical scheme for studying the BBM-KdV equation. The accurate simulations and deeper insights gained into the dynamics of nonlinear waves pave the way for further research and applications in areas such as fluid dynamics, coastal engineering, and nonlinear optics.

Bibliography

[1] Rouatbi A., Achouri T. and Omrani K. (2018) High-order conservative difference scheme for a model of nonlinear dispersive equations. Comp. Appl. Math. 37, 4169–4195. https://doi.org/10.1007/s40314-017-0567-1

[2] Suebcharoen T., Poochinapan K. and Wongsaijai B. (2022) Bifurcation Analysis and Numerical Study of Wave Solution for Initial-Boundary Value Problem of the KdV-BBM Equation. Mathematics, 10, 3825. https://doi.org/10.3390/math10203825

[3] Wongsaijai B. and Poochinapan K. (2022) High-performance computing of structure-preserving algorithm for the coupled BBM system formulated by weighted compact difference operators. Mathematics and Computers in Simulation 205 : 439-467

[4] Li S. (2016) Numerical analysis for fourth-order compact conservative difference scheme to solve the 3D Rosenau-RLW equation. Comp Math Appl 72:2388-2407

[5] Gray R.M. (2006) Toeplitz and Circulant Matrices: A Review, Now Publishers inc

[6] Wang T., Guo B. and Xu Q. (2013) Fourth-order compact and energy conservative difference schemes for the nonlinear Schrödinger equation in two dimensions. J Comput Phys 243:382-399

[7] C.M. Fonseca Da (2007) On the eigenvalues of some tridiagonal matrices, J. Comput. Appl. Math. 200(1) 283–286.

[8] Wang T. (2014) Optimal point-wise error estimate of a compact difference scheme for the coupled Gross–Pitaevskii equations in one dimension, J. Sci. Comput. 59 158–186.

[9] Wang T. and Guo B. (2011) Unconditional convergence of two conservative compact difference schemes for nonlinear Schroinger equation in one dimension, Sci. Sin. Math. 41 (3) 207–233.

[10] Yu.I. Dimitrienko, Shuguang Lia and Yi Niu (2021) Study on the dynamics of a nonlinear dispersion model in both 1D and 2D based on the fourth-order compact conservative difference scheme, Mathematics and Computers in Simulation 182 : 661–689. https://doi.org/10.1016/j.matcom.2020.11.012

[11] Zhou Y. (1990) Application of Discrete Functional Analysis to the Finite Difference Method. Inter Acad Publishers Beijing

[12] Hao Z.P., Sun Z.Z., Cao W.R. (2015) A three-level linearized compact difference scheme for the Ginzburg-Landau equation. Numer Methods Partial Differ Equ 31(3):876-899

[13] Rouatbi A. and Omrani K. (2017) Two conservative difference schemes for a model of nonlinear dispersive equations, Chaos, Solitons & Fractals, Volume 104, Pages 516-530, ISSN 0960-0779, https://doi.org/10.1016/j.chaos.2017.09.006.

Appendices

PERFORMANCE OF A COMPACT STRUCTURE-PRESERVING FINITE DIFFERENCE SCHEME FOR A MODEL OF NONLINEAR DISPERSIVE WAVE EQUATIONS

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Abstract

In this study, a high-order algorithm based on the finite difference method is developed to solve the nonlinear shallow water equation modeled by the BBM-KdV equation. The proposed scheme is implemented using compact difference operators and precisely conserves the mass preserving property on any time region. Numerical experiments are conducted to illustrate the performance and accuracy of the proposed method in comparison with other schemes in various benchmark problems.

Introduction

The study of nonlinear dispersive wave equations is an active area of research in applied mathematics and engineering due to its widespread applications in various physical phenomena, such as water waves, optics, and plasma physics. In particular, the BBM-KdV equation, which combines the characteristics of the Benjamin-Bona-Mahony (BBM) equation and the Korteweg-de Vries (KdV) equation, provides a versatile framework for analyzing the dynamics of nonlinear waves in different contexts. The BBM-KdV equation that we focus in this study can be expressed in the following form :

$$u_t - u_{xxt} + u_x + u_{xxx} + uu_x = 0,$$

Here, u represents the wave profile as a function of space x and time t. Furthermore, we consider the BBM-KdV equation with the following initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in [x_L, x_R],$$

$$u(x_L,t) = u(x_R,t) = 0, \quad u_x(x_L,t) = u_x(x_R,t) = 0, \quad t \in (0,T]$$

In the study of the BBM-KdV equation, various numerical methods have been employed to obtain solutions and analyze the wave dynamics. One widely utilized numerical technique is the finite difference method (FDM), which normally approximates the derivative terms in the BBM-KdV equation by expressing the wave profile u at discrete points in space and time. To enhance the accuracy and efficiency of the FDM, compact finite difference operators are often employed. Compact operators are difference operators that provide high-order accuracy while requiring fewer computational resources compared to traditional finite difference approximations. These operators can effectively capture the intricate behavior of the BBM-KdV equation while minimizing numerical artifacts. In this study, the authors focus on evaluating the performance of a specific compact structure-preserving finite difference scheme for the BBM-KdV equation. In particular, the study introduces a novel compact operator for the third-order derivative term in the equation. The developed compact operator for the third-order derivative is designed to achieve high-order accuracy, allowing for the precise representation of the wave profile u and its derivatives.

Structure preserving property

Theorem 3.8 Suppose that $U^n \in Z_{h,0}$ for $n = 0,1,2,\ldots,N$.Then, the scheme with initial and boundary conditions is mass conservative as appeared in the following :

$$Q^n = \frac{h}{2}\sum_{i=1}^{M-1}\left(U_i^{n+1} + U_i^n\right) = Q^{n-1} = \ldots = Q^0.$$

Notations

x_L is the initial point in one-dimentional,

x_R is the terminal point in one-dimentional,

T represents the final time,

M represents the number of grid points in space,

N represents the number of grid points in time,

x is space variable, h is time step size defined by $h = \dfrac{x_R - x_L}{M}$,

t is time variable, τ is time step size defined by $\tau = \dfrac{T}{N}$,

U_i^n denotes the numerical approximation to

$\|U^n\|_\infty$ denotes the maximum-norms $\|U^n\|$ denotes the L2-norms

Compact operators

First-order compact operator $\mathcal{A}_x(U_i^n) = (1 + \frac{h^2}{6}\delta_{x\bar{x}})U_i^n,$

Second-order compact operator $\mathcal{B}_x(U_i^n) = (1 + \frac{h^2}{12}\delta_{x\bar{x}})U_i^n,$

Third-order compact operator $\mathcal{C}_x(U_i^n) = (1 + \frac{h^2}{4}\delta_{x\bar{x}})U_i^n$

The proposed scheme

The scheme

$$\delta_{\hat{t}}(U_i^n) - \delta_{\hat{t}}\mathcal{B}_x^{-1}\delta_{x\bar{x}}(U_i^n) + \mathcal{A}_x^{-1}\delta_{\hat{x}}(\bar{U}_i^n) + \mathcal{C}_x^{-1}\delta_{x\bar{x}\hat{x}}(\bar{U}_i^n) + \frac{1}{2}\mathcal{A}_x^{-1}\delta_{\hat{x}}\left[(U_i^n)^2\right] = 0,$$

with initial condition

$$U_i^0 = U_0(x_i), \quad 0 \le i \le M,$$

and boundary conditions

$$U_0^n = U_M^n = 0, \quad \delta_{\hat{x}}(U_0^n) = \delta_{\hat{x}}(U_M^n) = 0,$$

Vector form

$$\delta_{\hat{t}}(U^n) - \mathbf{H}_2^{-1}\mathbf{D}_2\delta_{\hat{t}}(U^n) + \mathbf{H}_1^{-1}\mathbf{D}_1(\bar{U}^n) + \mathbf{H}_3^{-1}\mathbf{D}_3(\bar{U}^n) + \frac{1}{2}\mathbf{H}_1^{-1}\delta_x\left[(\mathrm{diag}U^n)U^n\right] = 0,$$

where

$$\mathbf{D}_1 = \frac{1}{2h}\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ -1 & 0 & 1 & \ldots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \ldots & -1 & 0 & 1 \\ 0 & \ldots & 0 & -1 & 0 \end{bmatrix}_{(M-1)\times(M-1)} \quad \mathbf{D}_2 = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & \ldots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \ldots & 1 & -2 & 1 \\ 0 & \ldots & 0 & 1 & -2 \end{bmatrix}_{(M-1)\times(M-1)} \quad \mathbf{D}_3 = \frac{1}{2h^3}\begin{bmatrix} 0 & -2 & 1 & 0 & \ldots & 0 \\ 2 & 0 & -2 & 1 & \ldots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & \ldots & -1 & 2 & 0 & -2 \\ 0 & \ldots & 0 & -1 & 2 & 0 \end{bmatrix}_{(M-1)\times(M-1)}.$$

Numerical experiments

In this section, we will look at the BBM-KdV equation

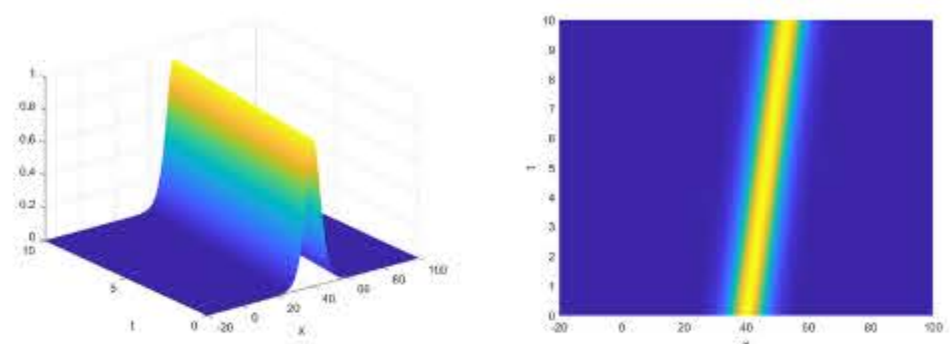
$$u_t - u_{xxt} + u_x + u_{xxx} + uu_x = 0,$$

with the initial condition

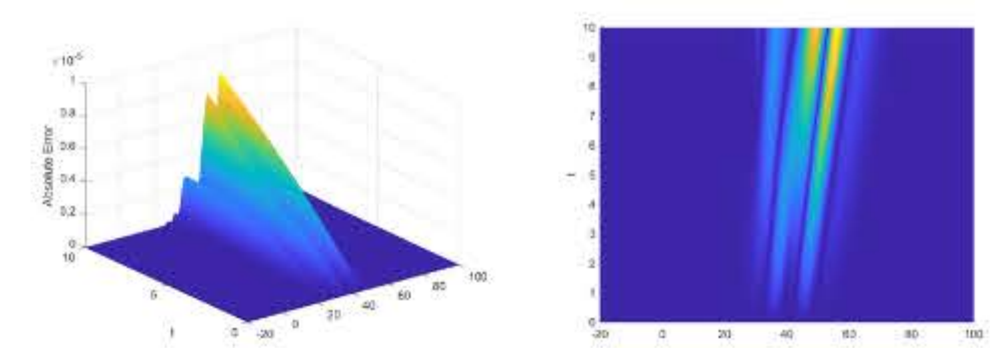
$$u(x,0) = 3csech^2[m(x - x_0)]$$

with the analytical solution to the BBM-KdV equation is as follows

$$u(x,t) = 3csech^2(m(x-(c+1)t - x_0))$$



Profiles of numerical solution.



Profiles of absolute error.

The error and convergence rate of numerical solutions for the BBM-KdV system at $T = 10$ using $\tau = h^2$.

Sceme	h	$\|\mathrm{Error}\|_\infty$	Rate	$\|\mathrm{Error}\|_{L_2}$	Rate
Presented scheme	0.5	2.1503×10^{-3}	-	5.9504×10^{-3}	-
	0.25	1.3593×10^{-4}	3.9795	3.7766×10^{-4}	3.9767
	0.125	8.5223×10^{-6}	3.9826	2.3681×10^{-5}	3.9842
	0.0625	5.3032×10^{-7}	3.9857	1.4790×10^{-6}	4.0012
scheme I	0.5	9.5002×10^{-3}	-	2.3928×10^{-2}	-
	0.25	2.0790×10^{-3}	2.1591	5.2421×10^{-3}	2.1577
	0.125	5.0023×10^{-4}	2.0568	1.2617×10^{-3}	2.0561
	0.0625	1.2385×10^{-4}	2.0084	3.1234×10^{-4}	2.0086
scheme II	0.5	1.3086×10^{-2}	-	3.3496×10^{-2}	-
	0.25	2.3102×10^{-3}	2.5019	5.8545×10^{-3}	2.5163
	0.125	5.1477×10^{-4}	2.1367	1.3001×10^{-3}	2.1491
	0.0625	1.2476×10^{-4}	2.0385	3.1475×10^{-4}	2.0405

	Present scheme		Scheme I		Scheme II	
t	Q^n	$\|Q^n - Q(0)\|$	Q^n	$\|Q^n - Q(0)\|$	Q^n	$\|Q^n - Q(0)\|$
0	9.9679486318	-	9.9679486355	-	9.9679486355	-
1	9.9679486352	3.3284×10^{-9}	9.9679486354	6.5263×10^{-12}	9.9679484074	2.2809×10^{-7}
2	9.9679486374	5.5818×10^{-9}	9.9679486354	1.3312×10^{-11}	9.9679481782	4.5728×10^{-7}
3	9.9679486398	7.9470×10^{-9}	9.9679486354	1.9893×10^{-11}	9.9679479606	6.7490×10^{-7}
4	9.9679486423	1.0484×10^{-8}	9.9679486354	2.6636×10^{-11}	9.9679477581	8.7734×10^{-7}
5	9.9679486450	1.3148×10^{-8}	9.9679486354	3.3239×10^{-11}	9.9679475727	1.0627×10^{-6}
6	9.9679486477	1.5873×10^{-8}	9.9679486354	3.9874×10^{-11}	9.9679474049	1.2306×10^{-6}
7	9.9679486504	1.8585×10^{-8}	9.9679486354	4.6422×10^{-11}	9.9679472541	1.3814×10^{-6}
8	9.9679486530	2.1200×10^{-8}	9.9679486354	5.2957×10^{-11}	9.9679471193	1.5161×10^{-6}
9	9.9679486554	2.3591×10^{-8}	9.9679486354	5.9490×10^{-11}	9.9679469994	1.6361×10^{-6}
10	9.9679486574	2.5599×10^{-8}	9.9679486354	6.6104×10^{-11}	9.9679468928	1.7427×10^{-6}

The quantities of Q^n and its errors using $h = 0.125$ and $\tau = h^2$.

Conclusion

In this study, we present a new compact operator for the third-derivative term that achieves a fourth-order accuracy. Furthermore, we have developed a specific scheme using finite difference method and compact operators to solve the numerical solution of the BBM-KdV equation. The scheme is a linear explicit system, providing second-order accuracy in time and fourth-order accuracy in space, matching the theoretical predictions. Moreover, the structural preserving properties have been proof.