Chapter 1

Infinite Series

1.1 Infinite series

Definition 1.1. An infinite series is an expression that can be written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

The number a_n is called the n^{th} term of the series.

Let s_n denote the sum of the initial terms of the series $\sum_{n=1}^{\infty} a_n$, i.e.,

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \end{aligned}$$

We call $\displaystyle\sum_{n=1}^{\infty} a_n$ **the infinite series** of the sequence $\{a_n\}$.The sequence s_n is called the nth partial sum of the series and the sequence and $\{s_n\}$ is called the sequence of partial sums.

Definition 1.2. *Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$. If $\lim_{n\to\infty} s_n = S$ where S is a real number , then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is said to converge to S and we can denote $S = \displaystyle\sum_{n=1}^{\infty} a_n$.*

*If $\lim_{n\to\infty} s_n$ does not exists, the sequence $\displaystyle\sum_{n=1}^{\infty} a_n$ is called **divergence** and has no sum.*

Ex. 1.1. *Determine whether the following series converges or diverges. If it converges, find the sum.*

1. $\sum_{n=1}^{\infty}(-1)^n$

2. $\ln\frac{1}{2}+\ln\frac{2}{3}+\ln\frac{3}{4}+\cdots+\ln\frac{n}{n+1}+\cdots$

3. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}+\cdots$

1.2 Geometric series

Theorem 1.3. *A geometric series* $\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r}, & \text{if} |r| < 1 \\ \text{diverges}, & \text{if} |r| \geq 1 \end{cases}$

Ex. 1.2. Determine whether the following series converges or diverges. If it converges, find the sum.

1. $1 + 2 + 4 + 8 + \cdots + 2^{n-1} + \cdots$

   ....

   ....

   ....

   ....

   ....

   ....

2. $3 + \dfrac{3}{10} + \dfrac{3}{10^2} + \cdots + \dfrac{3}{10^{n-1}} + \cdots$

   ....

   ....

   ....

   ....

   ....

   ....

3. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots + (-1)^{n+1}\frac{1}{2^n} + \cdots$

4. $1 + 1 + 1 + \cdots + 1 + \cdots$

5. $1 - 1 + 1 - 1 + \cdots + (-1)^{n+1} + \cdots$

6. $1 + \frac{3}{2} + \frac{9}{4} + \cdots + \frac{3^{n-1}}{2^{n-1}} + \cdots$

1.3 Convergence Tests

1.3.1 The divergence test

Theorem 1.4. *If* $\lim_{n\to\infty} a_n \neq 0$ *or* $\lim_{n\to\infty} a_n$ *does not exists then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Note 1. If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

2. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ may either converge or diverge.

Ex. 1.3. Determine whether the following series converges or diverges.

1. $\sum_{n=1}^{\infty} \sqrt[3]{n}$

....

....

....

....

....

....

2. $\sum_{n=1}^{\infty} \frac{n^2+1}{2n^2+3}$

....

....

....

....

....

....

3. $\sum_{n=1}^{\infty}(-1)^n$

....

....

....

....

....

....

4. $\sum_{n=1}^{\infty}(\ln 3)^n$

....

....

....

....

....

....

1.3.2 Algebraic properties of infinite series

Let A, B and k be any real numbers.

1. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to A and B, respectively, then $\sum_{n=1}^{\infty}(a_n \pm b_n)$ converges to $A \pm B$
2. The series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} ka_n$ both converge or both diverge. In the case of convergence, $\sum_{n=1}^{\infty} ka_n$ converges to kA
3. Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer M, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=M}^{\infty} a_n$ both converge or both diverge.

Ex. 1.4. Find $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$ and $\sum_{n=1}^{\infty} \frac{4}{2^{n-1}}$.

....

....

....

....

....

....

Ex. 1.5. Find $\sum_{n=5}^{\infty} \frac{3}{6^{n}}$.

....

....

....

....

....

....

1.4 Convergence tests for a series with positive terms

1.4.1 The Integral Test

Theorem 1.5. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series with positive terms. If* f *is a function that is decreasing and continuous on an interval* $[a, \infty)$ *such that* $f(n) = a_n$ *for all* $n \geq a$*, then* $\sum_{n=1}^{\infty} a_n$ *and* $\int_a^{\infty} f(x)dx$ *both converge or both diverge.*

Ex. 1.6. Show that the integral test applies, and use the integral test to determine whether the following series converge or diverge.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

   ....

   ....

   ....

   ....

   ....

   ....

2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

   ....

   ....

   ....

   ....

   ....

   ....

Definition 4.6. *A $p-$series is an infinite series of the form*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

where $p > 0$. In the case $p = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is called the ***harmonic series.***

Theorem 1.7. *A $p-$series converges if $p > 1$ and diverges if $0 < p \leq 1$.*

Ex. 1.7. Determine whether the following series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n^e}$

   ....

   ....

   ....

   ....

   ....

   ....

2. $\sum_{n=1}^{\infty} \frac{1}{n^{\sin 26^\circ}}$

   ....

   ....

   ....

   ....

   ....

   ....

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

1.4.2 The Comparison Test

Theorem 1.8. *Let* $\displaystyle\sum_{n=1}^{\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{\infty} b_n$ *be series with nonnegative terms and suppose that and* $a_1 \leq b_1, a_2 \leq b_2, a_3 \leq b_3, ..., a_k \leq b_k, ...$

1. *If* $\displaystyle\sum_{n=1}^{\infty} b_n$ *converge, then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converge.*
2. *If* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverge, then* $\displaystyle\sum_{n=1}^{\infty} b_n$ *diverge.*

Ex. 1.8. Determine whether the following series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n} - \frac{1}{2}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{5n-1}}$

1.4.3 The Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with nonnegative terms and define $\rho = \lim_{n\to\infty} \frac{a_n}{b_n}$.

1. If ρ is finite and $\rho > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
2. If $\rho = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
3. If $\rho = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex. 1.9. Determine whether the following series converges or diverges by using the limit comparison test .

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+2}$

   ....

   ....

   ....

   ....

   ....

   ....

2. $\sum_{n=1}^{\infty} \frac{n+2}{n^2}$

   ....

   ....

   ....

   ....

3. $\sum_{n=1}^{\infty} \frac{3n^3-4n^2+5}{n^5-n^3+2}$

   ....

   ....

   ....

   ....

   ....

   ....

4. $\sum_{n=1}^{\infty} \frac{2n^2+n}{n^4+\sqrt{n}}$

....

....

....

....

....

....

1.4.4 The Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

1. If $\rho < 1$, then the series converges.
2. If $\rho > 1$ or $\rho = \infty$, then the series diverges.
3. If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

Ex. 1.10. Determine whether the following series converges or diverges by using the ratio test.

1. $\sum_{n=1}^{\infty} \frac{1}{n!}$

....

....

....

....

2. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

....

....

....

....

....

....

3. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

....

....

....

....

....

....

4. $\sum_{n=1}^{\infty} \frac{(2n)!}{5^n}$

....

....

....

....

....

....

5. $\sum_{n=1}^{\infty} \frac{1}{4n-1}$

....

....

....

....

1.4.5 The Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and $\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{1/n}$.

1. If $\rho < 1$, $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\rho > 1$ or $\rho = \infty$, $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

Ex. 1.11. Determine whether the following series converges or diverges by using the root test.

1. $\sum_{n=2}^{\infty} (\frac{9n-1}{3n+2})^n$

....

....

....

....

....

....

2. $\sum_{n=1}^{\infty} \frac{1}{(\ln(2n+3))^n}$

....

....

....

....

....

....

3. $\sum_{n=1}^{\infty}(n^{1/n}-1)^n$

....

....

....

....

....

....

4. $\sum_{n=1}^{\infty}(\frac{n+1}{n})^{n^2}$

....

....

....

....

....

....

1.5 Alternating Series, Absolute and Conditional Convergence

1.5.1 Convergence Test for Alternating Series

An alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ ($a_n > 0$ for all n) converges if and only if

1. $a_n \geq a_{n+1}$ for all n
2. $\lim_{n\to\infty} a_n = 0$

Ex. 1.12. Determine whether the following alternating series converges or diverges.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

2. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{2n+1}$

3. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n+3}{n^2+n}$

1.5.2 Absolute Convergence and Conditionally Convergence

- A series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.
- A series $\sum_{n=1}^{\infty} a_n$ is said to **converge conditionally** if the series $\sum_{n=1}^{\infty} |a_n|$ diverges but the series $\sum_{n=1}^{\infty} a_n$ converges.

Remarks:

1. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.

Ex. 1.13. Determine that the following series converge absolutely or converge conditionally.

1. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2^n}$

   ....

   ....

   ....

   ....

2. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

   ....

   ....

   ....

   ....

   ....

   ....

3. $\sum_{n=1}^{\infty}(-1)^{n+1}(\frac{n}{n^2+1})^n$

....

....

....

....

....

....

4. $1-\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}-\frac{1}{2^5}-\frac{1}{2^6}+...$

....

....

....

....

....

....

5. $\sum_{n=1}^{\infty}(-1)^n\frac{3^n}{n!}$

....

....

....

....

....

....

6. $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$

....

....

....

....

Ratio Test for Absolute Convergence

Theorem: Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms and $\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$

- If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely and therefore converges.
- If $\rho > 1$ or $\rho = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\rho = 1$, no conclusion about convergence or absolute convergence can be drawn from this test.

Ex. 1.14. Use the ratio test for absolute convergence to determine whether the series converges.

1. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$

    ....

    ....

    ....

    ....

    ....

    ....

2. $\sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!}{3^n}$

    ....

    ....

    ....

    ....

    ....

    ....

1.5 Alternating Series, Absolute and Conditional Convergence

Exercise : Determine that the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+5}$ [converge]
2. $\displaystyle\sum_{n=1}^{\infty} \frac{5\sin^2 n}{n!}$ [diverge]
3. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ [diverge]
4. $\displaystyle\sum_{n=1}^{\infty} \frac{4n^2-2n+6}{8n^7+n-8}$ [converge]
5. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{3^n+1}$ [converge]
6. $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+3)}{(n+1)(n+2)(n+5)}$ [diverge]
7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{8n^2-3n}}$ [diverge]
8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+3)^{17}}$ [converge]
9. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n^2}$ [diverge]
10. $\displaystyle\sum_{n=1}^{\infty} n(\frac{1}{2})^n$ [converge]
11. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^3}$ [diverge]
12. $\displaystyle\sum_{n=1}^{\infty} \frac{7^n}{n!}$ [converge]
13. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5^n}$ [converge]
14. $\displaystyle\sum_{n=1}^{\infty} \frac{n!10^n}{3^n}$ [diverge]
15. $\displaystyle\sum_{n=1}^{\infty} n^{50}e^{-n}$ [converge]
16. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3+1}$ [converge]
17. $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{5^n}$ [converge]
18. $\displaystyle\sum_{n=1}^{\infty} \frac{4+|\cos n|}{n^3}$ [converge]

1.6 Power series in x

If $c_0, c_1, ...$ are constants and x is a variable, then the series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + ... + c_k x^k + ...$$

is called a power series in x

Some examples are

1. $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + ...$
2. $\sum_{k=0}^{\infty} \frac{x^k}{k!} =$ ....

1.6.1 Radius and Interval of Convergence

Theorem : For any power series $\sum_{k=0}^{\infty} c_k x^k$ exactly one of the following is true:

1. The series converges only for $x = 0$. (Radius of convergence $R = 0$)

2. The series converges absolutely (and hence converges) for all real values of x. (Radius of convergence $R = \infty$)

3. (a) The series converges absolutely (and hence converges) for all x in some finite open interval $(-R, R)$ (Radius of convergence =R) and;
   (b) diverges if $x < -R$ or $x > R$ and;
   (c) for $x = \pm R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

Ex. 1.15. Find the interval of convergence and radius of convergence of the following power series.

1. $\sum_{k=0}^{\infty} x^k$

....

....

....

....

....

....

1.6 Power series in x

2. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

....

....

....

....

....

....

....

....

....

....

....

....

3. $\sum_{k=0}^{\infty} k!x^k$

....

....

....

....

....

....

....

....

....

....

....

....

1.7 Power series in $x - a$

If $c_0, c_1, ...$ are constants and if we replace x by $(x-a)$, then the series of the form

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_k(x-a)^k + ...$$

is called a power series in $x - a$

Some examples are

$$\sum_{k=0}^{\infty} \frac{(x-1)^k}{k+1} = 1 + \frac{(x-1)}{2} + \frac{(x-1)^2}{3} + \frac{(x-1)^3}{4} + ... \qquad a = 1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k(x+3)^k}{k!} = 1 - (x+3) + \frac{(x+3)^2}{2!} - \frac{(x+3)^3}{3!} + ... \qquad a = -3$$

Theorem : For any power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ exactly one of the following is true:

1. The series converges only for $x = a$. (Radius of convergence $R = 0$)

2. The series converges absolutely (and hence converges) for all real values of x. (Radius of convergence $R = \infty$)

3. (a) The series converges absolutely (and hence converges) for all x in some finite open interval $(a - R, a + R)$ (Radius of convergence =R) and;
   (b) diverges if $x < a - R$ or $x > a + R$ and;
   (c) for $x = a \pm R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

Ex. 1.16. Find the interval of convergence and radius of convergence of the following power series.

1. $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$

....

....

....

....

....

....

2. $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{(x-2)^k}{4^k\sqrt{k}}$

....

....

....

....

....

....

....

....

....

....

....

....

Exercise: Find the interval of convergence and radius of convergence of the following power series.

1. $\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{k+1}$
2. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!}$
3. $\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{1+k^2}$
4. $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{2^k}(x-5)^k$
5. $\displaystyle\sum_{k=0}^{\infty}(\frac{3}{4})^k(x+5)^k$
6. $\displaystyle\sum_{k=0}^{\infty}(-1)^{k+1}\frac{(x+1)^k}{k}$

1.8 Taylor and Maclaurin Series

Definition : If $f(x)$ has derivaives of all orders at a , then we call the series

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

the **Taylor series** for f at a.
In the special case where $a = 0$, this series becomes

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

in which case we call it the **Maclaurin series** for f.

Ex. 1.17. Find the Taylor series for

1. $f(x) = 3x^3 - 5x^2 + x + 2$ about $a = 1$

....

....

....

....

....

....

2. $f(x) = \dfrac{1}{x}$ about $a = -1$

....

....

....

....

....

....

1.8 Taylor and Maclaurin Series

Ex. 1.18. Find the Maclaurin series for

1. $f(x) = e^x$

2. $f(x) = \sin x$

3. $f(x) = \cos x$

4. $f(x) = \dfrac{1}{1-x}$

Second-Order Linear Equations

A **second-order linear differential equation** has the form

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=G(x) \qquad (1)$$

where P, Q, R and G are continuous functions.

In this section we study the case where $G(x)=0$, for all x, in Equation (1). Such equations are called **homogeneous** linear equations. Thus, the form of a second-order linear homogeneous differential equation is

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0 \qquad (2)$$

If $G(x)\neq 0$ for some x, Equation (1) is **nonhomogeneous**.

Theorem If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation (2) and c_1 and c_2 are any constants, then the function

$$y(x)=c_1y_1(x)+c_2y_2(x)$$

is also a solution of Equation (2).

In general, it is not easy to discover particular solutions to a second-order linear equation. But it is always possible to do so if the coefficient functions P, Q and R are constant functions, that is, if the differential equation has the form

$$\boxed{ay''+by'+cy=0} \qquad (3)$$

where a, b and c are constants and $a\neq 0$.

It's not hard to think of some likely candidates for particular solutions of Equation (3) if we state the equation verbally. We are looking for a function y such that a constant times its second derivative y'' plus another constant times y' plus a third constant times y is equal to 0. We know that the exponential function $y=e^{rx}$ (where r is a constant) has the property that its derivative is a constant multiple of itself: $y'=re^{rx}$. Furthermore, $y''=r^2e^{rx}$. If we substitute these expressions into Equation (3), we see that $y=e^{rx}$ is a solution if

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

or

$$(ar^2 + br + c)e^{rx} = 0$$

But e^{rx} is never 0. Thus, $y = e^{rx}$ is a solution of Equation (3) if r is a root of the equation

$$\boxed{ar^2 + br + c = 0} \qquad (4)$$

Equation (4) is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$. Notice that it is an equation that is obtained from the differential equation by replacing y'' by r^2, y' by r, and y by 1.

Sometimes the roots r_1 and r_2 of the auxiliary equation can be found by factoring. In other cases they are found by using the quadratic formula:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad\qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We distinguish three cases according to the sign of the discriminant $b^2 - 4ac$.

Case I: $\boldsymbol{b^2 - 4ac > 0}$

If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$$

EXAMPLE 1 Solve the equation $y'' + y' - 6y = 0$.

EXAMPLE 2 Solve $3\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0$.

Case II: $b^2 - 4ac = 0$

If the auxiliary equation $ar^2+br+c=0$ has only one real root r, then the general solution of $ay''+by'+cy=0$ is

$$y(x)=c_1e^{rx}+c_2xe^{rx}$$

EXAMPLE 3 Solve the equation $4y''+12y'+9y=0$.

Case III: $b^2 - 4ac < 0$

> If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is
>
> $$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

EXAMPLE 4 Solve the equation $y'' - 6y' + 13y = 0$.

Initial-Value and Boundary-Value Problems

An **initial-value problem** for the second-order Equation (1) or (2) consists of finding a solution y of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0 \qquad\qquad y'(x_0) = y_1$$

where y_0 and y_1 are given constants.

EXAMPLE 5 Solve the initial-value problem

$$y'' + y' - 6y = 0 \qquad\qquad y(0) = 1 \qquad\qquad y'(0) = 0$$

EXAMPLE 6 Solve the initial-value problem

$$y'' + y' = 0 \qquad y(0) = 2 \qquad y'(0) = 3$$

A **boundary-value problem** for Equation (1) consists of finding a solution y of the differential equation that also satisfies boundary conditions of the form

$$y(x_0) = y_0 \qquad y(x_1) = y_1$$

EXAMPLE 7 Solve the boundary-value problem

$$y'' + 2y' + y = 0 \qquad y(0) = 1 \qquad y(1) = 3$$

Summary: Solutions of $ay'' + by' + cy = 0$

Roots of $ar^2 + br + c = 0$	General solution
r_1, r_2 real and distinct	$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$r_1 = r_2 = r$	$y(x) = c_1 e^{rx} + c_2 x e^{rx}$
r_1, r_2 complex: $\alpha \pm i\beta$	$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

EXERCISES

1–13 Solve the differential equation.

1. $y''-6y'+8y=0$
2. $y''-4y'+8y=0$
3. $y''+8y'+41y=0$
4. $2y''-y'-y=0$
5. $y''-2y'+y=0$
6. $3y''=5y'$
7. $4y''+y=0$
8. $16y''+24y'+9y=0$
9. $4y''+y'=0$
10. $9y''+4y=0$
11. $\dfrac{d^2y}{dt^2}-2\dfrac{dy}{dt}-y=0$
12. $\dfrac{d^2y}{dt^2}-6\dfrac{dy}{dt}+4y=0$
13. $\dfrac{d^2y}{dt^2}+\dfrac{dy}{dt}+y=0$

14–21 Solve the initial-value problem.

14. $2y''+5y'+3y=0, \quad y(0)=3, \quad y'(0)=-4$
15. $y''+3y=0, \quad y(0)=1, \quad y'(0)=3$
16. $4y''-4y'+y=0, \quad y(0)=1, \quad y'(0)=-1.5$
17. $2y''+5y'-3y=0, \quad y(0)=1, \quad y'(0)=4$
18. $y''+16y=0, \quad y(\pi/4)=-3, \quad y'(\pi/4)=4$
19. $y''-2y'+5y=0, \quad y(\pi)=0, \quad y'(\pi)=2$
20. $y''+2y'+2y=0, \quad y(0)=2, \quad y'(0)=1$
21. $y''+12y'+36y=0, \quad y(1)=0, \quad y'(1)=1$

22–29 Solve the boundary-value problem, if possible.

22. $4y''+y=0, \quad y(0)=3, \quad y(\pi)=-4$
23. $y''+2y'=0, \quad y(0)=1, \quad y(1)=2$
24. $y''-3y'+2y=0, \quad y(0)=1, \quad y(3)=0$
25. $y''+100y=0, \quad y(0)=2, \quad y(\pi)=5$
26. $y''-6y'+25y=0, \quad y(0)=1, \quad y(\pi)=2$
27. $y''-6y'+9y=0, \quad y(0)=1, \quad y(1)=0$
28. $y''+4y'+13y=0, \quad y(0)=2, \quad y(\pi/2)=1$
29. $9y''-18y'+10y=0, \quad y(0)=0, \quad y(\pi)=1$

Nonhomogeneous Linear Equations

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form

$$ay'' + by' + cy = G(x) \qquad (1)$$

where $a, b,$ and c are constants and G is a continuous function. The related homogeneous equation

$$ay'' + by' + cy = 0 \qquad (2)$$

is called the **complementary equation** and plays an important role in the solution of the original nonhomogeneous equation (1).

Theorem The general solution of the nonhomogeneous differential equation (1) can be written as

$$y(x) = y_p(x) + y_c(x)$$

where y_p is a particular solution of Equation (1) and y_c is the general solution of the complementary Equation (2).

The Method of Undetermined Coefficients

We first illustrate the method of undetermined coefficients for the equation

$$ay'' + by' + cy = G(x)$$

where $G(x)$ is a polynomial. It is reasonable to guess that there is a particular solution y_p that is a polynomial of the same degree as G because if y is a polynomial, then $ay'' + by' + cy$ is also a polynomial. We therefore substitute $y_p(x) =$ a polynomial (of the same degree as G) into the differential equation and determine the coefficients.

EXAMPLE 1 Solve the equation $y'' + y' - 2y = x^2$.

If $G(x)$ (the right side of Equation (1)) is of the form Ce^{kx}, where C and k are constants, then we take as a trial solution a function of the same form, $y_p(x) = Ae^{kx}$, because the derivatives of e^{kx} are constant multiples of e^{kx}.

EXAMPLE 2 Solve $y'' + 4y = e^{3x}$.

If $G(x)$ is either $C\cos kx$ or $C\sin kx$, then, because of the rules for differentiating the sine and cosine functions, we take as a trial particular solution a function of the form

$$y_p(x) = A\cos kx + B\sin kx$$

EXAMPLE 3 Solve $y'' + y' - 2y = \sin x$.

If $G(x)$ is **a product of functions of the preceding types**, then we take the trial solution to be a product of functions of the same type. For instance, in solving the differential equation

$$y'' + 2y' + 4y = x\cos 3x$$

we would try

$$y_p(x) = (Ax + B)\cos 3x + (Cx + D)\sin 3x$$

If $G(x)$ is a **sum of functions of these types**, we use the easily verified principle of superposition, which says that if y_{p_1} and y_{p_2} are solutions of

$$ay'' + by' + cy = G_1(x) \qquad\qquad ay'' + by' + cy = G_2(x)$$

respectively, then $y_{p_1} + y_{p_2}$ is a solution of

$$ay'' + by' + cy = G_1(x) + G_2(x)$$

EXAMPLE 4 Solve $y'' - 4y = xe^x + \cos 2x$.

Finally we note that the recommended trial solution y_p sometimes turns out to be a solution of the complementary equation and therefore can't be a solution of the nonhomogeneous equation. In such cases we multiply the recommended trial solution by x (or by x^2 if necessary) so that no term in $y_p(x)$ is a solution of the complementary equation.

EXAMPLE 5 Solve $y'' + y = \sin x$.

We summarize the method of undetermined coefficients as follows:

1. If $G(x) = e^{kx}P(x)$, where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth-degree polynomial (whose coefficients are determined by substituting in the differential equation.)

2. If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where P is an nth-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx$$

where Q and R are nth-degree polynomials.

Modification: If any term of y_p is a solution of the complementary equation, multiply y_p by x (or by x^2 if necessary).

EXAMPLE 6 Determine the form of the trial solution for the differential equation
$y'' - 4y' + 13y = e^{2x}\cos 3x$.

EXERCISES

1-10 Solve the differential equation or initial-value problem using the method of undetermined coefficients.

1. $y'' + 3y' + 2y = x^2$
2. $y'' + 9y = e^{3x}$
3. $y'' - 2y' = \sin 4x$
4. $y'' + 6y' + 9y = 1 + x$
5. $y'' - 4y' + 5y = e^{-x}$
6. $y'' + 2y' + y = xe^{-x}$
7. $y'' + y = e^x + x^3, \quad y(0) = 2, \quad y'(0) = 0$
8. $y'' - 4y = e^x \cos x, \quad y(0) = 1, \quad y'(0) = 2$
9. $y'' - y' = xe^x, \quad y(0) = 2, \quad y'(0) = 1$
10. $y'' + y' - 2y = x + \sin 2x, \quad y(0) = 1, \quad y'(0) = 0$

11-16 Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

11. $y'' + 9y = e^{2x} + x^2 \sin x$
12. $y'' + 9y' = xe^{-x} \cos \pi x$
13. $y'' + 9y' = 1 + xe^{9x}$
14. $y'' + 3y' - 4y = (x^3 + x)e^x$
15. $y'' + 2y' + 10y = x^2 e^{-x} \cos 3x$
16. $y'' + 4y = e^{3x} + x \sin 2x$

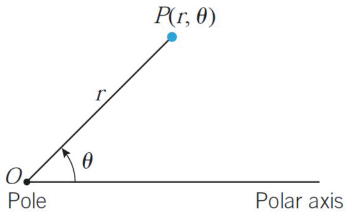
17-20 Solve the differential equation using undetermined coefficients.

17. $y'' + 4y = x$
18. $y'' - 3y' + 2y = \sin x$
19. $y'' - 2y' + y = e^{2x}$
20. $y'' - y' = e^x$

Polar Coordinates and Graphs

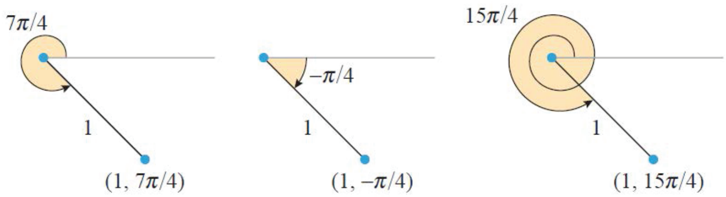
Polar Coordinate Systems

A *polar coordinate system* in a plane consists of a xed point O, called the *pole* (or *origin*), and a ray emanating from the pole, called the *polar axis*. In such a coordinate system, we can associate with each point P in the plane a pair of polar coordinates (r, θ), where r is the distance from P to the pole and θ is an angle from the polar axis to the ray OP. The number r is called the *radial coordinate* of P and the number θ the *angular coordinate* (or *polar angle*)of P.



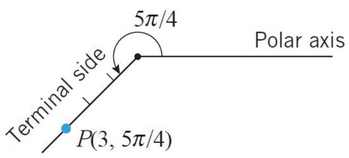
Remarks.

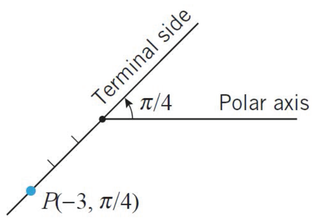
1. The polar coordinates of a point are not unique.



2. In general, if a point P has polar coordinates (r, θ), then $(r, \theta + 2n\pi)$ and $(r, \theta - 2n\pi)$ are also polar coordinates of P for any nonnegative integer n. Thus, every point has infinitely many pairs of polar coordinates.

3. In general, the terminal side of the angle $\theta + \pi$ is the extension of the terminal side of θ, so we define negative radial coordinates by agreeing that $(-r, \theta)$ and $(r, \theta + \pi)$ are polar coordinates of the same point.



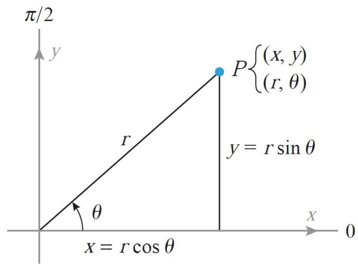


From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 706

Relationship between Polar and Rectangular coordinates

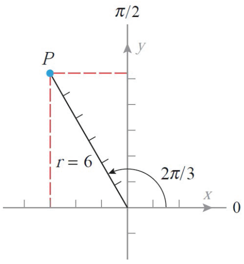
As suggested by below figure, these coordinates are related by the equations

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

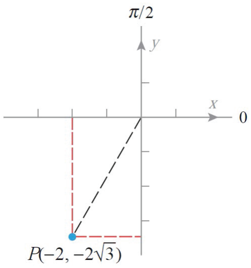


From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 707

Example 1. Find the rectangular coordinates of the point P whose polar coordinates are $(r, \theta) = (6, 2\pi/3)$.



Example 2. Find the rectangular coordinates of the point P whose rectangular coordinates are $(-2, -2\sqrt{3})$.



Graphs in Polar Coordinates

We will now consider the problem of graphing equations in r and θ, where θ is assumed to be measured in radians. In a rectangular coordinate system the graph of an equation in x and y consists of all points whose coordinates (x, y) satisfy the equation. However, in a polar coordinate system, points have innitely many different pairs of polar coordinates, so that a given point may have some polar coordinates that satisfy an equation and others that do not.

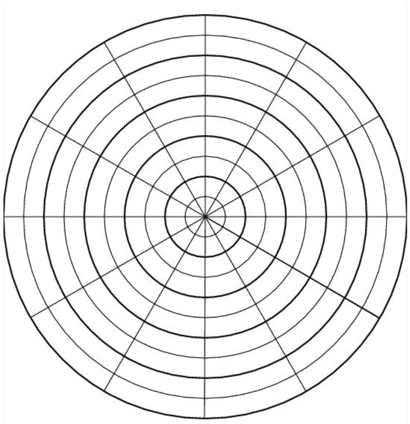
Given an equation in r and θ, we dene its graph in polar coordinates to consist of all points with at least one pair of coordinates (r, θ) that satisfy the equation.

Example 3. Sketch the graphs in polar coordinates

(a) $r = 1$ (b) $\theta = \dfrac{\pi}{4}$

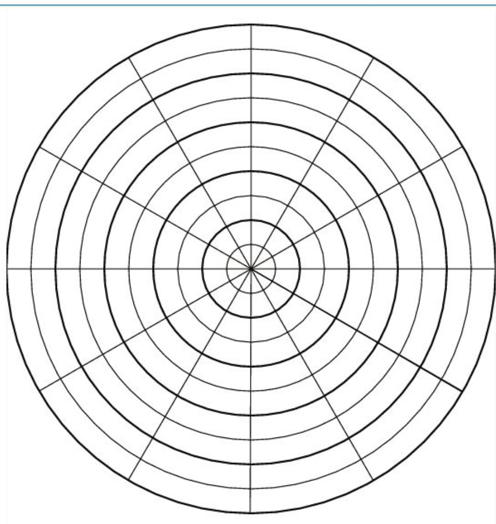
Equations $r = f(\theta)$ that express r as a function of θ are especially important. One way to graph such an equation is to choose some typical values of θ, calculate the corresponding values of r, and then plot the resulting pairs (r, θ) in a polar coordinate system. The next two examples illustrate this process.

Example 4. Sketch the graph of $r = \theta$ $(\theta \geq 0)$ in polar coordinates by plotting points.



Example 5. Sketch the graph of $r = \sin\theta$ in polar coordinates by plotting points.

θ (RADIANS)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = \sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0
(r, θ)	$(0, 0)$	$\left(\frac{1}{2}, \frac{\pi}{6}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$	$\left(1, \frac{\pi}{2}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$	$\left(\frac{1}{2}, \frac{5\pi}{6}\right)$	$(0, \pi)$	$\left(-\frac{1}{2}, \frac{7\pi}{6}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{4\pi}{3}\right)$	$\left(-1, \frac{3\pi}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{3}\right)$	$\left(-\frac{1}{2}, \frac{11\pi}{6}\right)$	$(0, 2\pi)$

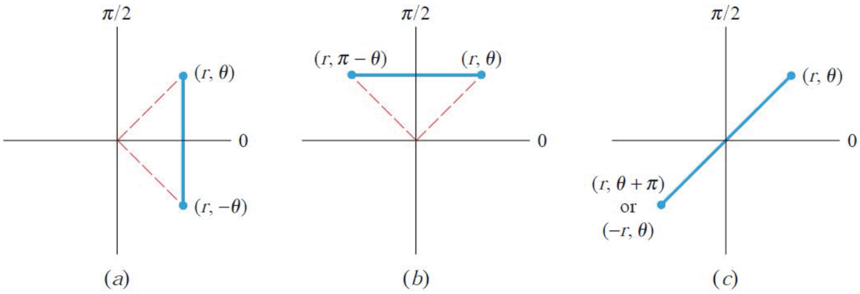


Symmetry Tests

1. A curve in polar coordinates is symmetric about the x-axis if replacing θ by $-\theta$ in its equation produces an equivalent equation (Figure a).

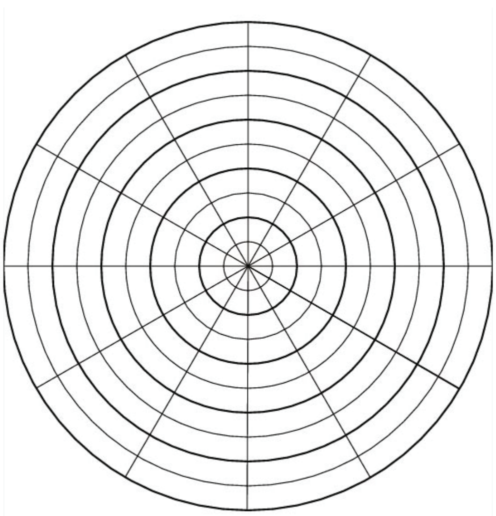
2. A curve in polar coordinates is symmetric about the y-axis if replacing θ by $\pi - \theta$ in its equation produces an equivalent equation (Figure b).

3. A curve in polar coordinates is symmetric about the origin if replacing θ by $\theta + \pi$, or replacing r by r in its equation produces an equivalent equation (Figure c).

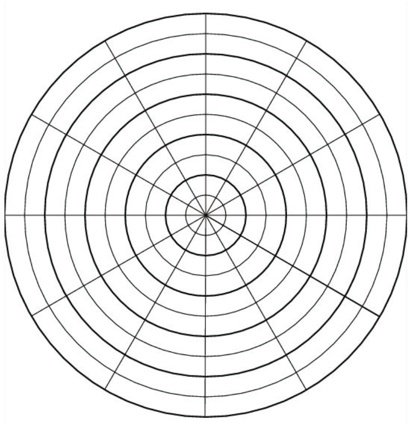


From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 710

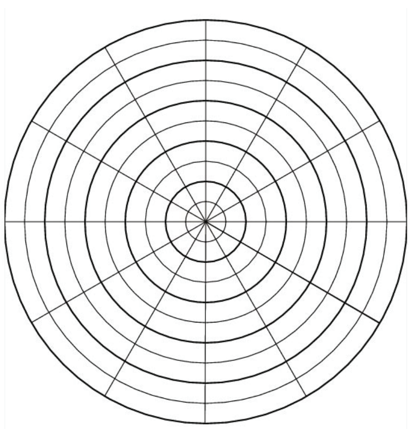
Example 6. Sketch the graph of $r = 2\cos\theta$ in polar coordinates by plotting points.



Example 7. Sketch the graph of $r = 2(1 - \cos\theta)$ in polar coordinates.



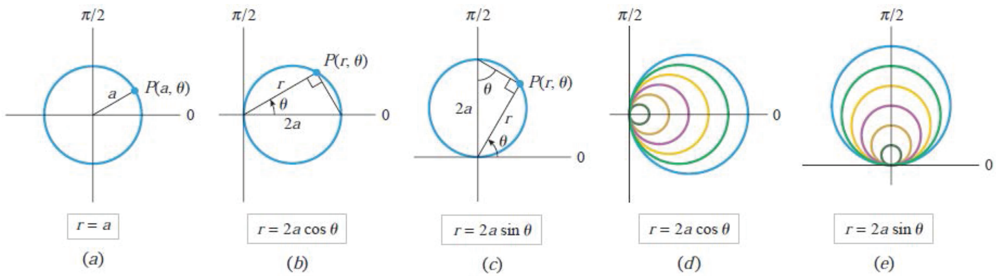
Example 8. Sketch the graph of $r^2 = 4\cos 2\theta$ in polar coordinates.



Families of Circles

We will consider three families of circles in which a is assumed to be a positive constant:

$$r = a, \qquad r = 2a\cos\theta, \qquad r = 2a\cos\theta.$$



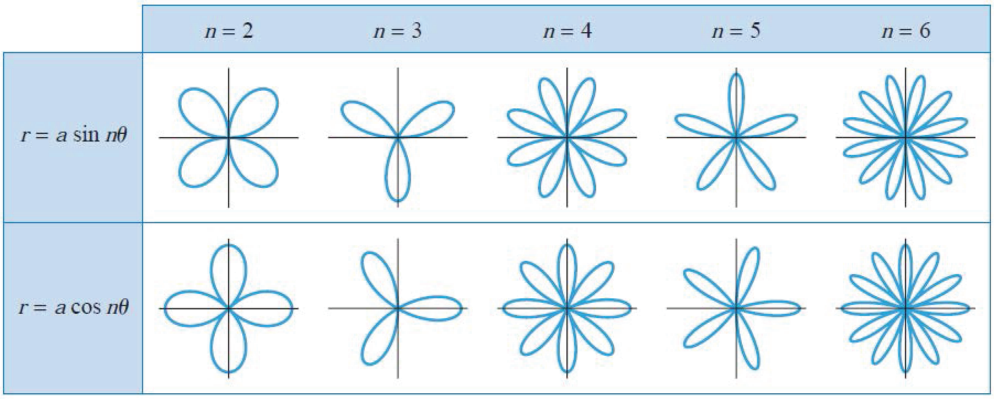
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 713

Families of Rose Curves

In polar coordinates, equations of the form

$$r = a\sin n\theta, \qquad r = a\cos n\theta$$

in which $a > 0$ and n is a positive integer represent families of flower-shaped curves called *roses*.



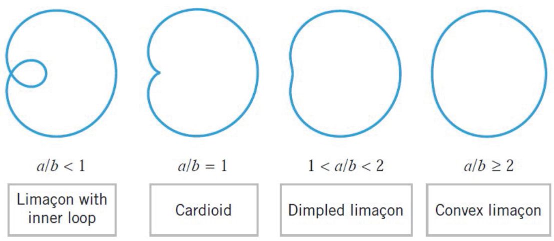
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 713

Families of Cardioids and limaons

In polar coordinates, equations of the form

$$r = a \pm b\sin\theta, \qquad r = a \pm b\cos\theta$$

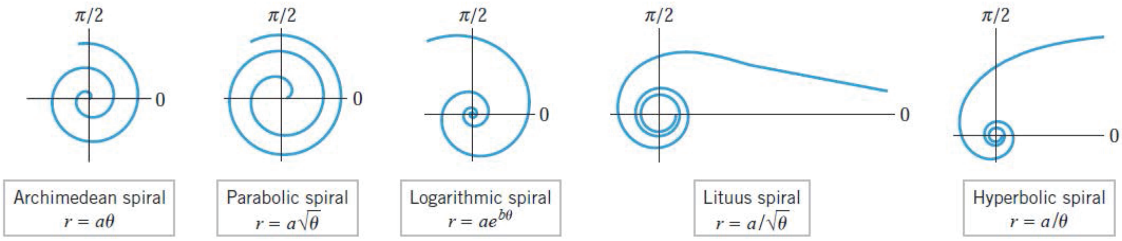
in which $a > 0$ and $b > 0$ represent polar curves called *limaons* (from the Latin word limax for a snail-like creature that is commonly called a slug). There are four possible shapes for a limaon that are determined by the ratio a/b. If $a = b$ (the case $a/b = 1$), then the limaon is called a *cardioid* because of its heart-shaped appearance.



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 714

Families of Spirals

A *spiral* is a curve that coils around a central point. Spirals generally have left-hand and right-hand versions that coil in opposite directions, depending on the restrictions on the polar angle and the signs of constants that appear in their equations.



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 714

EXERCISES

1. Rectangular coordinates of a point (x, y) may be recovered from its polar coordinates (r, θ) by means of the equations $x =$ .... and $y =$ ....

2. Polar coordinates (r, θ) may be recovered from rectangular coordinates (x, y) by means of the equations $r^2 =$ .... and $\tan\theta =$ ....

3. Find the rectangular coordinates of the points whose polar coordinates are given. (a) $(4, \pi/3)$ (b) $(2, -\pi/6)$ (c) $(6, -2\pi/3)$ (d) $(4, 5\pi/4)$

4. In each part, find polar coordinates satisfying the stated conditions for the point whose rectangular coordinates are $(1, \sqrt{3})$.
   (a) $r \geq 0$ and $0 \leq \theta < 2\pi$ (b) $r \leq 0$ and $0 \leq \theta < 2\pi$
   (c) $r \geq 0$ and $-2\pi < \theta \leq 0$ (d) $r \leq 0$ and $-\pi \leq \theta < \pi$

5. In each part, state the name that describes the polar curve most precisely: a rose, a line, a circle, a limaon, a cardioid, a spiral, a lemniscate, or none of these.
   (a) $r = 1 - \theta$ (b) $1 + 2\sin\theta$ (c) $r = \sin 2\theta$ (d) $r = \cos^2\theta$
   (e) $r = \csc\theta$ (f) $2 + 2\cos\theta$ (g) $r = -\sin\theta$ (h) $r = 2\theta$

6. Plot the points in polar coordinates.
   (a) $(3, \pi/4)$ (b) $(5, 2\pi/3)$ (c) $(1, \pi/2)$ (d) $(4, 7\pi/6)$
   (e) $(-6, -\pi)$ (f) $(-1, 9\pi/4)$ (g) $(2, -\pi/3)$ (h) $(3/2, -7\pi/4)$
   (i) $(-3, 3\pi/2)$ (j) $(-5, -\pi/6)$ (k) $(2, 4\pi/3)$ (l) $(0, \pi)$

7. Find the rectangular coordinates of the points whose polar coordinates are given.
   (a) $(6, \pi/6)$ (b) $(7, 2\pi/3)$ (c) $(-6, -5\pi/6)$ (d) $(0, -\pi)$
   (e) $(7, 17\pi/6)$ (f) $(-5, 0)$ (g) $(-2, \pi/4)$ (h) $(6, -\pi/4)$
   (i) $(4, 9\pi/4)$ (j) $(3, 0)$ (k) $(-4, -3\pi/2)$ (l) $(0, 3\pi)$

8. In each part, a point is given in rectangular coordinates. Find two pairs of polar coordinates for the point, one pair satisfying $r \geq 0$ and $0 \leq \theta < 2\pi$, and the second pair satisfying $r \geq 0$ and $-2\pi < \theta \leq 0$.
   (a) $(-5, 0)$ (b) $(2\sqrt{3}, -2)$ (c) $(0, -2)$ (d) $(-8, -8)$
   (e) $(-3, 3\sqrt{3})$ (f) $(1, 1)$ (g) $(-2, 2)$ (h) $(3, -3\sqrt{3})$

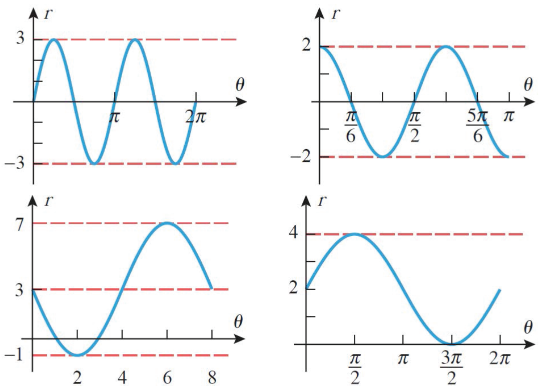
9. Identify the curve by transforming the given polar equa-tion to rectangular coordinates.

   (a) $r = 2$ (b) $r \sin\theta = 4$ (c) $r = 3\cos\theta$ (d) $r = \dfrac{6}{3\cos\theta + 2\sin\theta}$
   (e) $r = 5\sec\theta$ (f) $r = 2\sin\theta$ (g) $r = 4\cos\theta + 4\sin\theta$ (h) $r = \sec\theta\tan\theta$

10. Express the given equations in polar coordinates.

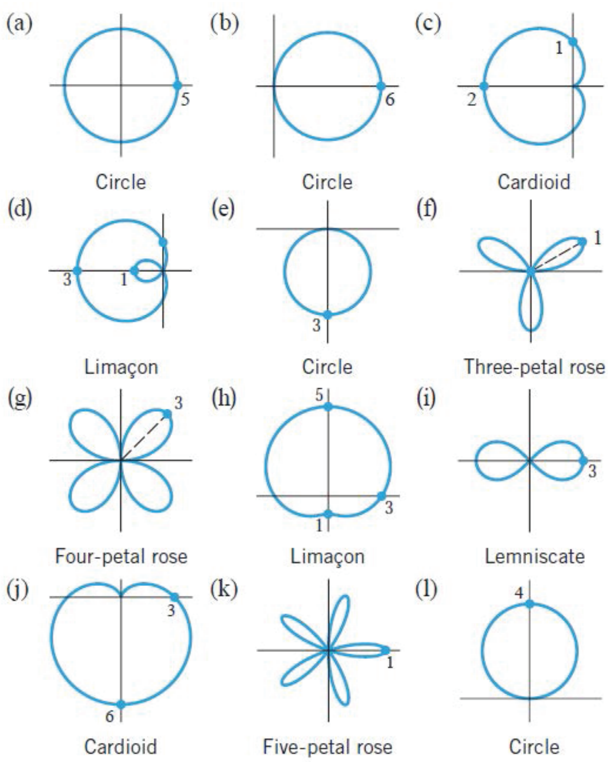
    (a) $x = 3$ (b) $x^2 + y^2 = 7$ (c) $x^2 + y^2 + 6y = 0$ (d) $9xy = 4$
    (e) $y = -3$ (f) $x^2 + y^2 = 5$ (g) $x^2 + y^2 + 4x = 0$ (h) $x^2(x^2 + y^2) = y^2$

11. A graph is given in a rectangular θr-coordinate system. Sketch the corresponding graph in polar coordinates.



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 717

12. Find an equation for the given polar graph. [Note: Numeric labels on these graphs represent distances to the origin.]



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 717

13. Identify the curve by transforming the given polar equation to rectangular coordinates.

(a) $\theta = \pi/3$ (b) $\theta = -3\pi/4$ (c) $r = 3$ (d) $r = 4\cos\theta$
(e) $r = 6\sin\theta$ (f) $r - 2 = 2\cos\theta$ (g) $r = 3(1 + \sin\theta)$ (h) $r = 5 - 5\sin\theta$
(i) $r = 4 - 4\cos\theta$ (j) $r = 1 + 2\sin\theta$ (k) $r = -1 - \cos\theta)$ (l) $r = 4 + 3\cos\theta$
(m) $r = 3 - \sin\theta$ (n) $r = 3 + 4\cos\theta$ (o) $r - 5 = 3\sin\theta$ (p) $r^2 = \cos 2\theta$
(q) $r^2 = 16\sin 2\theta$ (r) $r = 4\theta$, $(\theta \geq 0)$ (s) 4θ, $(\theta \leq 0)$ (t) $r = 4\theta$
(u) $r = -2\cos 2\theta$ (v) $r = 3\sin 2\theta$ (w) $r = 9 + \sin 4\theta$ (x) $r = 2\cos 3\theta$
(y) $r = 3 + \sin\theta$ (z) $r = 2\cos\theta$

See more exercises from : Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 716-719

Quadric surfaces

The equation in an xyz –coordinate system is

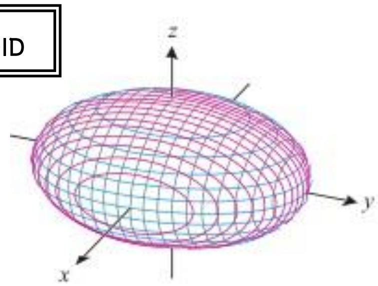
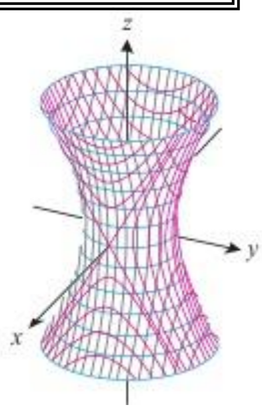
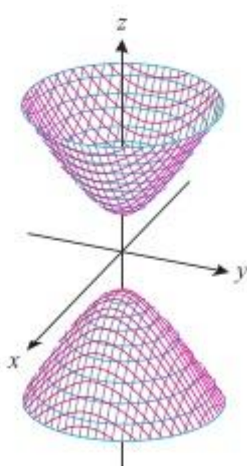
$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

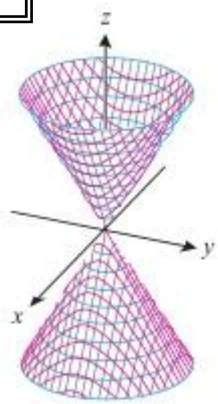
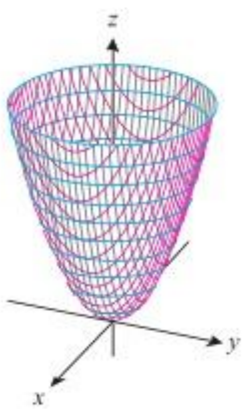
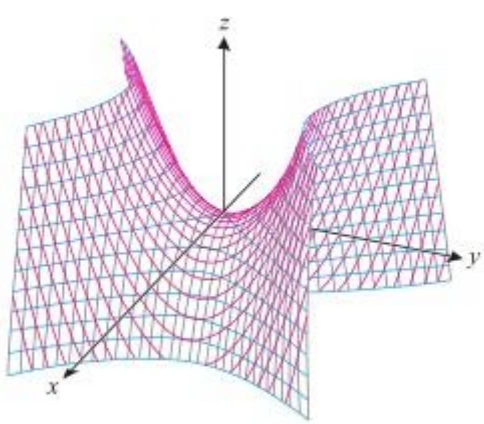
which is called a **second-degree equation in** x, y **and** z . The graphs of such equations are called **quadric surfaces** or sometimes **quadrics**.

Six common types of quadric surfaces are shown in Table 1—ellipsoids, hyperboloids of one sheet, hyperboloids of two sheets, elliptic cones, elliptic paraboloids, and hyperbolic paraboloids. (The constants a, b, and c that appear in the equations in the table are assumed to be positive.) Observe that none of the quadric surfaces in the table have cross-product terms in their equations. This is because of their orientations relative to the coordinate axes. Later in this section we will discuss other possible orientations that produce equations of the quadric surfaces with no cross-product terms. In the special case where the elliptic cross sections of an elliptic cone or an elliptic paraboloid are circles, the terms circular cone and circular paraboloid are used.

Table 1

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 823)

surface	equation
ELLIPSOID 	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ The traces in the coordinate planes are ellipses, as are the traces in those planes that are parallel to the coordinate planes and intersect the surface in more than one point.
HYPERBOLOID OF ONE SHEET 	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$ The trace in the xy –plane is an ellipse, as are the traces in planes parallel to the xy –plane. The traces in the yz –plane and xz –plane are hyperbolas, as are the traces in those planes that are parallel to these and do not pass through the x – or y –intercepts. At these intercepts the traces are pairs of intersecting lines.
HYPERBOLOID OF TWO SHEET 	$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$ There is no trace in the xy –plane. In planes parallel to the xy –plane that intersect the surface in more than one point the traces are ellipses. In the yz – and xz –planes, the traces are hyperbolas, as are the traces in those planes that are parallel to these.

surface	equation
ELLIPTIC CONE 	$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ The trace in the xy –plane is a point (the origin), and the traces in planes parallel to the xy –plane are ellipses. The traces in the yz –and xz –planes are pairs of lines intersecting at the origin. The traces in planes parallel to these are hyperbolas.
ELLIPTIC PARABOLOID 	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ The trace in the xy –plane is a point (the origin), and the traces in planes parallel to and above the xy –plane are ellipses. The traces in the yz –and xz –planes are parabolas, as are the traces in planes parallel to these.
HYPERBOLIC PARABOLOID 	$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$ The trace in the xy –plane is a pair of lines intersecting at the origin. The traces in planes parallel to the xy –plane are hyperbolas. The hyperbolas above the xy –plane open in the y –direction, and those below in the x –direction. The traces in the yz –and xz –planes are parabolas, as are the traces in planes parallel to these.

1. Techniques for graphing quadric surfaces

Accurate graphs of quadric surfaces are best left for graphing utilities. However, the techniques that we will now discuss can be used to generate rough sketches of these surfaces that are useful for various purposes.

A rough sketch of an **ellipsoid**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (a>0, b>0, c>0)$$

can be obtained by first plotting the intersections with the coordinate axes, and then sketching the elliptical traces in the coordinate planes. Example 7 illustrates this technique.

Example 7 Sketch the ellipsoid $\frac{x^2}{4}+\frac{y^2}{16}+\frac{z^2}{9}=1$

A rough sketch of a **hyperboloid of one sheet**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad (a>0, b>0, c>0)$$

can be obtained by first sketching the elliptical trace in the xy–plane, then the elliptical traces in the planes $z=\pm c$, and then the hyperbolic curves that join the endpoints of the axes of these ellipses. The next example illustrates this technique.

Example 8 Sketch the graph of the hyperboloid of one sheet $x^2+y^2-\frac{z^2}{4}=1$

A rough sketch of the **hyperboloid of two sheet**

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (a>0, b>0, c>0)$$

can be obtained by first plotting the intersections with the z –axis, then sketching the elliptical traces in the planes $z=\pm 2c$, and then sketching the hyperbolic traces that connect the z –axis intersections and the endpoints of the axes of the ellipses. (It is not essential to use the planes $z=\pm 2c$, but these are good choices since they simplify the calculations slightly and have the right spacing for a good sketch.) The next example illustrates this technique.

Example 9 Sketch the graph of the hyperboloid of two sheet $z^2-x^2-\frac{y^2}{4}=1$

A rough sketch of the **elliptic cone**

$$z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2} \qquad (a>0, b>0)$$

can be obtained by first sketching the elliptical traces in the planes $z=\pm 1$ and then sketching the linear traces that connect the endpoints of the axes of the ellipses. The next example illustrates this technique.

Example 10 Sketch the graph of the elliptic cone $z^2=x^2+\frac{y^2}{4}$

A rough sketch of the **elliptic paraboloid**

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2} \qquad (a>0,b>0)$$

can be obtained by first sketching the elliptical trace in the plane $z=1$ and then sketching the parabolic traces in the vertical coordinate planes to connect the origin to the ends of the axes of the ellipse. The next example illustrates this technique.

Example 11 Sketch the graph of the elliptic paraboloid $z=\frac{x^2}{4}+\frac{y^2}{9}$

A rough sketch of the **hyperbolic paraboloid**

$$z^2=\frac{y^2}{b^2}-\frac{x^2}{a^2} \qquad (a>0,b>0)$$

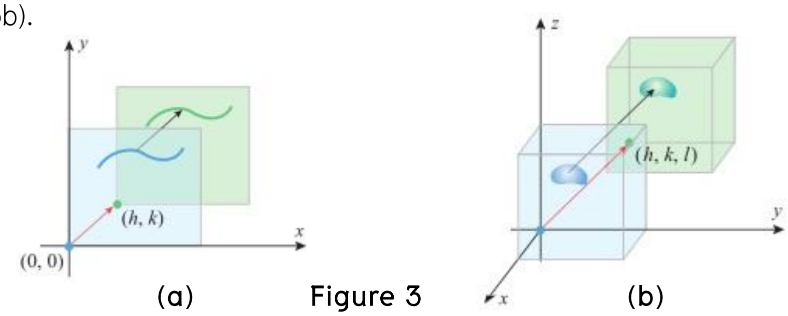
can be obtained by first sketching the two parabolic traces that pass through the origin (one in the plane $x=0$ and the other in the plane $y=0$). After the parabolic traces are drawn, sketch the hyperbolic traces in the planes $z=\pm 1$ and then fill in any missing edges. The next example illustrates this technique.

Example 12 Sketch the graph of the hyperbolic paraboloid $z=\frac{y^2}{4}-\frac{x^2}{9}$

2. Translations of quadric surfaces

A conic in an xy –coordinate system can be translated by substituting $x-h$ for x and $y-k$ for y in its equation. To understand why this works, think of the xy –axes as fixed and think of the plane as a transparent sheet of plastic on which all graphs are drawn. When the coordinates of points are modified by substituting $(x-h, y-k)$ for (x, y), the geometric effect is to translate the sheet of plastic (and hence all curves) so that the point on the plastic that was initially at $(0,0)$ is moved to the point (h,k) (see Figure 3a).

For the analog in three dimensions, think of the xyz –axes as fixed and think of 3–space as a transparent block of plastic in which all surfaces are embedded. When the coordinates of points are modified by substituting $(x-h, y-k, z-l)$ for (x, y, z) the geometric effect is to translate the block of plastic (and hence all surfaces) so that the point in the plastic block that was initially at $(0,0,0)$ is moved to the point (h,k,l) (see Figure 3b).



(a) **Figure 3** (b)

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 827)

Example 13 Describe the surface $z=(x-1)^2+(y+2)^2+3$

Example 14 Describe the surface $4x^2+4y^2+z^2+8y-4z=-4$

Techniques for identifying quadric surfaces

The equations of the quadric surfaces in Table 1 have certain characteristics that make it possible to identify quadric surfaces that are derived from these equations by reflections. These identifying characteristics, which are shown in Table 2, are based on writing the equation of the quadric surface so that all of the variable terms are on the left side of the equation and there is a 1 or a 0 on the right side. These characteristics do not change when the surface is reflected about a coordinate plane or planes of the form $x=y, x=z,$ or $y=z,$ thereby making it possible to identify the reflected quadric surface from the form of its equation.

Table 2

identifying a quadric surface from the form of its equation

equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$z^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{y^2}{b^2}+\frac{x^2}{a^2}=0$
characteristic	No minus signs	One minus sign	Two minus signs	No linear terms	One linear term; two quadratic terms with the same sign	One linear term; two quadratic terms with opposite signs
classification	Ellipsoid	Hyperboloid of one sheet	Hyperboloid of two sheets	Elliptic cone	Elliptic paraboloid	Hyperbolic paraboloid

Example 15 Identify the surfaces

(a) $3x^2-4y^2+12z^2+12=0$ (b) $4x^2-4y+z^2=0$

EXERCISES2

1–2 Identify the quadric surface as an ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid, or hyperbolic paraboloid by matching the equation with one of the forms given in Table 1. State the values of $a, b,$ and c in each case.

1. (a) $z = \frac{x^2}{4} + \frac{y^2}{9}$ (b) $z = \frac{y^2}{25} - x^2$

   (c) $x^2 + y^2 - z^2 = 16$ (d) $x^2 + y^2 - z^2 = 0$

   (e) $4z = x^2 + 4y^2$ (f) $z^2 - x^2 - y^2 = 1$

2. (a) $6x^2 + 3y^2 + 4z^2 = 12$ (b) $y^2 - x^2 - z = 0$

   (c) $9x^2 + y^2 - 9z^2 = 9$ (d) $4x^2 + y^2 - 4z^2 = -4$

   (e) $2z - x^2 - 4y^2 = 0$ (f) $12z^2 - 3x^2 = 4y^2$

3. Find an equation for and sketch the surface that results when the circular paraboloid $z = x^2 + y^2$ is reflected about the plane

   (a) $z = 0$ (b) $x = 0$ (c) $y = 0$

   (d) $y = x$ (e) $x = z$ (f) $y = z$

4. Find an equation for and sketch the surface that results when the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ is reflected about the plane

   (a) $z = 0$ (b) $x = 0$ (c) $y = 0$

   (d) $y = x$ (e) $x = z$ (f) $y = z$

5. The given equations represent quadric surfaces whose orientations are different from those in Table 1. In each part, identify the quadric surface, and give a verbal description of its orientation (e.g., an elliptic cone opening along the z–axis or a hyperbolic paraboloid straddling the y–axis).

   (a) $\frac{z^2}{c^2} - \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$ (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

   (c) $x = \frac{y^2}{b^2} + \frac{z^2}{c^2}$ (d) $x^2 = \frac{y^2}{b^2} + \frac{z^2}{c^2}$

   (e) $y = \frac{z^2}{c^2} - \frac{x^2}{a^2}$ (f) $y = -\left(\frac{x^2}{a^2} + \frac{z^2}{c^2}\right)$

6. For each of the surfaces in Exercise 5, find the equation of the surface that results if the given surface is reflected about the xz –plane and that surface is then reflected about the plane $z=0$.

7–18 Identify and sketch the quadric surface.

7. $x^2+\frac{y^2}{4}+\frac{z^2}{9}=1$

8. $x^2+4y^2+9z^2=36$

9. $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$

10. $x^2+y^2-z^2=9$

11. $4z^2=x^2+4y^2$

12. $9x^2+4y^2-36z^2=0$

13. $9z^2-4y^2-9x^2=36$

14. $y^2-\frac{x^2}{4}-\frac{z^2}{9}=1$

15. $z=y^2-x^2$

16. $16z=y^2-x^2$

17. $4z=x^2+2y^2$

18. $z-3y^2-3x^2=0$

19–24 The given equation represents a quadric surface whose orientation is different from that in Table 1. Identify and sketch the surface.

19. $x^2-3y^2-3z^2=0$

20. $x-y^2-4z^2=0$

21. $2y^2-x^2+2z^2=8$

22. $x^2-3y^2-3z^2=9$

23. $z=\frac{x^2}{4}-\frac{y^2}{9}$

24. $4x^2-y^2+4z^2=16$

25–28 Sketch the surface.

25. $z=\sqrt{x^2+y^2}$

25. $z=\sqrt{1-x^2-y^2}$

27. $z=\sqrt{x^2+y^2-1}$

28. $z=\sqrt{1+x^2+y^2}$

29–32 Identify the surface and make a rough sketch that shows its position and orientation.

29. $z=(x+2)^2+(y-3)^2-9$

30. $4x^2-y^2+16(z-2)^2=100$

31. $9x^2+y^2+4z^2-18x+2y+16z=10$

32. $z^2=4x^2+y^2+8x-2y+4z$

(See more exercises from: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 830–832)

Functions of several variables

1. Functions of several variables

There are many familiar formulas in which a given variable depends on two or more other variables. For example, the area A of a triangle depends on the base length b and height h by the formula $A=\frac{1}{2}bh$; the volume V of a rectangular box depends on the length l, the width w, and the height h by the formula $V=lwh$; and the arithmetic average $\bar{x}$ of n real numbers, $x_1, x_2, \ldots, x_n$, depends on those numbers by the formula

$$\bar{x}=\frac{1}{n}(x_1+x_2+\cdots x_n)$$

Thus, we say that

A is a function of the two variables b and h;

V is a function of the three variables l, w, and h;

$\bar{x}$ is a function of the n variables $x_1, x_2, \ldots, x_n$.

The terminology and notation for functions of two or more variables is similar to that for functions of one variable. For example, the expression

$$z=f(x,y)$$

means that is a function of x and y in the sense that a unique value of the depentdent variable is determined by specifying values for the independent variables x and y. Similarly

$$w=f(x,y,z)$$

expresses w as a function of and , and

$$u=f(x_1,x_2,\ldots,x_n)$$

expresses u as a function of $x_1, x_2, \ldots, x_n$.

Definition 1 A function f of two variables, x and y, is a rule that assigns a unique real number $f(x,y)$ to each point (x,y) in some set D in the xy–plane.

Definition 2 A function f of three variables, and , is a rule that assigns a unique real number $f(x,y,z)$ to each point (x,y,z) in some set D in three–dimensional space.

Example 1 Let $f(x,y)=5x^2\sqrt{y}-1$. Find $f(1,4)$ and sketch the natural domain of f.

Example 2 Let $f(x,y)=\sqrt{y+1}+\ln(x^2-y)$. Find $f(e,0)$ and sketch the natural domain of f.

Example 3 Let $f(x,y,z)=\sqrt{1-x^2-y^2-z^2}$. Find $f(0,\frac{1}{2},-\frac{1}{2})$ and the natural domain of f.

2. Graphs of functions of two variables

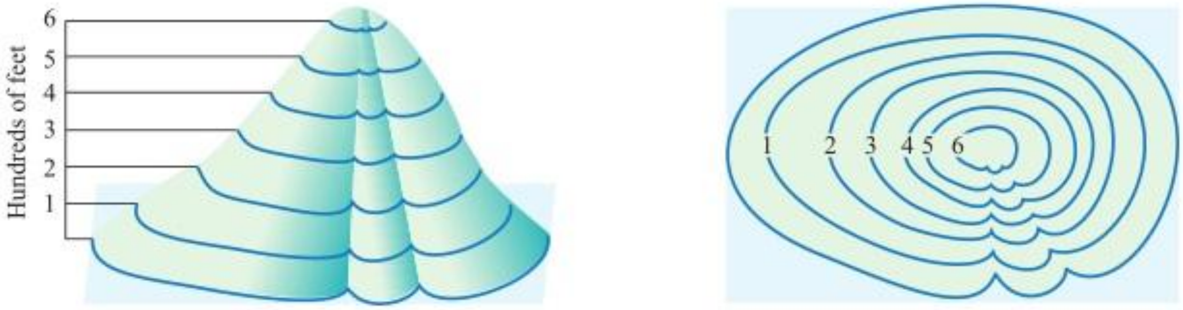
Recall that for a function f of one variable, the graph of $f(x)$ in the xy–plane was defined to be the graph of the equation $y = f(x)$. Similarly, if f is a function of two variables, we define the **graph** of $f(x,y)$ in –space to be the graph of the equation $z = f(x,y)$. In general, such a graph will be a surface in 3–space.

Example 4 In each part, describe the graph of the function in an –coordinate system.

(a) $f(x,y) = 1 - x - \frac{1}{2}y$ (b) $f(x,y) = \sqrt{1 - x^2 - y^2}$ (c) $f(x,y) = -\sqrt{x^2 + y^2}$

3. Level curve

We are all familiar with the topographic (or contour) maps in which a three–dimensional landscape, such as a mountain range, is represented by two–dimensional contour lines or curves of constant elevation. Consider, for example, the model hill and its contour map shown in Figure 1. The contour map is constructed by passing planes of constant elevation through the hill, projecting the resulting contours onto a flat surface, and labeling the contours with their elevations. In Figure 1, note how the two gullies appear as indentations in the contour lines and how the curves are close together on the contour map where the hill has a steep slope and become more widely spaced where the slope is gradual.



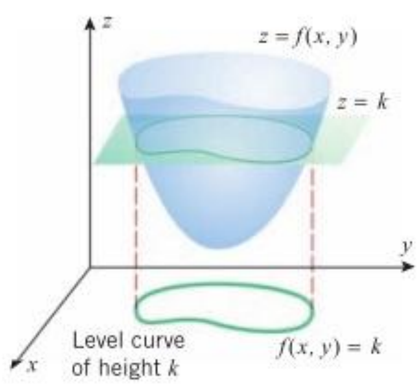
A perspective view of a model hill with two gullies

A contour map of the model hill

Figure 1

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 909)

Contour maps are also useful for studying functions of two variables. If the surface $z = f(x, y)$ is cut by the horizontal plane $z = k$, then at all points on the intersection we have $f(x, y) = k$. The projection of this intersection onto the xy –plane is called the **level curve of height** k or the **level curve with constant** k (Figure 2). A set of level curves for $z = f(x, y)$ is called a **contour plot** or **contour map** of f .



Level curve of height k

Figure 2

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 909)

Example 5 Sketch the contour plot of $f(x, y) = 4x^2 + y^2$ using level curves of height $k = 0, 1, 2, 3, 4, 5$.

4. Level surfaces

Observe that the graph of $y = f(x)$ is a curve in 2–space, and the graph of $z = f(x, y)$ is a surface in 3–space, so the number of dimensions required for these graphs is one greater than the number of independent variables. Accordingly, there is no “direct” way to graph a function of three variables since four dimensions are required. However, if k is a constant, then the graph of the equation $f(x, y, z) = k$ will generally be a surface in 3–space (e.g., $x^2 + y^2 + z^2 = 1$ the graph of is a sphere), which we call the **level surface with constant** k. Some geometric insight into the behavior of the function f can sometimes be obtained by graphing these level surfaces for various values of k.

Example 6 Describe the level surfaces of

(a) $f(x, y, z) = x^2 + y^2 + z^2$ (b) $f(x, y, z) = z^2 - x^2 - y^2$

EXERCISES 1

1–8 These exercises are concerned with functions of two variables.

1. Let $f(x,y)=x^2y+1$. Find (a) $f(2,1)$ (b) $f(1,2)$ (c) $f(0,0)$.
   (d) $f(1,-3)$ (e) $f(3a,a)$ (f) $f(ab,a-b)$.
2. Let $f(x,y)=x+\sqrt[3]{xy}$. Find (a) $f(t,t^2)$ (b) $f(x,x^2)$ (c) $f(2y^2,4y)$.
3. Let $f(x,y)=xy+3$. Find (a) $f(x+y,x-y)$ (b) $f(xy,3x^2y^3)$.
4. Let $g(x)=x\sin x$. Find (a) $g(x/y)$ (b) $g(xy)$ (c) $g(x-y)$.
5. Find $F(g(x),h(y))$ if $F(x,y)=xe^{xy}$, $g(x)=x^3$, and $h(y)=3y+1$.
6. Find $g(u(x,y),v(x,y))$ if $g(x,y)=y\sin(x^2y)$, $u(x,y)=x^2y^3$, and $v(x,y)=\pi xy$.
7. Let $f(x,y)=x+3x^2y^2$, $x(t)=t^2$, and $y(t)=t^3$. Find
   (a) $f(x(t),y(t))$ (b) $f(x(0),y(0))$ (c) $f(x(2),y(2))$
8. Let $g(x,y)=ye^{-3x}$, $x(t)=\ln(t^2+1)$, and $y(t)=\sqrt{t}$. Find $g(x(t),y(t))$.
9. Suppose that the concentration C in mg/L of medication in a patient's bloodstream is modeled by the function $C(x,t)=0.2x(e^{-0.2t}-e^{-t})$, where x is the dosage of the medication in mg and t is the number of hours since the beginning of administration of the medication.

   (a) Estimate the value of $C(25,3)$ to two decimal places. Include appropriate units and interpret your answer in a physical context.

   (b) If the dosage is 100 mg, give a formula for the concentration as a function of time t.

10–13 These exercises are concerned with functions of three variables.

10. Let $f(x,y,z)=xy^2z^3+3$. Find (a) $f(2,1,2)$ (b) $f(-3,2,1)$ (c) $f(0,0,0)$
    (d) $f(a,a,a)$ (e) $f(t,t^2,-t)$ (f) $f(a+b,a-b,b)$.
11. Let $f(x,y,z)=zxy+x$. Find (a) $f(x+y,x-y,x^2)$ (b) $f(xy,y/x,xz)$.
12. Find $F(f(x),g(y),h(z))$ if $F(x,y,z)=ye^{xyz}$, $f(x)=x^2$, $g(y)=y+1$, and $h(z)=z^2$.
13. Find $g(u(x,y,z),v(x,y,z),w(x,y,z))$ if $g(x,y,z)=z\sin xy$, $u(x,y,z)=x^2z^3$,
    $v(x,y,z)=\pi xyz$, and $w(x,y,z)=xy/z$.

14. Sketch the domain of f. Use solid for portions of the boundary included in the domain and dashed lines for portions not included.

(a) $f(x,y)=\ln(1-x^2-y^2)$ (b) $f(x,y)=\sqrt{x^2+y^2-4}$

(c) $f(x,y)=\dfrac{1}{x-y^2}$ (d) $f(x,y)=\ln xy$

15. Describe the domain of f in words.

(a) $f(x,y)=xe^{-\sqrt{y+2}}$ (b) $f(x,y)=\dfrac{\sqrt{4-x^2}}{y^2+3}$

(c) $f(x,y)=\ln(y-2x)$ (d) $f(x,y,z)=\sqrt{25-x^2-y^2-z^2}$

(e) $f(x,y,z)=e^{xyz}$ (f) $f(x,y,z)=\dfrac{xyz}{x+y+z}$

16. Sketch the graph of f.

(a) $f(x,y)=3$ (b) $f(x,y)=\sqrt{9-x^2-y^2}$

(c) $f(x,y)=\sqrt{x^2+y^2}$ (d) $f(x,y)=x^2+y^2$

(e) $f(x,y)=x^2-y^2$ (f) $f(x,y)=4-x^2-y^2$

(h) $f(x,y)=\sqrt{x^2+y^2+1}$ (i) $f(x,y)=\sqrt{x^2+y^2-1}$

(j) $f(x,y)=y+1$ (k) $f(x,y)=x^2$

17–18 In each part, select the term that best describes the level curves of the function f. Choose from the term lines, circles, noncircular ellipses, parabolas or hyperbolars.

17. (a) $f(x,y)=5x^2-5y^2$ (b) $f(x,y)=y-4x^2$

(c) $f(x,y)=x^2+3y^2$ (d) $f(x,y)=3x^2$

18. (a) $f(x,y)=x^2-2xy+y^2$ (b) $f(x,y)=2x^2+2y^2$

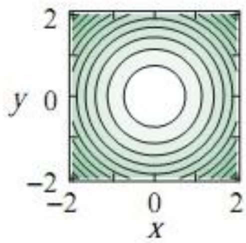
(c) $f(x,y)=x^2-2x-y^2$ (d) $f(x,y)=2y^2-x$

19. In each part, match the contour plot with one of the functions

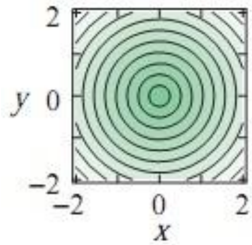
$$f(x,y)=\sqrt{x^2+y^2},\ f(x,y)=x^2+y^2,\ f(x,y)=1-x^2-y^2$$

by inspection, and explain your reasoning. Larger values of are indicated by lighter colors in the contour plot, and the concentric contours correspond to equally spaced values of .

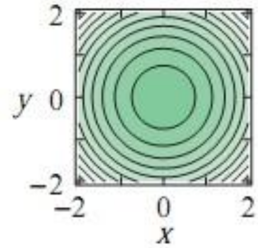
(a)



(b)



(c)



20. Sketch the level curve $z = k$ for the specified values of k.

(a) $z = x^2 + y^2;\ k = 0,1,2,3,4$

(b) $z = y/x;\ k = -2,-1,0,1,2$

(c) $z = x^2 + y;\ k = -2,-1,0,1,2$

(d) $z = x^2 + 9y^2;\ k = 0,1,2,3,4$

(e) $z = x^2 - y^2;\ k = -2,-1,0,1,2$

(f) $z = y\csc x;\ k = -2,-1,0,1,2$

(See more exercises from: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 914–917)

---

6. Limits and continuity

6.1 Limits along curves

For a function of one variable there are two one-sided limits at a point x_0, namely,

$$\lim_{x\to x_0^+} f(x) \quad \text{and} \quad \lim_{x\to x_0^-} f(x)$$

reflecting the fact that there are only two directions from which x can approach x_0, the right or the left. For functions of two or three variables the situation is more complicated because there are infinitely many different curves along which one point can approach another (Figure 4). Our first objective in this section is to define the limit of $f(x,y)$ as (x,y) approaches a point (x_0,y_0) along a curve C (and similarly for functions of three variables).

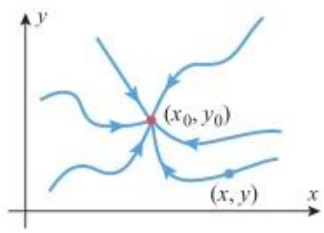


Figure 4

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 918)

If C is a smooth parametric curve in 2-space or 3-space that is represented by the equations

$$x = x(t), \quad y = y(t) \qquad \text{or} \qquad x = x(t), \quad y = y(t), \quad z = z(t)$$

and if $x_0 = x(t_0), y_0 = y(t_0)$ and $z_0 = z(t_0)$, then the limits

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ (\text{along } C)}} f(x,y) \qquad \text{and} \qquad \lim_{\substack{(x,y,z)\to(x_0,y_0,z_0) \\ (\text{along } C)}} f(x,y,z)$$

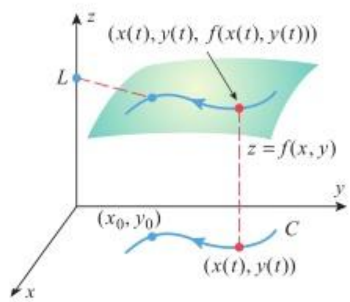
are defined by

$$\boxed{\lim_{\substack{(x,y)\to(x_0,y_0) \\ (\text{along } C)}} f(x,y) = \lim_{t\to t_0} f(x(t), y(t))}$$

$$\boxed{\lim_{\substack{(x,y,z)\to(x_0,y_0,z_0) \\ (\text{along } C)}} f(x,y,z) = \lim_{t\to t_0} f(x(t), y(t), z(t))}$$

In these formulas the limit of the function of t must be treated as a one-sided limit if (x_0, y_0) or (x_0, y_0, z_0) is an endpoint of C.

A geometric interpretation of the limit along a curve for a function of two variables is shown in Figure 5: As the point $(x(t), y(t))$ moves along the curve C in the xy-plane toward (x_0, y_0), the point $(x(t), y(t), f(x(t), y(t)))$ moves directly above it along the graph of $z = f(x, y)$ with $f(x(t), y(t))$ approaching the limiting value L. In the figure we followed a common practice of omitting the zero – coordinate for points in the xy-plane.



$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{(along } C)}} f(x,y) = L$$

Figure 5

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 918)

Example 16 Let $f(x,y) = -\dfrac{xy}{x^2+y^2}$. Find the limit of $f(x,y)$ as $(x,y)\to(0,0)$ along

(a) the x-axis

(b) the y-axis

(c) the line $y = x$

(d) the line $y = -x$

(e) the parabola $y = x^2$

6.2 General limits of functions of two variables

The statement

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

Is intended to convey the idea that the value of $f(x,y)$ can be made as close as we like to the number L by restricting the point (x,y) to be sufficiently close to (but different from) the point (x_0,y_0).

6.3 Relationships between general limits and limits along smooth curve

Theorem 1

(a) If $f(x,y)\to L$ as $(x,y)\to(x_0,y_0)$, then $f(x,y)\to L$ as $(x,y)\to(x_0,y_0)$ along any smooth curve.

(b) If the limit of $f(x,y)$ fails to exist as $(x,y)\to(x_0,y_0)$ along some smooth curve, or if $f(x,y)$ has different limits as $(x,y)\to(x_0,y_0)$ along two different smooth curves, then the limit of $f(x,y)$ does not exist as $(x,y)\to(x_0,y_0)$.

Example 17 The limit $\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2+y^2}$ does **not exist** because

6.4 Continuity

Definition 3 A function $f(x,y)$ is said to be **continuous at** (x_0,y_0) if $f(x_0,y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

In addition, if f is continuous at every point in an open set D, then we say that f is **continuous on** D, and if f is continuous at every point in the xy–plane, then we say that f is **continuous everywhere.**

Theorem 2

(a) If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y)=g(x)h(y)$ is continuous at (x_0,y_0).

(b) If $h(x,y)$ is continuous at (x_0,y_0) and $g(u)$ is continuous at $u=h(x_0,y_0)$, then the composition $f(x,y)=g(h(x,y))$ is continuous at (x_0,y_0).

(c) If $f(x,y)$ is continuous at (x_0,y_0), and $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0)=x_0$ and $y(t_0)=y_0$, then the composition $f(x(t),y(t))$ is continuous at t_0.

Example 18 Use the Theorem 2 to show that the functions $f(x,y)=3x^2y^5$ and $f(x,y)=\sin(3x^2y^5)$ are continuous everywhere.

Recognizing Continuous Functions

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

Example 19 Evaluate $\lim\limits_{(x,y)\to(-1,2)} \dfrac{xy}{x^2+y^2}$

Example 20 Since the function $f(x,y)=\dfrac{x^3y^2}{1-xy}$ is a quotient of continuous functions, it is continuous except

6.5 Limits at discontinuities

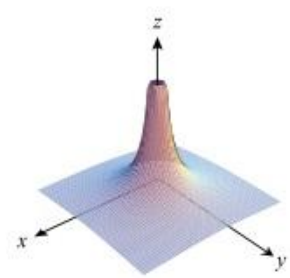
Sometimes it is easy to recognize when a limit does not exist. For example, it is evident that

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = +\infty$$

which implies that the values of the function approach $+\infty$ as $(x,y)\to(0,0)$ along any smooth curve (Figure 6). However, it is not evident whether the limit

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

exists because it is an indeterminate form of type $0.\infty$. Although L'Hôpital's rule cannot be applied directly, the following example illustrates a method for finding this limit by converting to polar coordinates.



$$z = \frac{1}{x^2+y^2}$$

Figure 6

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 923)

Example 21 Find $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

EXERCISES 3

1–6 Use limit laws and continuity properties to evaluate the limit.

1. $\lim\limits_{(x,y)\to(1,3)} (4xy^2 - x)$

2. $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x-y}{\sin y - 1}$

3. $\lim\limits_{(x,y)\to(-1,2)} \dfrac{xy^3}{x+y}$

4. $\lim\limits_{(x,y)\to(1,-3)} e^{2x-y^2}$

5. $\lim\limits_{(x,y)\to(0,0)} \ln(1+x^2y^3)$

6. $\lim\limits_{(x,y)\to(4,-2)} x\sqrt[3]{y^3+2x}$

7–8 Show that the limit does not exist by considering the limits as $(x,y)\to(0,0)$ along the coordinate axes.

7. (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{3}{x^2+2y^2}$ (b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{2x^2+y^2}$

8. (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y}{x^2+y^2}$ (b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\cos xy}{x^2+y^2}$

9–12 Evaluate the limit using the substitution $z = x^2 + y^2$ and observing that $z \to 0^+$ if and only if $(x,y)\to(0,0)$.

9. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

10. $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-\cos(x^2+y^2)}{x^2+y^2}$

11. $\lim\limits_{(x,y)\to(0,0)} e^{-1/(x^2+y^2)}$

12. $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{-1/\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$

13–22 Determine whether the limit exists. If so, find its value.

13. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-y^4}{x^2+y^2}$

14. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-16y^4}{x^2+4y^2}$

15. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{3x^2+2y^2}$

16. $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-x^2-y^2}{x^2+y^2}$

17. $\lim\limits_{(x,y,z)\to(2,-1,2)} \dfrac{xz^2}{\sqrt{x^2+y^2+z^2}}$

18. $\lim\limits_{(x,y,z)\to(2,0,-1)} \ln(2x+y-z)$

19. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\sin(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}$

20. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\sin\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}$

21. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}$

22. $\lim\limits_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\dfrac{1}{x^2+y^2+z^2}\right]$

23–26 Evaluate the limits by converting to polar coordinates, as in Example 21.

23. $\lim\limits_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\,\ln(x^2+y^2)$

24. $\lim\limits_{(x,y)\to(0,0)} y\ln(x^2+y^2)$

25. $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{\sqrt{x^2+y^2}}$ 26. $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+2y^2}}$

27. The accompanying figure shows a portion of the graph of $f(x,y)=\dfrac{x^2y}{x^4+y^2}$

(a) Based on the graph in the figure, does $f(x,y)$ have a limit as $(x,y)\to(0,0)$? Explain your reasoning.

(b) Show that $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along any line $y=mx$. Does this imply that $f(x,y)\to 0$ as $(x,y)\to(0,0)$? Explain.

(c) Show that $f(x,y)\to \dfrac{1}{2}$ as $(x,y)\to(0,0)$ along the parabola $y=x^2$, and confirm visually that this is consistent with the graph of $f(x,y)$.

(d) Based on parts (b) and (c), does $f(x,y)$ have a limit as $(x,y)\to(0,0)$? Is this consistent with your answer to part (a) ?

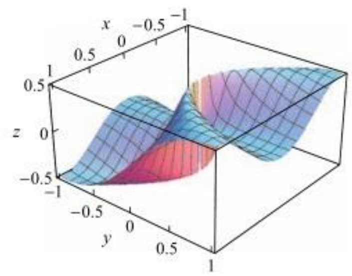


Figure Ex. 27

28. (a) Show that the value of $\dfrac{x^3y}{2x^6+y^2}$ approaches 0 as $(x,y)\to(0,0)$ along any straight line $y=mx$, or along any parabola $y=kx^2$.

(b) Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^3y}{2x^6+y^2}$ does not exist by letting $(x,y)\to(0,0)$ along the curve $y=x^3$.

29. (a) Show that the value of $\dfrac{xyz}{x^2+y^4+z^4}$ approaches 0 as $(x,y,z)\to(0,0,0)$ along any line $x=at, y=bt, z=ct$.

(b) Show that $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^4+z^4}$ does not exist by letting $(x,y,z)\to(0,0,0)$ along the curve $x=t^2, y=t, z=t$.

30–37 Sketch the largest region on which the function f is continuous.

30. $f(x,y)=y\ln(1+x)$ 31. $f(x,y)=\sqrt{x-y}$

32. $f(x,y)=\dfrac{x^2y}{\sqrt{25-x^2-y^2}}$ 33. $f(x,y)=\ln(2x-y+1)$

34. $f(x,y)=\dfrac{y}{11x^2+3}$

35. $f(x,y)=e^{1-xy}$

36. $f(x,y)=\sin^{-1}(xy)$

37. $f(x,y)=\tan^{-1}(y-x)$

(See more exercises from: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 925–927)

---

7. Partial derivatives

7.1 Partial derivatives of functions of two variables

Definition 4 If $z=f(x,y)$ and (x_0,y_0) is a point in the domain of f, then the **partial derivative of f with respect to x at (x_0,y_0)** [also called the **partial derivative of with respect to x at (x_0,y_0)**] is the derivative at x_0 of the function that results when $y=y_0$ is held fixed and x is allowed to vary. This partial derivative is denoted by $f_x(x_0,y_0)$ and is given by

$$f_x(x_0,y_0)=\frac{d}{dx}[f(x,y_0)]\Big|_{x=x_0} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x} \qquad (1)$$

Similarly, the **partial derivative of f with respect to y at (x_0,y_0)** [also called the **partial derivative of with respect to y at (x_0,y_0)**] is the derivative at y_0 of the function that results when $x=x_0$ is held fixed and y is allowed to vary. This partial derivative is denoted by $f_y(x_0,y_0)$ and is given by

$$f_y(x_0,y_0)=\frac{d}{dy}[f(x_0,y)]\Big|_{y=y_0} = \lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y} \qquad (2)$$

Example 22 Find $f_x(1,3)$ and $f_y(1,3)$ for the function $f(x,y)=2x^3y^2+2y+4x$.

7.2 The partial derivative functions

Formulas (1) and (2) define the partial derivatives of a function at a specific point (x_0, y_0). However, often it will be desirable to omit the subscripts and think of the partial derivatives as functions of the variables x and y. These functions are

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Example 23 Find $f_x(x,y)$ and $f_y(x,y)$ for $f(x,y) = 2x^3y^2 + 2y + 4x$, and use those partial derivatives to compute $f_x(1,3)$ and $f_y(1,3)$.

7.3 Partial derivative notation

If $z = f(x,y)$, then the partial derivatives f_x and f_y are also denoted by the symbols

$$\frac{\partial f}{\partial x}, \frac{\partial z}{\partial x} \qquad \text{and} \qquad \frac{\partial f}{\partial y}, \frac{\partial z}{\partial y}$$

Some typical notations for the partial derivatives of $z = f(x,y)$ at a point (x_0, y_0) are

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0, y=y_0}, \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}, \quad \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}, \quad \frac{\partial f}{\partial x}(x_0, y_0), \quad \frac{\partial z}{\partial x}(x_0, y_0)$$

Example 24 Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x,y) = \sqrt{5x + 2y}$.

Example 25 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^4 \sin(xy^3)$.

7.4 Partial derivatives viewed as rates of change and slopes

Recall that if $y = f(x)$, then the value of $f'(x_0)$ can be interpreted either as the rate of change of y with respect to x at x_0 or as the slope of the tangent line to the graph of f at x_0 . Partial derivatives have analogous interpretations. To see that this is so, suppose that C_1 is the intersection of the surface $z = f(x, y)$ with the plane $y = y_0$ and that C_2 is its intersection with the plane $x = x_0$ (Figure 7). Thus, $f_x(x, y_0)$ can be interpreted as the rate of change of with respect to x along the curve C_1, and $f_y(x_0, y)$ can be interpreted as the rate of change of with respect to y along the curve C_2. In particular, $f_x(x_0, y_0)$ is the rate of change of with respect to x along the curve C_1 at the point (x_0, y_0), and $f_y(x_0, y_0)$ is the rate of change of with respect to y along the curve C_2 at the point (x_0, y_0).

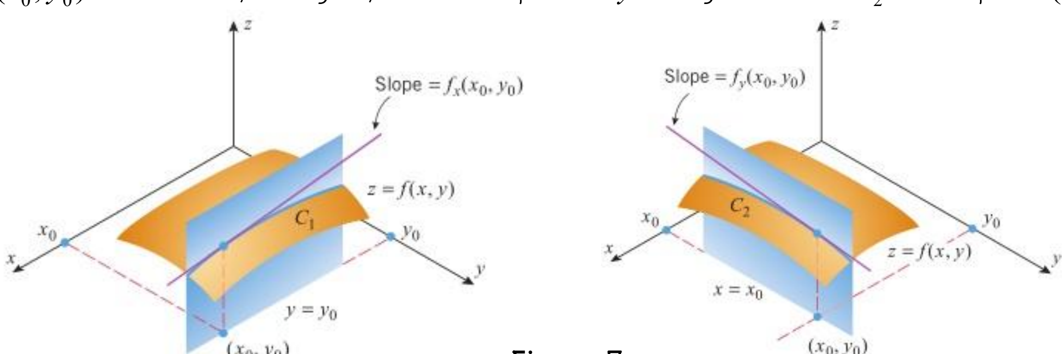


Figure 7

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 929)

Geometrically, $f_x(x_0, y_0)$ can be viewed as the slope of the tangent line to the curve C_1 at the point (x_0, y_0), and $f_y(x_0, y_0)$ can be viewed as the slope of the tangent line to the curve C_2 at the point (x_0, y_0) (Figure 7). We will call $f_x(x_0, y_0)$ the **slope of the surface in the x–direction at** (x_0, y_0) and $f_y(x_0, y_0)$ the **slope of the surface in the y–direction at** (x_0, y_0).

Example 26 Let $f(x,y)=x^2y+5y^3$.

(a) Find the slope of the surface $z=f(x,y)$ in the x –direction at the point (1, −2).

(b) Find the slope of the surface $z=f(x,y)$ in the y –direction at the point (1, −2).

7.5 Implicit partial differentiation

Example 27 Find the slope of the sphere $x^2+y^2+z^2=1$ in the y –direction at the points $(\frac{2}{3},\frac{1}{3},\frac{2}{3})$ and $(\frac{2}{3},\frac{1}{3},-\frac{2}{3})$ (Figure 8).

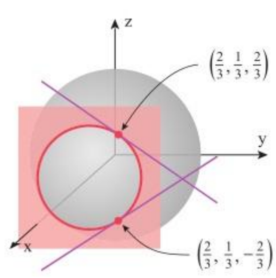


Figure 8

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 931)

7.6 Partial derivatives and continuity

Example 28 Let

$$f(x,y)=\begin{cases} -\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

(a) Show that $f_x(x,y)$ and $f_y(x,y)$ exist at all points (x,y).

(b) Explain why f is not continuous at (0,0).

7.7 Partial derivatives of functions with more than two variables

For a function $f(x,y,z)$ of three variables, there are three **partial derivatives**:

$$f_x(x,y,z),\quad f_y(x,y,z),\quad f_z(x,y,z)$$

The partial derivative f_x is calculated by holding y and constant and differentiating with respect to x. For f_y the variables x and are held constant, and for f_z the variables x and y are held constant. If a dependent variable

$$w=f(x,y,z)$$

is used, then the three partial derivatives of f can be denoted by

$$\frac{\partial w}{\partial x},\quad \frac{\partial w}{\partial y} \quad \text{and} \quad \frac{\partial w}{\partial z}$$

Example 29 Let $f(x,y,z)=x\sin(y+3z)$. Find f_x, f_y and f_z.

Example 30 If $f(x,y,z) = x^3y^2z^4 + 2xy + z$, then

$$f_x(x,y,z) =$$

$$f_y(x,y,z) =$$

$$f_z(x,y,z) =$$

$$f_z(-1,1,2) =$$

Example 31 If $f(\rho,\theta,\phi) = \rho^2 \cos\phi \sin\theta$, then

$$f_\rho(\rho,\theta,\phi) =$$

$$f_\theta(\rho,\theta,\phi) =$$

$$f_\phi(\rho,\theta,\phi) =$$

In general, if $f(v_1, v_2, \ldots, v_n)$ is a function of n variables, there are n partial derivatives of f , each of which is obtained by holding $n-1$ of the variables fixed and differentiating the function f with respect to the remaining variable. If $w = f(v_1, v_2, \ldots, v_n)$, then these partial derivatives are denoted by

$$\frac{\partial w}{\partial v_1}, \quad \frac{\partial w}{\partial v_2}, \quad \ldots, \quad \frac{\partial w}{\partial v_n}$$

where $\dfrac{\partial w}{\partial v_i}$ is obtained by holding all variables except v_i fixed and differentiating with respect to v_i .

7.8 Higher – order partial derivatives

Suppose that f is a function of two variables x and y . Since the partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ are also functions of x and y , these functions may themselves have partial derivatives. This gives rise to four possible **second–order partial derivatives** of f , which are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

Differentiate twice with respect to x.

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Differentiate twice with respect to y.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Differentiate first with respect to x and then with respect to y.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Differentiate first with respect to y and then with respect to x.

The last two cases are called the **mixed second-order partial derivatives** or the **mixed second partials**. Also, the derivatives $\partial f / \partial x$ and $\partial f / \partial y$ are often called the **first-order partial derivatives** when it is necessary to distinguish them from higher-order partial derivatives. Similar conventions apply to the second-order partial derivatives of a function of three variables.

Example 32 Find the second-order partial derivatives of $f(x,y) = x^2y^3 + x^4y$.

Third-order, fourth-order, and higher-order partial derivatives can be obtained by successive differentiation. Some possibilities are

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = f_{xxx} \qquad \frac{\partial^4 f}{\partial y^4} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y^3}\right) = f_{yyyy}$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = f_{xyy} \qquad \frac{\partial^4 f}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y \partial x^2}\right) = f_{xxyy}$$

Example 33 Let $f(x,y) = y^2e^x + y$. Find f_{xyy}.

EXERCISES 4

1. Let $f(x,y)=3x^3y^2$. Find (a) $f_x(x,y)$ (b) $f_y(x,y)$ (c) $f_x(1,y)$ (d) $f_x(x,1)$ (e) $f_y(1,y)$ (f) $f_y(x,1)$ (g) $f_x(1,2)$ (h) $f_y(1,2)$.

2. Let $f(x,y)=e^{2x}\sin y$. Find

(a) $\partial z/\partial x$ (b) $\partial z/\partial y$ (c) $\partial z/\partial x\big|_{(0,y)}$ (d) $\partial z/\partial x\big|_{(x,0)}$

(e) $\partial z/\partial y\big|_{(0,y)}$ (f) $\partial z/\partial y\big|_{(x,0)}$ (g) $\partial z/\partial x\big|_{(\ln 2,0)}$ (h) $\partial z/\partial y\big|_{(\ln 2,0)}$.

3–10 Evaluate the indicated partial derivatives.

3. $z=9x^2y-3x^5y;\quad \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

4. $f(x,y)=10x^2y^4-6xy^2+10x^2;\quad f_x(x,y),\ f_y(x,y)$

5. $z=(x^2+5x-2y)^8;\quad \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

6. $f(x,y)=\dfrac{1}{xy^2-x^2y};\quad f_x(x,y),\ f_y(x,y)$

7. $\dfrac{\partial}{\partial p}(e^{-7p/q}),\ \dfrac{\partial}{\partial q}(e^{-7p/q})$

8. $\dfrac{\partial}{\partial x}(xe^{\sqrt{15xy}}),\ \dfrac{\partial}{\partial y}(xe^{\sqrt{15xy}})$

9. $z=\sin(5x^3y+7xy^2);\quad \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

10. $f(x,y)=\cos(2xy^2-3x^2y^2);\quad f_x(x,y),\ f_y(x,y)$

11. Let $f(x,y)=xe^{-y}+5y$.

(a) Find the slope of the surface $z=f(x,y)$ in the x–direction at the point (3, 0).

(b) Find the slope of the surface $z=f(x,y)$ in the y–direction at the point (3, 0).

12. Let $z=\sin(y^2-4x)$.

(a) Find the rate of change of with respect to x at the point (2, 1) with y held fixed.

(b) Find the rate of change of with respect to y at the point (2, 1) with x held fixed.

13. Use the information in the accompanying figure to find the values of the first-order partial derivatives of f at the point (1, 2).

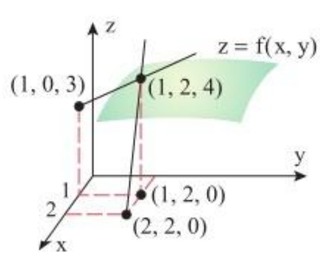


Figure Ex–13

14–19 Find $\partial z/\partial x$ and $\partial z/\partial y$.

14. $z=4e^{x^2y^3}$

15. $z=\cos(x^5y^4)$

16. $z=x^3\ln(1+xy^{-3/5})$

17. $z=e^{xy}\sin 4y^2$

18. $z=\dfrac{xy}{x^2+y^2}$

19. $z=\dfrac{x^2y^3}{\sqrt{x+y}}$

20–25 Find $f_x(x,y)$ and $f_y(x,y)$.

20. $f(x,y)=\sqrt{3x^5y-7x^3y}$ 21. $f(x,y)=\dfrac{x+y}{x-y}$

22. $f(x,y)=y^{-3/2}\tan^{-1}(x/y)$ 23. $f(x,y)=x^{-3}e^{-y}+y^3\sec\sqrt{x}$

24. $f(x,y)=(y^2\tan x)^{-4/3}$ 25. $f(x,y)=\cosh(\sqrt{x})\sinh^2(xy^2)$

26–29 Evaluate the indicated partial derivatives.

26. $f(x,y)=9-x^2-7y^3;\quad f_x(3,1),\ f_y(3,1)$ 27. $f(x,y)=x^2ye^{xy};\quad \partial f/\partial x(1,1),\ \partial f/\partial y(1,1)$

28. $z=\sqrt{x^2+4y^2};\quad \partial z/\partial x(1,2),\ \partial z/\partial y(1,2)$ 29. $w=x^2\cos(xy);\quad \partial w/\partial x(\frac{1}{2},\pi),\ \partial w/\partial y(\frac{1}{2},\pi)$

30. Let $f(x,y,z)=x^2y^4z^3+xy+z^2+1$. Find (a) $f_x(x,y,z)$ (b) $f_y(x,y,z)$ (c) $f_z(x,y,z)$ (d) $f_x(1,y,z)$ (e) $f_y(1,2,z)$ (f) $f_z(1,2,3)$

31. Let $w=x^2y\cos z$. Find (a) $\partial w/\partial x\,(x,y,z)$ (b) $\partial w/\partial y\,(x,y,z)$ (c) $\partial w/\partial z\,(x,y,z)$ (d) $\partial w/\partial x\,(2,y,z)$ (e) $\partial w/\partial y\,(2,1,z)$ (f) $\partial w/\partial z\,(1,2,0)$

32–35 Find f_x, f_y and f_z.

32. $f(x,y,z)=z\ln(x^2y\cos z)$ 33. $f(x,y,z)=y^{-3/2}\sec\left(\dfrac{xz}{y}\right)$

34. $f(x,y,z)=\tan^{-1}\left(\dfrac{1}{xy^2z^3}\right)$ 35. $f(x,y,z)=\cosh(\sqrt{z})\sinh^2(x^2yz)$

36–39 Find $\partial w/\partial x$, $\partial w/\partial y$ and $\partial w/\partial z$.

36. $w=ye^z\sin xz$ 37. $w=\dfrac{x^2-y^2}{y^2+z^2}$

38. $w=\sqrt{x^2+y^2+z^2}$ 39. $w=y^3e^{2x+3z}$

40. Let $f(x,y,z)=y^2e^{xz}$. Find $\partial f/\partial x\big|_{(1,1,1)}$, $\partial f/\partial y\big|_{(1,1,1)}$ and $\partial f/\partial z\big|_{(1,1,1)}$.

41. Let $w=\sqrt{x^2+4y^2-z^2}$. Find $\partial w/\partial x\big|_{(2,1,-1)}$, $\partial w/\partial y\big|_{(2,1,-1)}$ and $\partial w/\partial z\big|_{(2,1,-1)}$.

42. A point moves along the intersection of the elliptic paraboloid $z=x^2+3y^2$ and the plane $y=1$. At what rate is changing with respect to x when the point is at (2, 1, 7)?

43. A point moves along the intersection of the elliptic paraboloid $z=x^2+3y^2$ and the plane $x=2$. At what rate is changing with respect to y when the point is at (2, 1, 7)?

44–47 Calculate $\partial z/\partial x$ and $\partial z/\partial y$ using implicit differentiation. Leave your answers in terms of $x, y,$ and .

44. $(x^2+y^2+z^2)^{3/2}=1$ 45. $\ln(2x^2+y-z^3)=x$

46. $x^2 + z\sin xyz = 0$

47. $e^{xy}\sinh z - z^2 x + 1 = 0$

48–51 Find $\partial w/\partial x, \partial w/\partial y$ and $\partial w/\partial z$ using implicit differentiation. Leave your answers in terms of x, y, z and w.

48. $(x^2 + y^2 + z^2 + w^2)^{3/2} = 4$

49. $\ln(2x^2 + y - z^3 + 3w) = z$

50. $w^2 + w\sin xyz = 1$

51. $e^{xy}\sinh w - z^2 w + 1 = 0$

52. Let $z = \sqrt{x}\cos y$. Find $\partial^2 z/\partial x^2$, $\partial^2 z/\partial y^2$, $\partial^2 z/\partial x\partial y$ and $\partial^2 z/\partial y\partial x$.

53. Let $f(x,y) = 4x^2 - 2y + 7x^4 y^5$. Find f_{xx}, f_{yy}, f_{xy} and f_{yx}.

54. Let $f(x,y) = \sin(3x^2 + 6y^2)$. Find f_{xx}, f_{yy}, f_{xy} and f_{yx}.

55. Let $f(x,y) = xe^{2y}$. Find f_{xx}, f_{yy}, f_{xy} and f_{yx}.

56–63 Confirm that the mixed second-order partial derivatives of f are the same.

56. $f(x,y) = 4x^2 - 8xy^4 + 7y^5 - 3$

57. $f(x,y) = \sqrt{x^2 + y^2}$

58. $f(x,y) = e^x \cos y$

59. $f(x,y) = e^{x-y^2}$

60. $f(x,y) = \ln(4x - 5y)$

61. $f(x,y) = \ln(x^2 + y^2)$

62. $f(x,y) = (x-y)/(x+y)$

63. $f(x,y) = (x^2 - y^2)/(x^2 + y^2)$

64. Express the following derivatives in "∂" notation.

(a) f_{xxx} (b) f_{xyy} (c) f_{yyxx} (d) f_{xyyy}

65. Express the derivatives in "subscript" notation.

(a) $\dfrac{\partial^3 f}{\partial y^2 \partial x}$ (b) $\dfrac{\partial^4 f}{\partial x^4}$ (c) $\dfrac{\partial^4 f}{\partial y^2 \partial x^2}$ (d) $\dfrac{\partial^5 f}{\partial x^2 \partial y^3}$

66. Given $f(x,y) = x^3 y^5 - 2x^2 y + x$, find f_{xxy}, f_{yxy} and f_{yyy}.

67. Given $z = (2x - y)^5$, find $\dfrac{\partial^3 z}{\partial y \partial x \partial y}$, $\dfrac{\partial^3 z}{\partial x^2 \partial y}$ and $\dfrac{\partial^4 z}{\partial x^2 \partial y^2}$.

68. Given $f(x,y) = y^3 e^{-5x}$, find $f_{xyy}(0,1)$, $f_{xxx}(0,1)$ and $f_{yyxx}(0,1)$.

69. Given $w = e^y \cos x$, find $\left.\dfrac{\partial^3 w}{\partial y^2 \partial x}\right|_{(\pi/4,0)}$ and $\left.\dfrac{\partial^3 w}{\partial x^2 \partial y}\right|_{(\pi/4,0)}$.

70. Let $f(x,y,z) = x^3 y^5 z^7 + xy^2 + y^3 z$. Find $f_{xy}, f_{yz}, f_{xz}, f_{zz}, f_{zyy}, f_{xxy}, f_{zyx}$ and f_{xxyz}.

71. Let $w = (4x - 3y + 2z)^5$. Find $\dfrac{\partial^2 w}{\partial x \partial z}$, $\dfrac{\partial^3 w}{\partial x \partial y \partial z}$ and $\dfrac{\partial^4 w}{\partial z^2 \partial y \partial x}$.

72–75 Find the indicated partial derivatives.

72. $f(v,w,x,y)=4v^2w^3x^4y^5; \quad \partial f/\partial v,\ \partial f/\partial w,\ \partial f/\partial x,\ \partial f/\partial y$

73. $w=r\cos st+e^u\sin ur; \quad \partial w/\partial r,\ \partial w/\partial s,\ \partial w/\partial t,\ \partial w/\partial u$

74. $f(v_1,v_2,v_3,v_4)=\dfrac{v_1^{\,2}-v_2^{\,2}}{v_3^{\,2}-v_4^{\,2}}; \quad \partial f/\partial v_1,\ \partial f/\partial v_2,\ \partial f/\partial v_3,\ \partial f/\partial v_4$

75. $V=xe^{2x-y}+we^{zw}+yw; \quad \partial V/\partial x,\ \partial V/\partial y,\ \partial V/\partial z,\ \partial V/\partial w$

76. Let $u(w,x,y,z)=xe^{yw}\sin^2 z$. Find

(a) $\dfrac{\partial u}{\partial x}(0,0,1,\pi)$ (b) $\dfrac{\partial u}{\partial y}(0,0,1,\pi)$ (c) $\dfrac{\partial u}{\partial w}(0,0,1,\pi)$

(d) $\dfrac{\partial u}{\partial z}(0,0,1,\pi)$ (e) $\dfrac{\partial^4 u}{\partial x\partial y\partial w\partial z}$ (f) $\dfrac{\partial^4 u}{\partial w\partial z\partial y^2}$.

77. Let $f(v,w,x,y)=2v^{1/2}w^4x^{1/2}y^{2/3}$. Find $f_v(1,-2,4,8),\ f_w(1,-2,4,8),\ f_x(1,-2,4,8)$ and $f_y(1,-2,4,8)$.

78–79 Find $\dfrac{\partial w}{\partial x_i}$ for $i=1,2,\ldots,n$.

78. $w=\cos(x_1+2x_2+\cdots+nx_n)$

79. $w=\left(\sum_{k=1}^{n}x_k\right)^{1/n}$

(See more exercises from: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 936–940)

8. The Chain rule

8.1 Chain rules for derivatives

Theorem 3 (Chain rules for derivatives)

If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

If each of the functions $x = x(t), y = y(t)$, and $z = z(t)$ is differentiable at t, and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(t), y(t), z(t))$, then the function $w = f(x(t), y(t), z(t))$ is differentiable at t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y, z).

Example 34 Suppose that $z = x^2 y, \quad , x = t^2, \quad y = t^3.$ Use the chain rule to find $\frac{dz}{dt}$, and check the result by expressing as a function of t and differentiating directly.

Example 35 Suppose that $w = \sqrt{x^2 + y^2 + z^2}, \quad x = \cos\theta, \quad y = \sin\theta, \quad z = \tan\theta.$ Use the chain rule to find $\frac{dw}{d\theta}$ when $\theta = \frac{\pi}{4}$.

8.2 Chain rules for partial derivatives

Theorem 4 **(Chain rules for partial derivatives)**

If $x = x(u,v)$ and $y = y(u,v)$ have first-order partial derivatives at the point (u,v), and if $z = f(x,y)$ is differentiable at the point $(x,y) = (x(u,v), y(u,v))$, then $z = f(x(u,v), y(u,v))$ has first-order partial derivatives at the point (u,v) given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

If $x = x(u,v), y = y(u,v)$ and $z = z(u,v)$ have first-order partial derivatives at the point (u,v), and if the function $w = f(x,y,z)$ is differentiable at the point $(x,y,z) = (x(u,v), y(u,v), z(u,v))$, then $z = f(x(u,v), y(u,v), z(u,v))$ has first-order partial derivatives at the point (u,v) given by

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

Example 36 Given that $z = e^{xy}, \quad , x = 2u + v, \quad y = \frac{u}{v}$, find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using the chain rule.

Example 36 Suppose that $w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2 v$. Use appropriate forms of the chain rule to find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

8.3 Other versions of the chain rule

Although we will not prove it, the chain rule extends to functions $w = f(v_1, v_2, \ldots, v_n)$ of n variables. For example, if each v_i is a function of t, $i = 1, 2, \ldots, n$, the relevant formula is

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial v_1}\frac{dv_1}{dt} + \frac{\partial w}{\partial v_2}\frac{dv_2}{dt} + \cdots + \frac{\partial w}{\partial v_n}\frac{dv_n}{dt}}$$

Example 37 Suppose that $w = x^2 + y^2 - z^2, \quad x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$.

Use appropriate forms of the chain rule to find $\dfrac{\partial w}{\partial \rho}$ and $\dfrac{\partial w}{\partial \theta}$.

Example 38 Suppose that $w = xy + yz, \quad y = \sin x, \quad z = e^x$.

Use appropriate forms of the chain rule to find $\dfrac{dw}{dx}$.

8.4 Implicit differentiation

Consider the special case where $z = f(x, y)$ is a function of x and y and y is a differentiable function of x. Then

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \qquad (*)$$

This result can be used to find derivatives of functions that are defined implicitly. For example, suppose that the equation

$$f(x, y) = c \qquad (**)$$

defines y implicitly as a differentiable function of x and we are interested in finding $\frac{dy}{dx}$. Differentiating both sides of (**) with respect to x and applying (*) yields

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0.$$

Thus, if $\frac{\partial f}{\partial y} \neq 0$, we obtain

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\frac{f_x}{f_y}.$$

In summary, we have the following result.

Theorem 5 If the equation $f(x, y) = c$ defines y implicitly as a differentiable function of x, and if $\frac{\partial f}{\partial y} \neq 0$, then

$$\boxed{\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\frac{f_x}{f_y}}$$

Example 39 Given that $x^3 + y^2 x - 3 = 0.$

find $\frac{dy}{dx}$ using Theorem 5 and check the result using implicit differentiation.

The chain rule also applies to implicit partial differentiation. Consider the case where $w = f(x, y, z)$ is a function of , and and is a differentiable function of x and y. It follows from Theorem 5 that

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}. \qquad (***)$$

If the equation

$$f(x, y, z) = c \qquad (****)$$

defines implicitly as a differentiable function of x and y, then taking the partial derivative of each side of (****) with respect to x and applying (***) gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0.$$

If $\frac{\partial f}{\partial z} \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f / \partial x}{\partial f / \partial z} = -\frac{f_x}{f_z}.$$

A similar result holds for $\frac{\partial z}{\partial y}$.

Theorem 6 If the equation $f(x, y, z) = c$ defines implicitly as a differentiable function of x and y, and if $\frac{\partial f}{\partial z} \neq 0$, then

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\partial f / \partial x}{\partial f / \partial z} = -\frac{f_x}{f_z}} \quad \text{and} \quad \boxed{\frac{\partial z}{\partial y} = -\frac{\partial f / \partial y}{\partial f / \partial z} = -\frac{f_y}{f_z}}.$$

Example 40 Consider the sphere $x^2 + y^2 + z^2 = 1$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$.

EXERCISES 5

1–6 Use an appropriate form of the chain rule to find $\frac{dz}{dt}$.

1. $z=3x^2y^3;\ x=t^4,\ y=t^2$
2. $z=\ln(2x^2+y);\ x=\sqrt{t},\ y=t^{2/3}$
3. $z=3\cos x-\sin xy;\ x=1/t,\ y=3t$
4. $z=\sqrt{1+x-2xy^4};\ x=\ln t,\ y=t$
5. $z=e^{1-xy};\ x=t^{1/3},\ y=t^3$
6. $z=\cosh^2 xy;\ x=t/2,\ y=e^t$

7–10 Use an appropriate form of the chain rule to find $\frac{dw}{dt}$.

7. $w=5x^2y^3z^4;\ x=t^2,\ y=t^3,\ z=t^5$
8. $w=\ln(3x^2-2y+4z^3);\ x=t^{1/2},\ y=t^{2/3},\ z=t^{-2}$
9. $w=5\cos xy-\sin xz;\ x=1/t,\ y=t,\ z=t^3$
10. $w=\sqrt{1+x-2yz^4x};\ x=\ln t,\ y=t,\ z=4t$

11. Suppose that $w=5x^2y^3z^4;\ x=t^2,\ y=t^3,\ z=t^5$.

Find the rate of change of w with respect to t at $t=1$ by using the chain rule, and then check your work by expressing w as a function of t and differentiating.

12. Suppose that $w=x\sin yz^2;\ x=\cos t,\ y=t^2,\ z=e^t$

Find the rate of change of w with respect to t at $t=0$ by using the chain rule, and then check your work by expressing w as a function of t and differentiating.

13. Suppose that $z=f(x,y)$ is differentiable at the point $(4,8)$ with $f_x(4,8)=3$ and $f_y(4,8)=-1$. If $x=t^2$ and $y=t^3$, find dz/dt when $t=2$.

14. Suppose that $w=f(x,y,z)$ is differentiable at the point $(1,0,2)$ with $f_x(1,0,2)=1$, $f_y(1,0,2)=2$, and $f_z(1,0,2)=3$. If $x=t, y=\sin(\pi t)$, and $z=t^2+1$, find dw/dt when $t=1$.

15–20 Use appropriate forms of the chain rule to find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

15. $z=8x^2y-2x+3y;\ x=uv,\ y=u-v$
16. $z=x^2-y\tan x;\ x=u/v,\ y=u^2v^2$
17. $z=x/y;\ x=2\cos u,\ y=3\sin v$
18. $z=3x-2y;\ x=u+v\ln u,\ y=u^2-v\ln v$
19. $z=e^{x^2y};\ x=\sqrt{uv},\ y=1/v$
20. $z=\cos x\sin y;\ x=u-v,\ y=u^2+v^2$

21–28 Use appropriate forms of the chain rule to find the derivatives.

21. Let $T=x^2y-xy^3+2;\ x=r\cos\theta,\ y=r\sin\theta$. Find $\partial T/\partial r$ and $\partial T/\partial\theta$.

22. Let $R=e^{2s-t^2};\ s=3\phi,\ t=\phi^{1/2}$. Find $dR/d\phi$.

23. Let $t=u/v;\ u=x^2-y^2,\ v=4xy^3$. Find $\partial t/\partial x$ and $\partial t/\partial y$.

24. Let $w = rs/(r^2+s^2);\ r = uv,\ s = u-2v$. Find $\partial w/\partial u$ and $\partial w/\partial v$.

25. Let $z = \ln(x^2+1)$, where $x = r\cos\theta$. Find $\partial z/\partial r$ and $\partial z/\partial\theta$.

26. Let $u = rs^2\ln t;\ r = x^2,\ s = 4y+1,\ t = xy^3$. Find $\partial u/\partial x$ and $\partial u/\partial y$.

27. Let $w = 4x^2+4y^2+z^2;\ x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$. Find $\partial w/\partial\rho, \partial w/\partial\phi$ and $\partial w/\partial\theta$.

28. Let $w = 3xy^2z^3;\ y = 3x^2+2,\ z = \sqrt{x+1}$. Find dw/dx.

29. Use a chain rule to find the value of $\left.\dfrac{dw}{ds}\right|_{s=1/4}$ if $w = r^2 - r\tan\theta;\ r = \sqrt{s},\ \theta = \pi s$.

30. Use a chain rule to find the value of $\left.\dfrac{\partial f}{\partial u}\right|_{u=1,v=-2}$ and $\left.\dfrac{\partial f}{\partial v}\right|_{u=1,v=-2}$ if $f(x,y) = x^2y^2 - x + 2y;\ x = \sqrt{u},\ y = uv^3$.

31. Use a chain rule to find the value of $\left.\dfrac{\partial z}{\partial r}\right|_{r=2,\theta=\pi/6}$ and $\left.\dfrac{\partial z}{\partial\theta}\right|_{r=2,\theta=\pi/6}$ if $z = xye^{x/y};\ x = r\cos\theta,\ y = r\sin\theta$.

32. Use a chain rule to find $\left.\dfrac{dz}{dt}\right|_{t=3}$ if $z = x^2y;\ x = t^2,\ y = t+7$.

33–36 Use Theorem 5 to find dy/dx and check your result using implicit differentiation.

33. $x^2y^3 + \cos y = 0$

34. $x^3 - 3xy^2 + y^3 = 5$

35. $e^{xy} + ye^y = 1$

36. $x - \sqrt{xy} + 3y = 4$

37–40 Find $\partial z/\partial x$ and $\partial z/\partial y$ by implicit differentiation, and confirm that the results obtained agree with those predicted by the formulas in Theorem 6.

37. $x^2 + 3yz^2 + xyz - 2 = 0$

38. $\ln(1+z) + xy^2 + z = 1$

39. $ye^x - 5\sin 3z = 3z$

40. $e^{xy}\cos yz - e^{yz}\sin xz + 2 = 0$

(See more exercises from: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 956–959)

9. Differentials and local linearity

9.1 Differentials

As with the one-variable case, the approximations

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

for a function of two variables and the approximation

$$\Delta f \approx f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z$$

for a function of three variables have a convenient formulation in the language of differentials. If $z = f(x, y)$ is differentiable at a point (x_0, y_0), we let

$$\boxed{dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy}$$

denote a new function with dependent variable dz and independent variables dx and dy. We refer to this function (also denoted df) as the **total differential of** **at** (x_0, y_0) or as the **total differential of** f **at** (x_0, y_0). Similarly, for a function $w = f(x, y, z)$ of three variables we have the **total differential of** w **at** (x_0, y_0, z_0),

$$\boxed{dw = f_x(x_0, y_0, z_0)dx + f_y(x_0, y_0, z_0)dy + f_z(x_0, y_0, z_0)dz}$$

which is also referred to as the **total differential of** f **at** (x_0, y_0, z_0). It is common practice to omit the subscripts and write as

$$\boxed{dz = f_x(x, y)dx + f_y(x, y)dy}$$

and

$$\boxed{dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz}$$

In the two-variable case, the approximation

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

can be written in the form

$$\boxed{\Delta f \approx df}$$

for $dx = \Delta x$ and $dy = \Delta y$. Equivalently, we can write approximation $\Delta f \approx df$ as

$$\boxed{\Delta z \approx dz} \qquad (*)$$

In other words, we can estimate the change Δz in by the value of the differential dz where dx is the change in x and dy is the change in y. Furthermore, if Δx and Δy are close to 0, then the magnitude of the error in approximation (*) will be much smaller than the distance $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ between (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$.

Example 41 Use (*) to approximate the change in $z = xy^2$ from its value at (0.5, 1.0) to its value at (0.503, 1.004). Compare the magnitude of the error in this approximation with the distance between the points (0.5, 1.0) and (0.503, 1.004).

9.2 Local linear approximations

We now show that if a function f is differentiable at a point, then it can be very closely approximated by a linear function near that point. For example, suppose that $f(x,y)$ is differentiable at the point (x_0, y_0). Then approximation can be written in the form

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0)+f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y$$

If we let $x = x_0+\Delta x$ and $y = y_0+\Delta y$, this approximation becomes

$$f(x,y) \approx f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \qquad (**)$$

which yields a linear approximation of $f(x,y)$. Since the error in this approximation is equal to the error in approximation, we conclude that for (x,y) close to (x_0,y_0) ,the error in (**) will be much smaller than the distance between these two points. When $f(x,y)$ is differentiable at (x_0,y_0) we get

$$\boxed{L(x,y) = f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)}$$

and refer to $L(x,y)$ as **the local linear approximation to f at (x_0,y_0).**

Example 42 Let $L(x,y)$ denote the local linear approximation to $f(x,y)=\sqrt{x^2+y^2}$ at the point (3, 4). Compare the error in approximating

$$f(3.04,3.98)=\sqrt{(3.04)^2+(3.98)^2}$$

by $L(3.04,3.98)$ with the distance between the points (3, 4) and (3.04, 3.98).

EXERCISES 6

1. Suppose that a function $f(x,y)$ is differentiable at the point $(3,4)$ with $f_x(3,4)=2$ and $f_y(3,4)=-1$. If $f(3,4)=5$, estimate the value of $f(3.01,3.98)$.

2. Suppose that a function $f(x,y)$ is differentiable at the point $(-1,2)$ with $f_x(-1,2)=1$ and $f_y(-1,2)=3$. If $f(-1,2)=2$, estimate the value of $f(-0.99,2.02)$.

3. Suppose that a function $f(x,y,z)$ is differentiable at the point $(1,2,3)$ with $f_x(1,2,3)=1$, $f_y(1,2,3)=2$ and $f_z(1,2,3)=3$. If $f(1,2,3)=4$, estimate the value of $f(1.01,2.02,3.03)$.

4. Suppose that a function $f(x,y,z)$ is differentiable at the point $(2,1,-2)$ with $f_x(2,1,-2)=-1$, $f_y(2,1,-2)=1$ and $f_z(2,1,-2)=-2$. If $f(2,1,-2)=0$, estimate the value of $f(1.98,0.99,-1.97)$.

5–16 Compute the differential dz or dw of the function.

5. $z=7x-2y$
6. $z=e^{xy}$
7. $z=x^3y^2$
8. $z=5x^2y^5-2x+4y+7$
9. $z=\tan^{-1}xy$
10. $z=e^{-3x}\cos 6y$
11. $w=8x-3y+4z$
12. $w=e^{xyz}$
13. $w=x^3y^2z$
14. $w=4x^2y^3z^7-3xy+z+5$
15. $w=\tan^{-1}(xyz)$
16. $w=\sqrt{x}+\sqrt{y}+\sqrt{z}$

17–22 Use a total differential to approximate the change in the values of f from P to Q. Compare your estimate with the actual change in f.

17. $f(x,y)=x^2+2xy-4x;\ P(1,2),\ Q(1.01,2.04)$
18. $f(x,y)=x^{1/3}y^{1/2};\ P(8,9),\ Q(7.78,9.03)$
19. $f(x,y)=\dfrac{x+y}{xy};\ P(-1,-2),\ Q(-1.02,-2.04)$
20. $f(x,y)=\ln\sqrt{1+xy};\ P(0,2),\ Q(-0.09,1.98)$
21. $f(x,y,z)=2xy^2z^3;\ P(1,-1,2),\ Q(0.99,-1.02,2.02)$
22. $f(x,y,z)=\dfrac{xyz}{x+y+z};\ P(-1,-2,4),\ Q(-1.04,-1.98,3.97)$

23. In the accompanying figure a rectangle with initial length x_0 and initial width y_0 has been enlarged, resulting in a rectangle with length $x_0+\Delta x$ and width $y_0+\Delta y$. What portion of the figure

represents the increase in the area of the rectangle? What portion of the figure represents an approximation of the increase in area by a total differential?

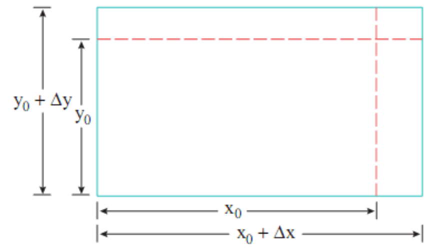


Figure Ex. 23

24. The volume V of a right circular cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$. Suppose that the height decreases from 20 in to 19.95 in and the radius increases from 4 in to 4.05 in. Compare the change in volume of the cone with an approximation of this change using a total differential.

25–32 (a) Find the local linear approximation L to the specified function f at the designated point P.

(b) Compare the error in approximating f by L at the specified point Q with the distance between P and Q.

25. $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$; $P(4,3)$, $Q(3.92, 3.01)$
26. $f(x,y) = x^{0.5}y^{0.3}$; $P(1,1)$, $Q(1.05, 0.97)$
27. $f(x,y) = x\sin y$; $P(0,0)$, $Q(0.003, 0.004)$
28. $f(x,y) = \ln xy$; $P(1,2)$, $Q(1.01, 2.02)$
29. $f(x,y,z) = xyz$; $P(1,2,3)$, $Q(1.001, 2.002, 3.003)$
30. $f(x,y,z) = \dfrac{x+y}{y+z}$; $P(-1,1,1)$, $Q(-0.99, 0.99, 1.01)$
31. $f(x,y,z) = xe^{yz}$; $P(1,-1,-1)$, $Q(0.99, -1.01, -0.99)$
32. $f(x,y,z) = \ln(x+yz)$; $P(2,1,-1)$, $Q(2.02, 0.97, -1.01)$

33. Suppose that a function $f(x,y)$ is differentiable at the point $(1,1)$ with $f_x(1,1) = 2$ and $f(1,1) = 3$. Let $L(x,y)$ denote the local linear approximation of f at $(1,1)$. If $L(1.1, 0.9)$, find the value of $f_y(1,1)$.

(See more exercises from: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 947–949)

10. Maxima and minima of functions of two variables

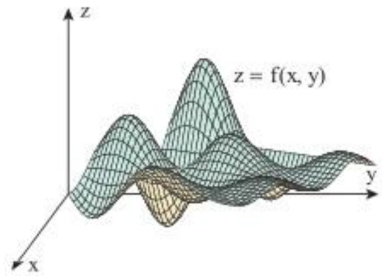


Figure 9

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 977)

10.1 Extrema

Definition 5 A function f of two variables is said to have a **relative maximum** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an **absolute maximum** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.

Definition 6 A function f of two variables is said to have a **relative minimum** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an **absolute minimum** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.

If f has a relative maximum or a relative minimum at (x_0, y_0), then we say that f has a **relative extremum** at (x_0, y_0), and if f has an absolute maximum or absolute minimum at (x_0, y_0), then we say that f has an **absolute extremum** at (x_0, y_0).

Figure 10 shows the graph of a function f whose domain is the square region in the xy–plane whose points satisfy the inequalities $0 \leq x \leq 1, 0 \leq y \leq 1$. The function f has relative minima at the points A and C and a relative maximum at B. There is an absolute minimum at A and an absolute maximum at D.

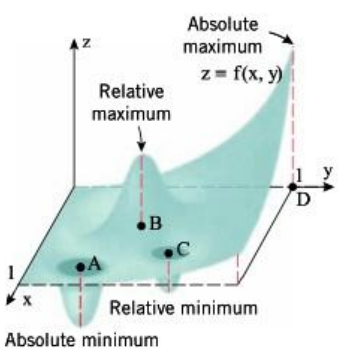


Figure 10

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 977)

10.2 Finding relative extrema

Recall that if a function g of one variable has a relative extremum at a point x_0 where g is differentiable, then $g'(x_0) = 0$. To obtain the analog of this result for functions of two variables, suppose that $f(x, y)$ has a relative maximum at a point (x_0, y_0) and that the partial derivatives of f exist at (x_0, y_0). It seems plausible geometrically that the traces of the surface $z = f(x, y)$ on the planes $x = x_0$ and $y = y_0$ have horizontal tangent lines at (x_0, y_0) (Figure 11), so

$$f_x(x_0, y_0) = 0 \qquad \text{and} \qquad f_y(x_0, y_0) = 0 .$$

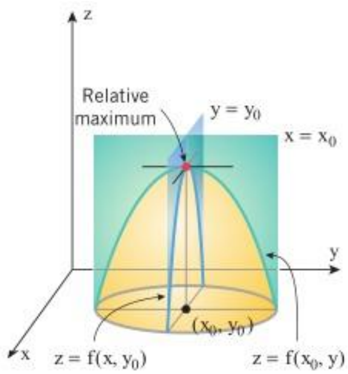


Figure 11

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 979)

The same conclusion holds if f has a relative minimum at (x_0, y_0), all of which suggests the following result, which we state without formal proof.

Theorem 7 If f has a relative extremum at a point (x_0, y_0), and if the first-order partial derivatives of f exist at this point, then

$$f_x(x_0, y_0) = 0 \qquad \text{and} \qquad f_y(x_0, y_0) = 0 .$$

Recall that the *critical points* of a function f of one variable are those values of x in the domain of f at which $f'(x) = 0$ or f is not differentiable. The following definition is the analog for functions of two variables.

Definition 7 A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a **critical point** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

It follows from this definition and Theorem 7 that relative extrema occur at critical points, just as for a function of one variable. However, recall that for a function of one variable a relative extremum need not occur at every critical point. For example, the function might have an inflection point with a horizontal tangent line at the critical point (see Figure 12). Similarly, a function of two variables need not have a relative extremum at every critical point. For example, consider the function

$$f(x,y) = y^2 - x^2$$

This function, whose graph is the hyperbolic paraboloid shown in Figure , has a critical point at $(0,0)$, since

$$f_x(x,y) = -2x \qquad \text{and} \qquad f_y(x,y) = 2y$$

from which it follows that

$$f_x(0,0) = 0 \qquad \text{and} \qquad f_y(0,0) = 0.$$

However, the function f has neither a relative maximum nor a relative minimum at $(0,0)$.

For obvious reasons, the point $(0,0)$ is called a saddle point of f. In general, we will say that a surface $z = f(x,y)$ has a **saddle point** at (x_0, y_0) if there are two distinct vertical planes through this point such that the trace of the surface in one of the planes has a relative maximum at (x_0, y_0) and the trace in the other has a relative minimum at (x_0, y_0).

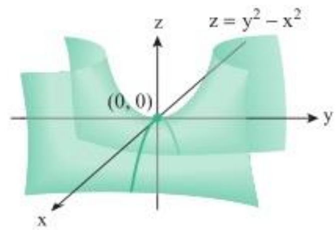
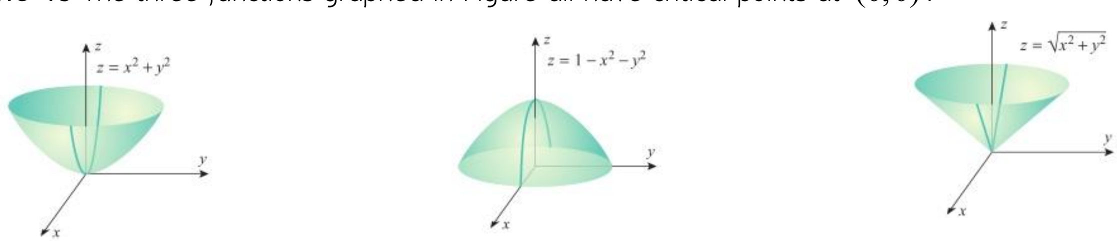


Figure 12

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 979)

Example 43 The three functions graphed in Figure all have critical points at $(0,0)$.



$f_x(0,0) = f_y(0,0) = 0$	$f_x(0,0) = f_y(0,0) = 0$	$f_x(0,0)$ and $f_y(0,0)$ do not exist
relative and absolute min at $(0,0)$	relative and absolute max at $(0,0)$	relative and absolute min at $(0,0)$

Figure 13

(From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 980)

10.3 The second partial test

> **Theorem 8 (The Second Partials Test)** Let f be a function of two variables with continuous second–order partial derivatives in some disk centered at a critical point (x_0, y_0), and let
>
> $$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^{\,2}(x_0, y_0)$$
>
> (a) If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).
>
> (b) If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).
>
> (c) If $D < 0$, then f has a saddle point at (x_0, y_0).
>
> (d) If $D = 0$, then no conclusion can be drawn.

Example 44 Locate all relative extrema and saddle points of $f(x, y) = 3x^2 - 2xy + y^2 - 8y$.

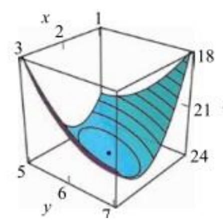


Figure Example 44

Example 45 Locate all relative extrema and saddle points of $f(x, y) = 4xy - x^4 - y^4$.

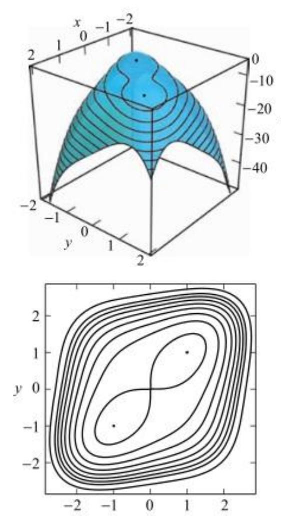


Figure Example 45

10.4 Finding absolute extrema on closed and bounded sets

> ***How to Find the Absolute Extrema of a Continuous Function f of Two Variables on a Closed and Bounded Set R***
>
> **Step 1.** Find the critical points of f that lie in the interior of R.
>
> **Step 2.** Find all boundary points at which the absolute extrema can occur.
>
> **Step 3.** Evaluate $f(x,y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

Example 46 Find the absolute maximum and minimum values of $f(x,y)=3xy-6x-3y+7$ on the closed triangular region R with vertices (0, 0), (3, 0), and (0, 5).

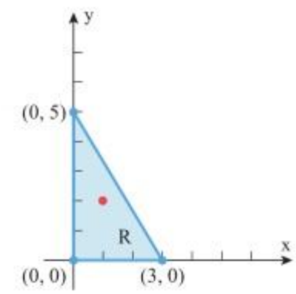


Figure Example 46

Example 47 Determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

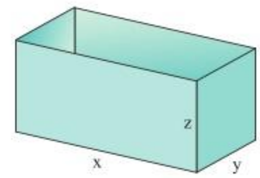


Figure Example 47

EXERCISES 7

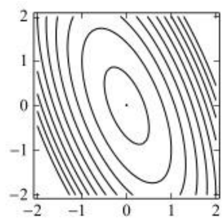
1–2 Locate all absolute maxima and minima, if any, by inspection. Then check your answers using calculus.

1. (a) $f(x,y)=(x-2)^2+(y+1)^2$ (b) $f(x,y)=1-x^2-y^2$ (c) $f(x,y)=x+2y-5$
2. (a) $f(x,y)=1-(x+1)^2-(y-5)^2$ (b) $f(x,y)=e^{xy}$ (c) $f(x,y)=x^2-y^2$

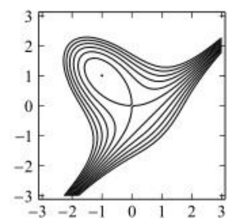
3–4 Complete the squares and locate all absolute maxima and minima, if any, by inspection. Then check your answers using calculus.

3. $f(x,y)=13-6x+x^2+4y+y^2$
4. $f(x,y)=1-2x-x^2+4y-2y^2$

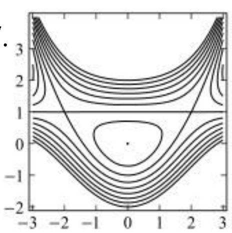
5–8 The contour plots show all significant features of the function. Make a conjecture about the number and the location of all relative extrema and saddle points, and then use calculus to check your conjecture.

5. 

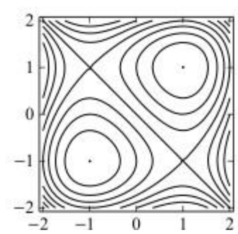
$f(x,y)=3x^2+2xy+y^2$

6. 

$f(x,y)=x^3-3xy-y^3$

7. 

$f(x,y)=x^2+2y^2-x^2y$

8. 

$f(x,y)=x^3+y^3-3x-3y$

9–20 Locate all relative maxima, relative minima, and saddle points, if any.

9. $f(x,y)=y^2+xy+3y+2x+3$
10. $f(x,y)=x^2+xy-2y-2x+1$
11. $f(x,y)=x^2+xy+y^2-3x$
12. $f(x,y)=xy-x^3-y^2$
13. $f(x,y)=x^2+y^2+\dfrac{2}{xy}$
14. $f(x,y)=xe^y$
15. $f(x,y)=x^2+y-e^y$
16. $f(x,y)=xy+\dfrac{2}{x}+\dfrac{4}{y}$
17. $f(x,y)=e^x\sin y$
18. $f(x,y)=y\sin x$
19. $f(x,y)=e^{-(x^2+y^2+2x)}$
20. $f(x,y)=xy+\dfrac{a^3}{x}+\dfrac{b^3}{y}$ $(a\neq 0, b\neq 0)$

21. (a) Show that the second partials test provides no information about the critical points of the function $f(x,y)=x^4+y^4$.

(b) Classify all critical points of f as relative maxima, relative minima, or saddle points.

22. (a) Show that the second partials test provides no information about the critical points of the function $f(x,y)=x^4-y^4$.

(b) Classify all critical points of f as relative maxima, relative minima, or saddle points.

23–28 Find the absolute extrema of the given function on the indicated closed and bounded set R.

23. $f(x,y)=xy-x-3y$; R is the triangular region with vertices (0, 0), (0, 4), and (5, 0).

24. $f(x,y)=xy-2x$; R is the triangular region with vertices (0, 0), (0, 4), and (4, 0).

25. $f(x,y)=x^2-3y^2-2x+6y$; R is the region bounded by the square with vertices (0, 0), (0, 2), (2, 2), and (2, 0).

26. $f(x,y)=xe^y-x^2-e^y$; R is the rectangular region with vertices (0, 0), (0, 1), (2, 1), and (2, 0).

27. $f(x,y)=x^2+2y^2-x$; R is the disk $x^2+y^2\le 4$.

28. $f(x,y)=xy^2$; R is the region that satisfies the inequalities $x\ge 0, y\ge 0$, and $x^2+y^2\le 1$.

29. Find three positive numbers whose sum is 48 and such that their product is as large as possible.

30. Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible.

31. Find all points on the portion of the plane $x+y+z=5$ in the first octant at which $f(x,y,z)=xy^2z^2$ has a maximum value.

32. A closed rectangular box with a volume of 16 ft^3 is made from two kinds of materials. The top and bottom are made of material costing 10c per square foot and the sides from material costing 5c per square foot. Find the dimensions of the box so that the cost of materials is minimized.

33. A manufacturer makes two models of an item, standard and deluxe. It costs $40 to manufacture the standard model and $60 for the deluxe. A market research firm estimates that if the standard model is priced at x dollars and the deluxe at y dollars, then the manufacturer will sell $500(y-x)$ of the standard items and $45,000+500(x-2y)$ of the deluxe each year. How should the items be priced to maximize the profit?

(See more exercises from: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Bevens, Stephen Davis, page 985–989)

CHAPTER 4
Multiple Integrals

4.1 Double Integrals

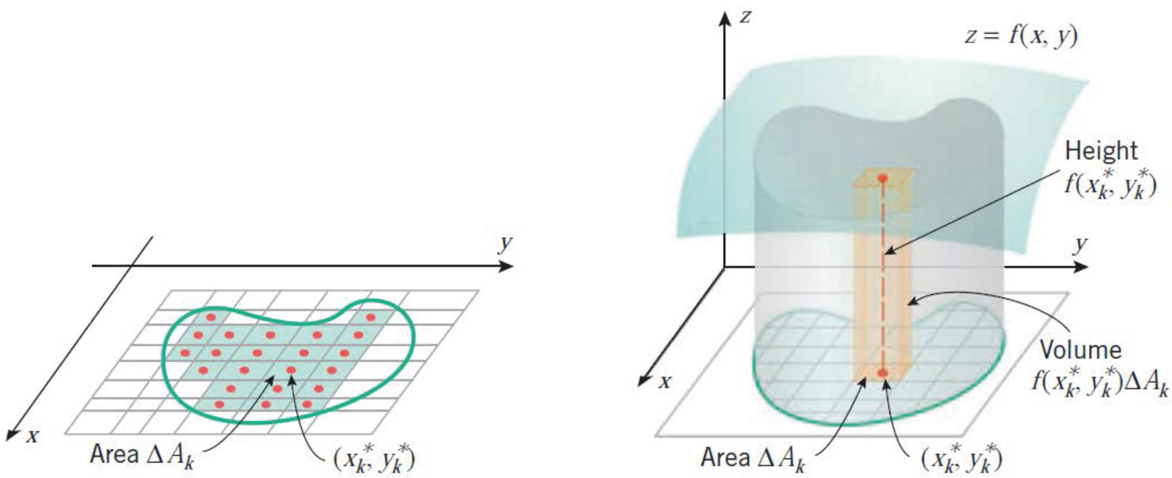
Recall that the definite integral of a function of one variable arose from the problem of nding areas under curves.

$$\int_a^b f(x)\,dx \;=\; \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k \;=\; \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k\,.$$

Integrals of functions of two variables arise from the problem of finding volumes under surfaces.

The Volume Problem

Given a function f of two variables that is continuous and nonnegative on a region R in the xy-plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R



From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1001

Definition : Volume Under a Surface

If f is a function of two variables that is *continuous and nonnegative* on a region R in the xy-plane, then the **volume of the solid enclosed between the surface** $z = f(x, y)$ **and the region** R is dened by

$$V \;=\; \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k\,.$$

Here,$n \to \infty$ indicates the process of increasing the number of subrectangles of the rectangle enclosing R in such a way that both the lengths and the widths of the subrectangles approach zero.

Definition : Double Integral

If f is a function of two variables that is continuous on a region R in the xy-plane, the inetgral

$$\iint_R f(x,y)\,dA \;=\; \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*,y_k^*)\Delta A_k\,.$$

is called the **double integral** of $f(x,y)$ over R.

Remarks. 1. If f is continuous and nonnegative on a region R, then

$$V \;=\; \iint_R f(x,y)\,dA.$$

2. If f has both positive and negative values on R, then a positive value for the double integral of f over R means that there is more volume above R than below, a negative value for the double integral means that there is more volume below R than above, and a value of zero means that the volume above R is the same as the volume below R.

Evaluating Double Integrals

Example 1. $\displaystyle\int_0^1 xy^2\,dx \;=$

$\displaystyle\int_0^1 xy^2\,dy \;=$

Iterated (or Repeated) Double Integrals

$$\int_c^d\int_a^b f(x,y)\,dx\,dy \;=\; \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$
$$\int_a^b\int_c^d f(x,y)\,dy\,dx \;=\; \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$$

Example 2. Evaluate

(a) $\displaystyle\int_1^3\int_2^4 (40-2xy)\,dy\,dx$ (b) $\displaystyle\int_1^3\int_2^4 (40-2xy)\,dy\,dx$

Fubini's Theorem

Let R be the rectangle defined by the inequalities $a \le x \le b,\ c \le y \le d$. If $f(x, y)$ is continuous on this rectangle, then

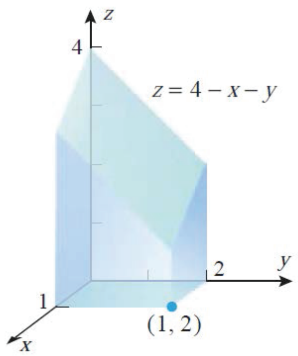
$$\iint_R f(x,y)\,dA \;=\; \int_c^d \int_a^b f(x,y)\,dx\,dy \;=\; \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Example 3. Evaluate $\iint_R y^2 x\,dA$ over the rectangle $R = \{(x, y) \mid -3 \le x \le 2,\ 0 \le y \le 1\}$

Solution.

Example 4. Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$

Solution.



From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1005

Properties of Double Integrals

1. $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$ (c is a constant)
2. $\iint_R [f(x,y) + g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$
3. $\iint_R [f(x,y) - g(x,y)]\,dA = \iint_R f(x,y)\,dA - \iint_R g(x,y)\,dA$
4. If $R = R_1 \cup R_2$, then $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

EXERCISES 4.1.1

1-12 Evaluate the iterated integrals.

1. $\displaystyle\int_0^1 \int_0^2 (x+3)\, dy\, dx$

2. $\displaystyle\int_1^3 \int_{-1}^1 (2x-4y)\, dy\, dx$

3. $\displaystyle\int_2^4 \int_0^1 x^2 y\, dx\, dy$

4. $\displaystyle\int_{-2}^0 \int_{-1}^2 (x^2+y^2)\, dx\, dy$

5. $\displaystyle\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y}\, dy\, dx$

6. $\displaystyle\int_0^2 \int_0^1 y \sin x\, dy\, dx$

7. $\displaystyle\int_{-1}^0 \int_2^5 dx\, dy$

8. $\displaystyle\int_4^6 \int_{-3}^7 dy\, dx$

9. $\displaystyle\int_0^1 \int_0^1 \frac{x}{(xy+1)^2}\, dy\, dx$

10. $\displaystyle\int_{\pi/2}^{\pi} \int_1^2 x \cos xy\, dy\, dx$

11. $\displaystyle\int_0^{\ln 2} \int_0^1 xye^{y^2 x}\, dy\, dx$

12. $\displaystyle\int_3^4 \int_1^2 \frac{1}{(x+y)^2}\, dy\, dx$

13-16 Evaluate the double integral over the rectangular region R.

13. $\displaystyle\iint_R 4xy^3\, dA;\ R = \{(x,y) \mid -1 \le x \le 1,\ -2 \le y \le 2\}$

14. $\displaystyle\iint_R \frac{xy}{x^2+y^2+1}\, dA;\ R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$

15. $\displaystyle\iint_R x\sqrt{1-x^2}\, dA;\ R = \{(x,y) \mid 0 \le x \le 1,\ 2 \le y \le 3\}$

16. $\displaystyle\iint_R (x \sin y - y \sin x)\, dA;\ R = \{(x, y) \mid 0 \le x \le \pi/2,\ 0 \le y \le \pi/3\}$

17. (a) Let $f(x, y) = x^2 + y$ and let $R = [0, 2] \times [0, 2]$ be subdivided into 16 subrectangles. Take (x_k^*, y_k^*) to be the center of the kth rectangle and approximate the double integral of f over R by the resulting Riemann sum.

    (b) Compare the result in part (a) to the exact value of the integral.

18. (a) Let $f(x, y) = x - 2y$ and let $R = [0, 2] \times [0, 2]$ be subdivided into 16 subrectangles. Take (x_k^*, y_k^*) to be the center of the kth rectangle and approximate the double integral of f over R by the resulting Riemann sum.

    (b) Compare the result in part (a) to the exact value of the integral.

19-20 Each iterated integral represents the volume of a solid. Make a sketch of the solid. Use geometry to find the volume of the solid, and then evaluate the iterated integrals.

19. $\displaystyle\int_0^5 \int_1^2 4\, dx\, dy$

20. $\displaystyle\int_0^1 \int_0^1 (2 - x - y)\, dx\, dy$

21-22 Each iterated integral represents the volume of a solid. Make a sketch of the solid. (You do not have to find the volume.

21. $\displaystyle\int_0^3 \int_0^4 \sqrt{25 - x^2 - y^2}\, dx\, dy$

22. $\displaystyle\int_{-2}^2 \int_{-2}^2 (x^2 + y^2)\, dx\, dy$

23-26 True-False Determine whether the statement is true or false. Explain your answer.

23. In the definition of a double integral

$$\iint_R f(x, y)\, dA \;=\; \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k\,.$$

    the symbol ΔA_k represents a rectangular region within R from which the point (x_k^*, y_k^*) is taken.

24. If R is the rectangle $\{(x, y) \mid 1 \le x \le 4,\ 0 \le y \le 3\}$ and $\displaystyle\int_0^3 f(x, y)\, dy = 2x$, then $\displaystyle\iint_R f(x, y)\, dA = 15.$

25. If R is the rectangle $\{(x, y) \mid 1 \le x \le 5,\ 2 \le y \le 4\}$, then

$$\iint_R f(x, y)\, dA = \int_1^5 \int_2^4 f(x, y)\, dx\, dy.$$

26. Suppose that for some region R in the xy-plane $\iint_R cf(x,y)\,dA = 0$. If R is subdivided into two region R_1 and R_2, then

$$\iint_{R_1} f(x,y)\,dA = -\iint_{R_2} f(x,y)\,dA.$$

27-30 Use a double integral to find the volume.

23. The volume under the plane $z = 2x + y$ and over the rectangle $R = \{(x,y) \mid 3 \leq x \leq 5,\ 1 \leq y \leq 2\}$.

24. The volume under the surface $z = 3x^3 + 3x^2y$ and over the rectangle $R = \{(x,y) \mid 1 \leq x \leq 3,\ 0 \leq y \leq 2\}$.

25. The volume of the solid enclosed by the surface $z = x^2$ and the planes $x = 0$, $x = 2$, $y = 3$, $y = 0$ and $z = 0$.

26. The volume in the first octant bounded by the coordinate planes, the plane $y = 4$ and the plane $(x/3) + (z/5) = 1$.

See more exercises from : Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1007-1008

Double Integrals over nonrectangular regions

Iterated Integrals with Nonconstant Limits of Integration

Later in this section we will see that double integrals over nonrectangular regions can often be evaluated as iterated integrals of the following types:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$
$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy$$

We begin with an example that illustrates how to evaluate such integrals.

Example 5. Evaluate

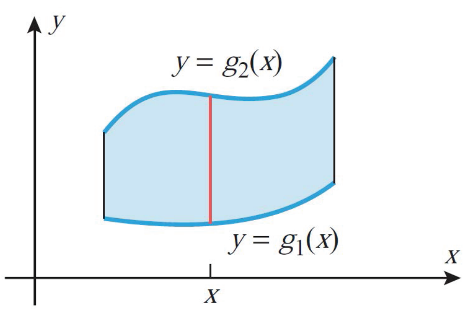
(a) $\int_0^1 \int_{-x}^{x^2} xy^2\,dy\,dx =$

(b) $\int_0^{\pi/3} \int_0^{\cos y} x \sin y\,dx\,dy =$

We will limit our study of double integrals to two basic type of regions.

Theorem.

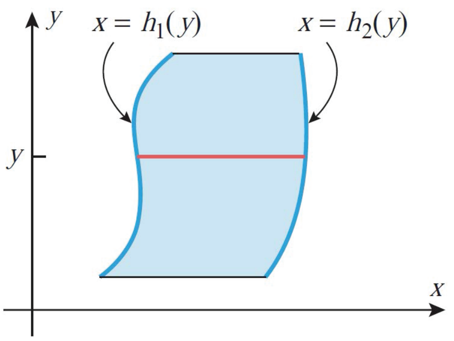
(a) If R is a region which is bounded on the left and the right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \le g_2(x)$ for $a \le x \le b$.



Suppose that $f(x,y)$ is continuous on R, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

(b) If R is a region which is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and the right by continuous curves $x = h_1(y)$ and $x = h_2(y)$, where $h_1(y) \leq h_2(y)$ for $c \leq x \leq d$.



Suppose that $f(x, y)$ is continuous on R, then

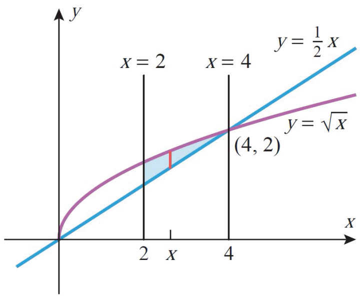
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Example 6. Each of the iterated integrals is equal to a double integral over a regio R. Identify the region R in each case.

(a) $\displaystyle\int_0^1 \int_{-x}^{x^2} xy^2\, dy\, dx$

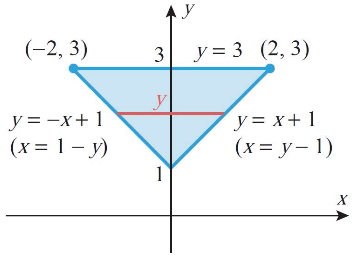
(b) $\displaystyle\int_0^{\pi/3} \int_0^{\cos y} x \sin y\, dx\, dy$

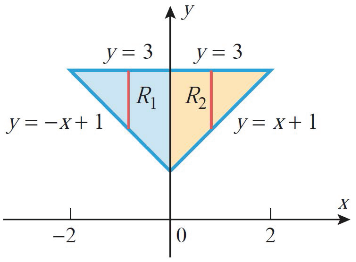
Example 7. Evaluate $\displaystyle\iint_R xy\, dA$ over the region R enclosed between $y = x/2$, $y = \sqrt{x}$, $x = 2$ and $x = 4$.



From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1011

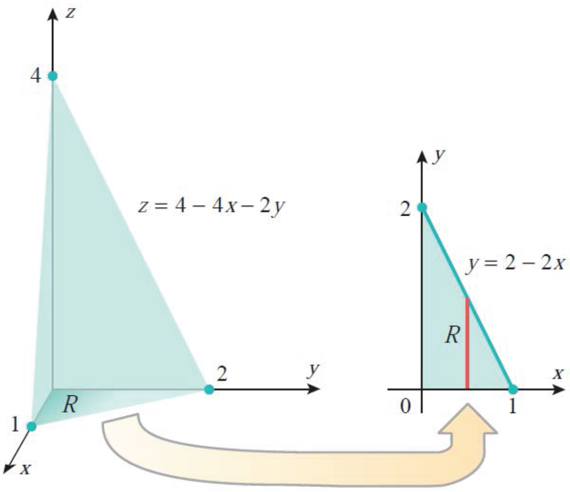
Example 8. Evaluate $\iint_R (2x - y^2)\, dA$ over the rectangle R enclosed between the lines $y = -x + 1$, $y = x + 1$ and $y = 3$.





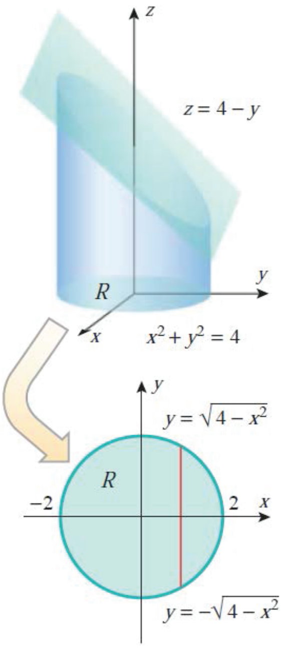
From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1012

Example 9. Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.



From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1013

Example 10. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.



From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1013

Reversing the Order of Integration

Sometimes the evaluation of an iterated integral can be simplified by reversing the order of integration. The next example illustrates how this is done.

Example 11. Since there is no elementary antiderivative of e^{x^2} , the integral

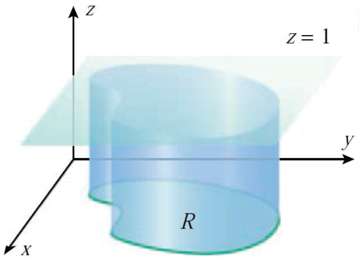
$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$$

cannot be evaluated by performing the x-integration rst. Evaluate this integral by express-ing it as an equivalent iterated integral with the order of integration reversed.

Solution.

Area calculated as a double integral

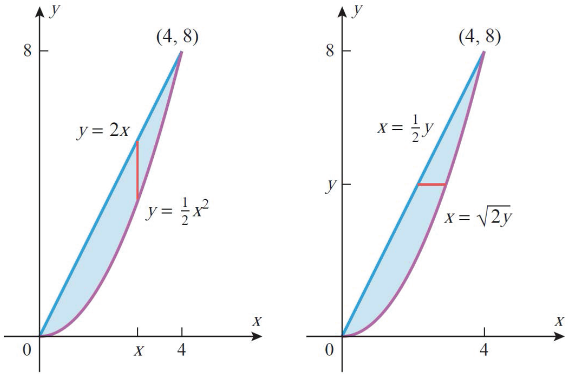
$$\text{area of } R = \iint_R 1\,dA = \iint_R dA$$



Cylinder with base R and height 1

From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1014

Example 12. Use a double integral to find the area of the region R enclosed between the parabola $y = \dfrac{x^2}{2}$ and the line $y = 2x$



From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1014

EXERCISES 4.1.2

1-8 Evaluate the iterated integrals.

1. $\displaystyle\int_0^1 \int_{x^2}^{x} xy^2 \, dy \, dx$

2. $\displaystyle\int_1^{3/2} \int_{y}^{3-y} y \, dx \, dy$

3. $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} y \, dx \, dy$

4. $\displaystyle\int_{1/4}^1 \int_{x^2}^{x} \sqrt{\frac{x}{y}} \, dy \, dx$

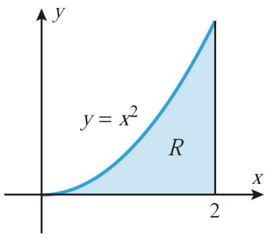
5. $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_0^{x^2} \sin\frac{y}{x} \, dy \, dx$

6. $\displaystyle\int_{-1}^1 \int_{-x^2}^{x^2} (x^2 - y) \, dy \, dx$

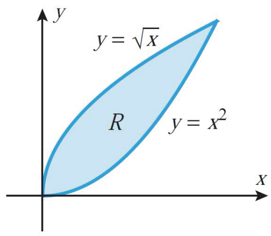
7. $\displaystyle\int_0^1 \int_0^{x} y\sqrt{x^2 - y^2} \, dy \, dx$

8. $\displaystyle\int_1^2 \int_0^{y^2} e^{x/y^2} \, dx \, dy$

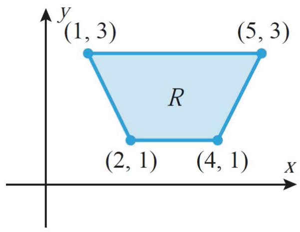
9-12 Let R be the region shown in the accompanying figure. Write $\iint_R f(x,y)\,dA$ in the form of iterated integrals.



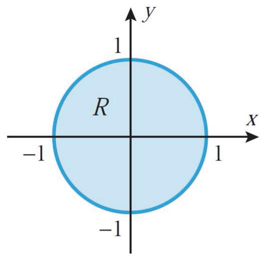
▲ Figure Ex-9



▲ Figure Ex-10



▲ Figure Ex-11



▲ Figure Ex-12

From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1015

13-16 Evaluate the double integral in two ways using iterated integrals.

13. $\iint_R x^2\,dA$; R is the region bounded by $y = 16/x$, $y = x$ and $x = 8$.

14. $\iint_R xy^2\,dA$; R is the region bounded by $y = 1$, $y = 2$, $x = 0$ and $y = x$.

15. $\iint_R (3x - 2y)\,dA$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

16. $\iint_R y\,dA$; R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 25$ and the line $x + y = 5$.

17-22 Evaluate the double integral.

17. $\iint_R x(1 + y^2)^{-1/2}\,dA$; R is the region in the first quadrant enclosed by $y = x^2$, $y = 4$ and $x = 0$.

18. $\iint_R x\cos y\,dA$; R is the triangular region bounded by the line $y = x$, $y = 0$ and $x = \pi$.

19. $\iint_R xy\,dA$; R is the region enclosed by $y = \sqrt{x}$, $y = 6 - x$ and $y = 0$.

20. $\iint_R x\,dA$; R is the region enclosed by $y = \sin^{-1} x$, $x = 1/\sqrt{2}$ and $y = 0$.

21. $\iint_R (x - 1)\,dA$; R is the region in the first quadrant enclosed between $y = x$ and $y = x^3$.

22. $\iint_R x^2\,dA$; R is the region in the first quadrant enclosed between $xy = 1$, $y = x$ and $y = 2x$.

23-25 Use double integration to nd the area of the plane region enclosed by the given curves.

23. $y = \sin x$ and $y = \cos x$, for $0 \leq x \leq \pi/4$

24. $y^2 = -x$ and $3y - x = 4$

25. $y^2 = 9 - x$ and $y^2 = 9 - 9x$

26-29 True or False Determine whether the statement is true or false. Explain your answer.

26. $\displaystyle\int_0^1 \int_{x^2}^{2x} f(x,y)\, dy\, dx = \int_{x^2}^{2x} \int_0^1 f(x,y)\, dx\, dy.$

27. If a region R is bounded below by $y = g_1(x)$ and above by $y = g_2(x)$ for $a \leq x \leq b$, then

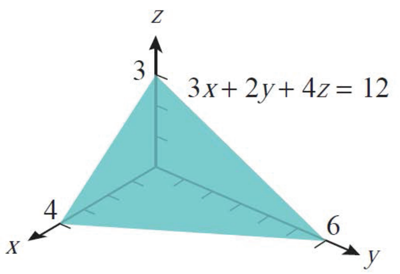
$$\iint_R f(x,y)\, dA \;=\; \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$$

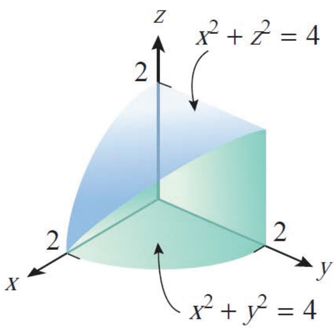
28. If a region R is the region in the xy-plane enclosed by $y = x^2$ and $y = 1$, then

$$\iint_R f(x,y)\, dA \;=\; 2\int_0^1 \int_{x^2}^1 f(x,y)\, dy\, dx.$$

29. The area of a region R in the xy-plane is given by $\displaystyle\iint_R xy\, dA.$

30-31 Use double integration to nd the volume of the solid.





From: Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1016

32-37 Use double integration to nd the volume of each solid.

32. The solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 3x$.

33. The solid in the rst octant bounded above by the paraboloid $z = x^2 + 3y^2$, below by the plane $z = 0$, and laterally by $y = x^2$ and $y = x$.

34. The solid bounded above by the paraboloid $z = 9x^2 + y^2$, below by the plane $z = 0$, and laterally by the planes $x = 0$, $y = 0$, $x = 3$, and $y = 2$.

35. The solid enclosed by $y^2 = x$, $z = 0$, and $x + z = 1$.

36. The wedge cut from the cylinder $4x^2 + y^2 = 9$ by the planes $z = 0$ and $z = y + 3$.

37. The solid in the rst octant bounded above by $z = 9x^2$, below by $z = 0$, and laterally by $y^2 = 3x$.

38-43 Express the integral as an equivalent integral with the order of integration reversed.

38. $\int_0^2 \int_0^{\sqrt{x}} f(x, y)\, dy\, dx$

39. $\int_0^4 \int_{2y}^{8} f(x, y)\, dx\, dy$

40. $\int_0^2 \int_1^{e^y} f(x, y)\, dx\, dy$

41. $\int_1^e \int_0^{\ln x} f(x, y)\, dy\, dx$

42. $\int_0^1 \int_{sin^{-1} y}^{\pi/2} f(x, y)\, dx\, dy$

43. $\int_0^1 \int_{y^2}^{\sqrt{y}} f(x, y)\, dx\, dy$

44-47 Evaluate the integral by first reversing the order of integration.

44. $\int_0^1 \int_{4x}^{4} e^{-y^2}\, dy\, dx$

45. $\int_0^2 \int_{y/2}^{1} \cos(x^2)\, dx\, dy$

46. $\int_0^4 \int_{\sqrt{y}}^{2} e^{x^3}\, dx\, dy$

47. $\int_1^3 \int_0^{\ln x} x\, dy\, dx$

See more exercises from : Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1015-1017

4.2 Double Integrals in Polar Coordinates

Simple Polar Regions

Some double integrals are easier to evaluate if the region of integration is expressed in polar coordinates. This is usually true if the region is bounded by a cardioid, a rose curve, a spiral, or, more generally, by any curve whose equation is simpler in polar coordinates than in rectangular coordinates. For example, the quarter-disk is described in rectangular coordinates by

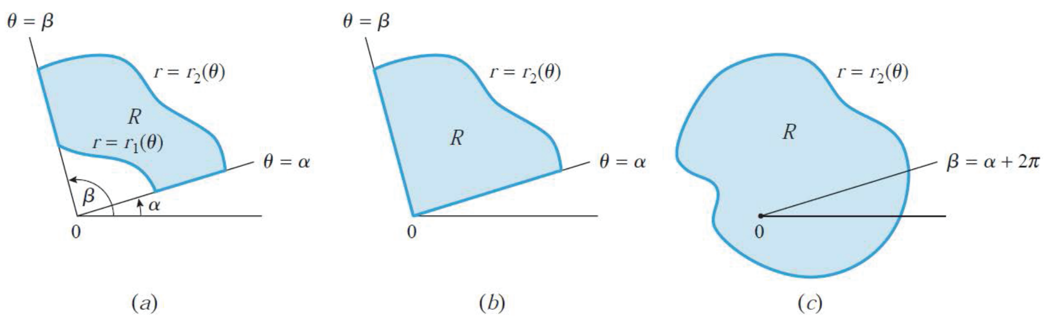
$$0 \leq y \leq \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$$

However, in polar coordinates the region is described more simply

$$0 \leq r \leq 2, \quad 0 \leq \theta \leq \pi/2$$

A *simple polar region* in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves,$r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

(i) $\alpha \leq \beta$ (ii) $\beta - \alpha \leq 2\pi$ (iii) $0 \leq r_1(\theta) \leq r_2(\theta)$.

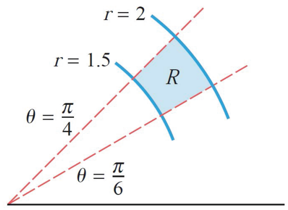


Simple polar regions

From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1018

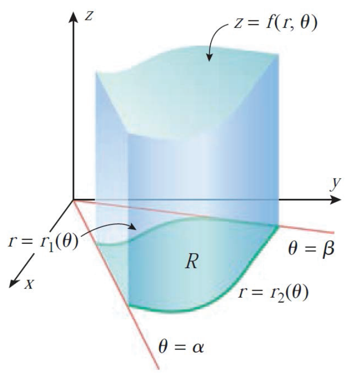
A *polar rectangle* is a simple polar region for which the bounding polar curves are circular arcs. For example, in below figure shows the polar rectangle R given by

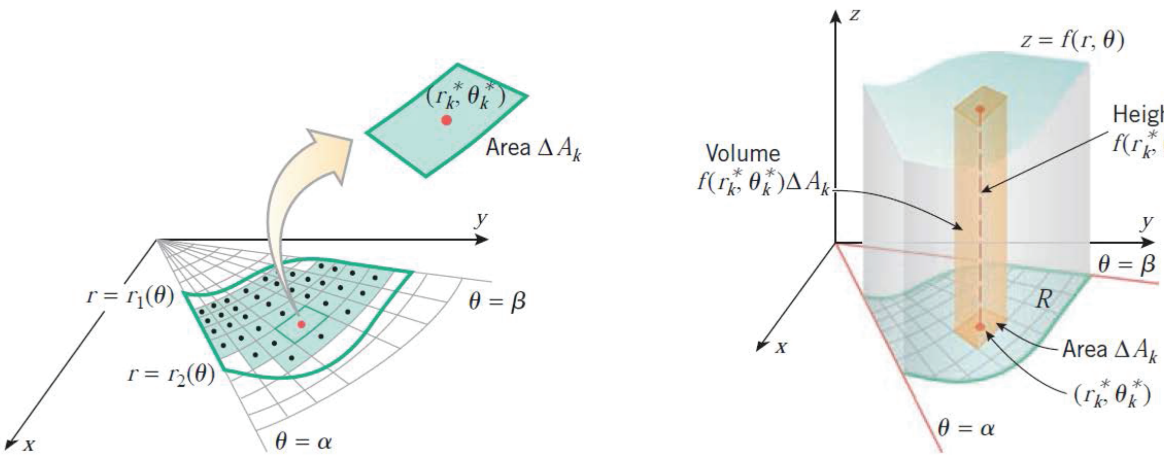
$$1.5 \le r \le 2, \quad \pi/6 \le \theta \le \pi/4.$$



The Volume Problem in Polar Coordinates

Given a function $f(r,\theta)$ that is continuous and nonnegative on a simple polar region R, find the volume of the solid that is enclosed between the region R and the surface whose equation in cylindrical coordinates is $z = f(r,\theta)$.





From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1019-1020

The volume of the solid is

$$V = \lim_{n\to+\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k.$$

If $f(r,\theta)$ is continuous on R and has both positive and negative values, then the limit

$$\lim_{n\to+\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k$$

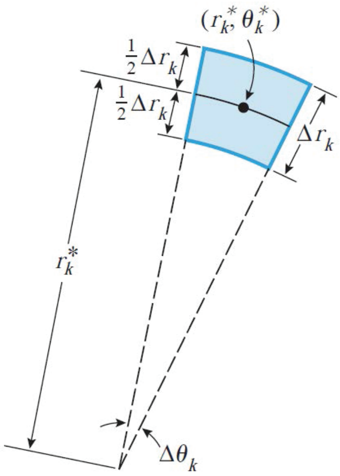
represents the net signed volume between the region R and the surface $z = f(r, \theta)$ (as with double integrals in rectangular coordinates). These sums are called *polar Riemann sums*, and the limit of the polar Riemann sums is denoted by

$$\iint_R f(r,\theta)\, dA \;=\; \lim_{n\to+\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k$$

which is called the *polar double integral* of $f(r, \theta)$ over R. If $f(r, \theta)$ is continuous and nonnegative on R, then the volume formula can be expressed as

$$V \;=\; \iint_R f(r,\theta)\, dA$$

Evaluating Polar Double Integrals



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1020

Treating the area ΔA_k of this polar rectangle as the difference in area of two sectors, we obtain

$$\Delta A_k \;=\; \frac{1}{2}(r_k^* + \frac{1}{2}\Delta r_k)^2 \Delta\theta_k - \frac{1}{2}(r_k^* - \frac{1}{2}\Delta r_k)^2 \Delta\theta_k$$

which simplifies to

$$\Delta A_k \;=\; r_k^* \Delta r_k \Delta\theta_k\,.$$

Thus

$$V \;=\; \iint_R f(r,\theta)\, dA \;=\; \lim_{n\to+\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) r_k^* \Delta r_k \Delta\theta_k$$

which suggests that the volume V can be expressed as the iterated integral

$$V \;=\; \iint_R f(r,\theta)\, dA \;=\; \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta\,.$$

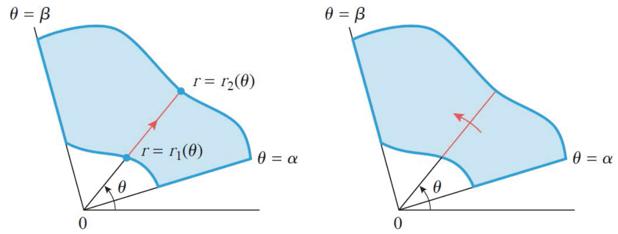
Theorem. If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ and if $f(r, \theta)$ is continuous on R, then

$$\iint_R f(r,\theta)\, dA \;=\; \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta\,.$$

Determining Limits of Integration for a Polar Double Integral: Simple Polar Region

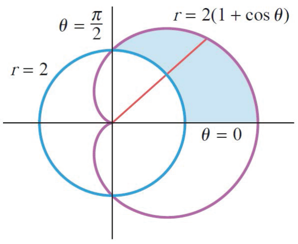
Step 1. Since θ is held fixed for the first integration, draw a radial line from the origin through the region R at a xed angle θ. This line crosses the boundary of R at most twice. The innermost point of intersection is on the inner boundary curve $r = r_1(\theta)$ and the outermost point is on the outer boundary curve $r = r_2(\theta)$. These intersections determine the r-limits of integration.

Step 2. Imagine rotating the radial line from Step 1 about the origin, thus sweeping out the region R. The least angle at which the radial line intersects the region R is $\theta = \alpha$ and the greatest angle is $\theta = \beta$. This determines the θ-limits of integration.



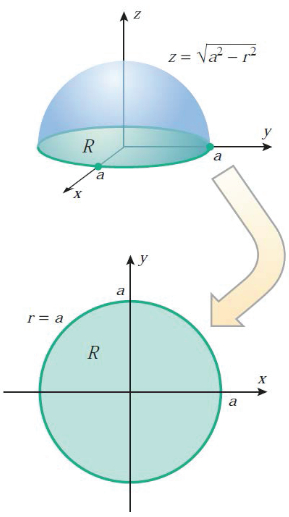
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1021

Example 9. Evaluate $\displaystyle\iint_R \sin\theta\, dA$ where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$.



From:Calculus Early Transcendentals,
10th edition,Howard Anton,Irl C. Beven,
Stephen Deavis, page 1022

Example 10. The sphere of radius a centered at the origin is expressed in rectangu-lar coordinates as $x^2 + y^2 + z^2 = a^2$, and hence its equation in cylindrical coordinates is $r^2 + z^2 = a^2$. Use this equation and a polar double integral to nd the volume of the sphere.



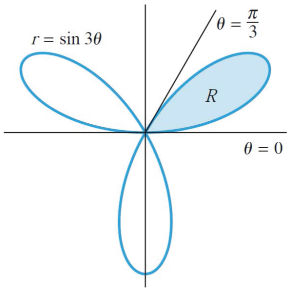
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1022

Finding Areas Using Polar Double Integrals

Recall that the area of a region R in the xy-plane can be expressed as

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA\,.$$

Example 11. Use a polar double integral to find the area enclosed by the three-petaled rose $r = \sin 3\theta$.



From:Calculus Early Transcendentals,
10th edition,Howard Anton,Irl C. Beven,
Stephen Deavis, page 1023

Converting Double Integrals from Rectangle to Polar Coordinates Sometimes a double integral that is difficult to evaluate in rectangular coordinates can be evaluated more easily in polar coordinates by making the substitution $x = r\cos\theta,\ y = r\sin\theta$ and expressing the region of integration in polar form; that is, we rewrite the double integral in rectangular coordinates as

$$\iint_R f(x,y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, dA = \iint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta)\ r\, dr\, d\theta\,.$$

Example 12. Use polar coordinates to evaluate $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$

EXERCISES 4.2.2

1. The polar region inside the circle $r = 2\sin\theta$ and outside the circle $r = 1$ is a simple polar region given by the inequalities

$$\ldots\ldots\ldots\ldots\ldots \leq r \leq \ldots\ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots \leq \theta \leq \ldots\ldots\ldots\ldots\ldots$$

2. Let R be the region in the rst quadrant enclosed between the circles $x^2 + y^2 = 9$ and $x^2 + y^2 = 100$. Supply the missing limits of integration.

$$\iint_R f(r,\theta)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(r,\theta)\, r\, dr\, d\theta\,.$$

3. Let V be the volume of the solid bounded above by the hemisphere $z = \sqrt{1-r^2}$ and bounded below by the disk enclosed within the circle $r = \sin\theta$. Expressed as a double integral in polar coordinates,

$$V = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

4. Express the iterated integral as a double integral in polar coordinates.

$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} \left(\frac{1}{x^2+y^2}\right) dy\, dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

5-10 Evaluate the iterated integral.

5. $\displaystyle\int_0^{\pi/2} \int_0^{\sin\theta} r\cos\theta\, dr\, d\theta$

6. $\displaystyle\int_0^{\pi} \int_0^{1+\cos\theta} r\, dr\, d\theta$

7. $\displaystyle\int_0^{\pi/2} \int_0^{a\sin\theta} r^2\, dr\, d\theta$

8. $\displaystyle\int_0^{\pi/6} \int_0^{\cos 3\theta} r\, dr\, d\theta$

9. $\displaystyle\int_0^{\pi} \int_0^{1-\sin\theta} r^2\cos\theta\, dr\, d\theta$

10. $\displaystyle\int_0^{\pi/2} \int_0^{\cos\theta} r^3\, dr\, d\theta$

11-14 Use a double integral in polar coordinates to nd the area of the region described.

11. The region enclosed by the cardioid $r = 1 - \cos\theta$.

12. The region enclosed by the rose $r = \sin 2\theta$

13. The region in the first quadrant bounded by $r = 1$ and $r = \sin 2\theta$, with $\pi/4 \leq \theta \leq \pi/2$.

14. The region inside the circle $x^2 + y^2 = 4$ and to the right of the line $x = 1$.

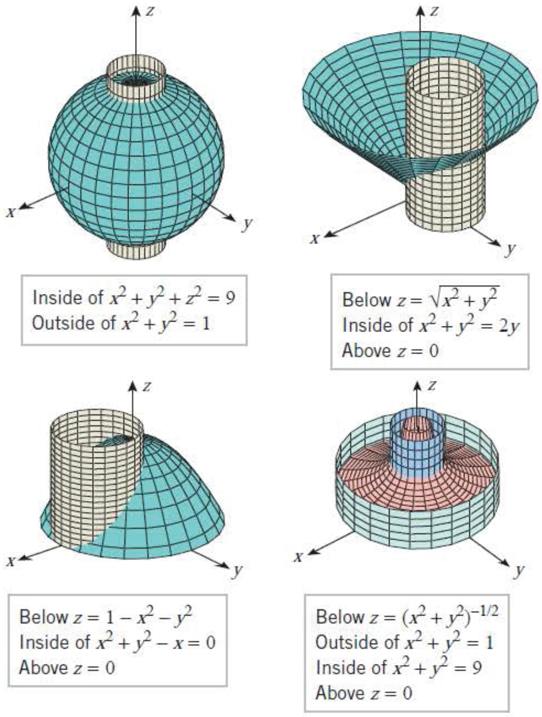
15-16 Let R be the region described. Sketch the region R and fill in the missing limits of integration.

$$\iint_R f(r,\theta)\,dA = \int_{\square}^{\square}\int_{\square}^{\square} f(r,\theta)\,r\,dr\,d\theta\,.$$

15. The region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$.

16. The region inside the circle $r = 1$ and outside the cardioid $r = 1 + \cos\theta$.

17. Express the volume of the solid described as a double integral in polar coordinates.



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1024

18. Find the volume of the solid in the rst octant bounded above by the surface $z = r\sin\theta$, below by the xy-plane, and laterally by the plane $x = 0$ and the surface $r = 3\sin\theta$.

19. Find the volume of the solid inside the surface $r^2 + z^2 = 4$ and outside the surface $r = 2\cos\theta$.

20-23 Use polar coordinates to evaluate the double integral.

20. $\displaystyle\iint_R \sin(x^2 + y^2)\,dA$ where R is the region enclosed by the circle $x^2 + y^2 = 9$.

21. $\displaystyle\iint_R \sqrt{9 - x^2 - y^2}\,dA$ where R is the region in the rst quadrant within the circle $x^2 + y^2 = 9$.

22. $\displaystyle\iint_R \frac{1}{1+x^2+y^2}\, dA$ where R is the sector in the rst quadrant bounded by $y = 0$, $y = x$, and $x^2 + y^2 = 4$.

23. $\displaystyle\iint_R 2y\, dA$ where R is the region in the first quadrant bounded above by the circle $(x-1)^2 + y^2 = 1$ and below by the line $y = x$.

24-31 Evaluate the iterated integral by converting to polar co-ordinates.

24. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$

25. $\displaystyle\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\, dx\, dy$

26. $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

27. $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2+y^2)\, dx\, dy$

28. $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dy\, dx}{(1+x^2+y^2)^{3/2}} \qquad (a > 0)$

29. $\displaystyle\int_0^1 \int_y^{\sqrt{y}} \sqrt{x^2+y^2}\, dx\, dy$

30. $\displaystyle\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{}{\sqrt{1+x^2+y^2}}\, dy\, dx$

31. $\displaystyle\int_{-4}^0 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 3x\, dy\, dx$

32-35 True-False Determine whether the statement is true or false. Explain your answer.

32. The disk of radius 2 that is centered at the origin is a polar rectangle.

33. If f is continuous and nonnegative on a simple polar region R, then the volume of the solid enclosed between R and the surface $z = f(r, \theta)$ is expressed as

$$\iint_R f(r,\theta) r\, dA.$$

34. If R is the region in the rst quadrant between the circles $r = 1$ and $r = 2$, and if f is continuous on R, then

$$\iint_R f(r,\theta)\, dA = \int_0^{\pi/2} \int_1^2 f(r,\theta)\, dr\, d\theta\,.$$

35. The area enclosed by the circle $r = \sin\theta$ is given by

$$A = \int_0^{2\pi} \int_0^{\sin\theta} r \, dr \, d\theta \, .$$

36. Find the area of the region enclosed by the lemniscate $r^2 = 2a^2 \cos 2\theta$.

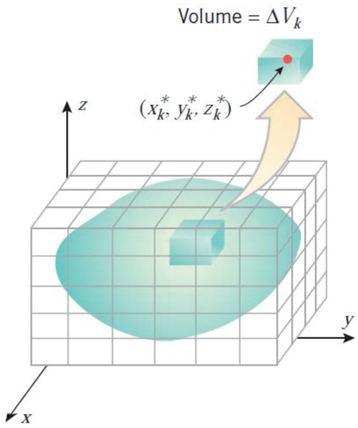
37. Find the area in the first quadrant that is inside the circle $r = 4\sin\theta$ and outside the lemniscate $r^2 = 8\cos 2\theta$.

See more exercises from : Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1024-1025

4.3 Triple Integrals in Rectangular Coordinates

Definition of a Triple Integral

To define the triple integral of $f(x, y, z)$ over G, we rst divide the box into n subboxes by planes parallel to the coordinate planes. We then discard those subboxes that contain any points outside of G and choose an arbitrary point in each of the remaining subboxes. We denote the volume of the kth remaining subbox by ΔV_k and the point selected in the kth subbox by (x_k^*, y_k^*, z_k^*).



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1039

Next we form the product

$$f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

for each subbox, then add the products for all of the subboxes to obtain the *Riemann sum*

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k\,.$$

Finally, we repeat this process with more and more subdivisions in such a way that the length, width, and height of each subbox approach zero, and n approaches $+\infty$. The limit

$$\iiint_G f(x, y, z)\,dV \;=\; \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k\,.$$

Properties of Triple Integrals

Triple integrals enjoy many properties of single and double integrals:

$$\iiint_G cf(x, y, z)\,dV \;=\; c\iiint_G f(x, y, z)\,dV \qquad (c \text{ a constant})$$

$$\iiint_G [f(x, y, z) + g(x, y, z])\,dV \;=\; \iiint_G f(x, y, z)\,dV + \iiint_G g(x, y, z)\,dV$$

$$\iiint_G [f(x, y, z) - g(x, y, z])\,dV \;=\; \iiint_G f(x, y, z)\,dV - \iiint_G g(x, y, z)\,dV$$

Moreover, if the region G is subdivided into two subregions G_1 and G_2 , then

$$\iiint_G f(x, y, z)\,dV \;=\; \iiint_{G_1} f(x, y, z)\,dV + \iiint_{G_2} f(x, y, z)\,dV$$

Evaluating Triple Integrals over Rectangular Boxes

Just as a double integral can be evaluated by two successive single integrations, so a triple integral can be evaluated by three successive integrations.

Fubini's Theorem Let G be the rectangular box defined by the inequalities

$$a \le x \le b\,, \quad c \le y \le d\,, \quad k \le z \le l\,.$$

If f is continuous on the region G, then

$$\iiint_G f(x,y,z)\,dV \;=\; \int_a^b \int_c^d \int_k^l f(x,y,z)\,dz\,dy\,dx\,.$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration:

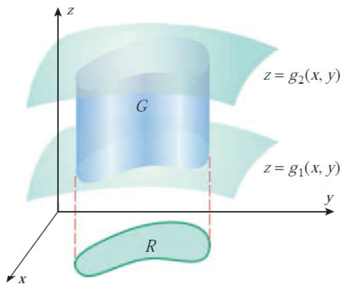
$$dydzdx,\ dzdxdy,\ dxdzdy,\ dzdydx,\ dydxdz$$

Example 1. Evaluate the triple integral $\iiint_G 12xy^2z^3\,dV$ over the rectangular box G dened by the inequalities $-1 \le x \le 2$, $0 \le y \le 3$, $0 \le z \le 2$.

Evaluating Triple Integrals over More general Regions

Theorem. Let G be a simple xy-solid with upper surface $z = g_2(x,y)$ and lower surface $z = g_1(x,y)$, and let R be the projection of G on the xy-plane. If $f(x,y,z)$ is continuous on G, then

$$\iiint_G f(x,y,z)\,dV \;=\; \iint_R \Big[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\Big]\,dA\,.$$



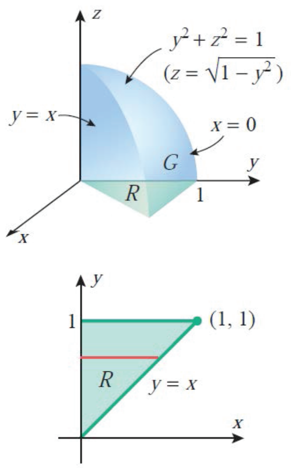
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1041

Determining Limits of Integration: Simple xy-Solid

Step 1. Find an equation $z = g_2(x, y)$ for the upper surface and an equation $z = g_1(x, y)$ for the lower surface of G. The functions $g_1(x, y)$ and $g_2(x, y)$ determine the lower and upper z-limits of integration.

Step 2. Make a two-dimensional sketch of the projection R of the solid on the xy-plane. From this sketch determine the limits of integration for the double integral over R

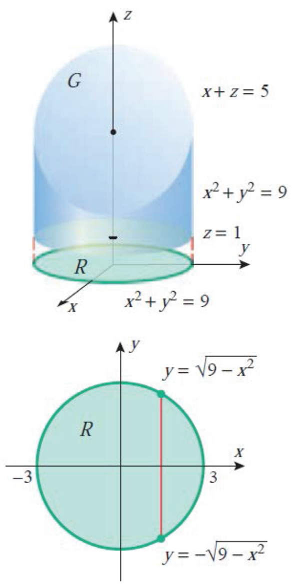
Example 2. Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \leq 1$ by the planes $y = x$ and $x = 0$. Evaluate $\iiint\limits_G z\, dV$



From:Calculus Early Transcendentals,
10th edition,Howard Anton,Irl C. Beven,
Stephen Deavis, page 1042

Volume calculated as a triple Integral : volume of $G = \iiint\limits_G dV$

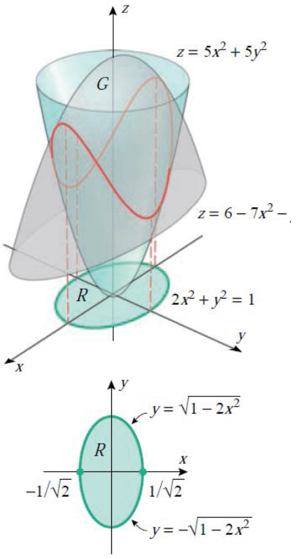
Example 3. Use a triple integral to nd the volume of the solid within the cylinder $x^2+y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1043

Example 4. Find the volume of the solid enclosed between the paraboloids

$$z = 5x^2 + 5y^2 \quad \text{and} \quad z = 6 - 7x^2 - y^2$$



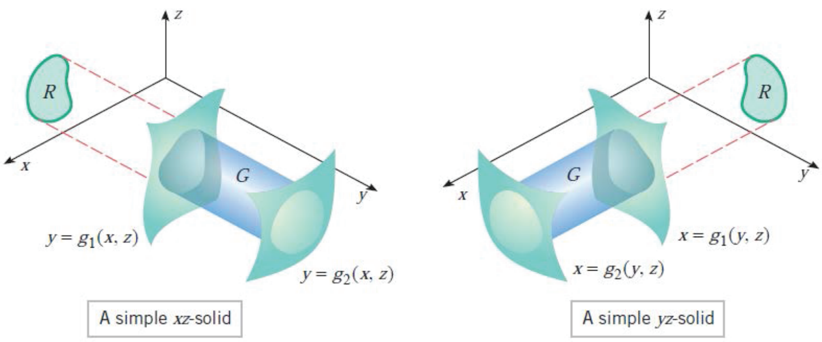
From:Calculus Early Transcendentals,
10th edition,Howard Anton,Irl C. Beven,
Stephen Deavis, page 1043

Integration in Other Orders

For integrating over a simple xy-solid, the z-integration was performed first. However, there are situations in which it is preferable to integrate in a different order. For a simple xz-solid it is usually best to integrate with respect to y first, and for a simple yz-solid it is usually best to integrate with respect to x first:

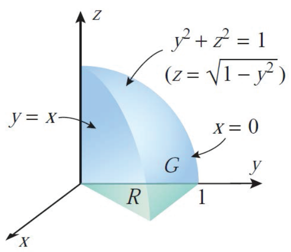
$$\underset{\substack{G \\ \text{simple } xz\text{-solid}}}{\iiint} f(x,y,z)\,dV = \underset{R}{\iint}\left[\int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z)\,dy\right] dA$$

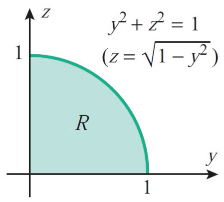
$$\underset{\substack{G \\ \text{simple } xz\text{-solid}}}{\iiint} f(x,y,z)\,dV = \underset{R}{\iint}\left[\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\,dx\right] dA.$$



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1043

Example 5. Evaluate this integral $\iiint\limits_G z\,dV$ in Example 2 by integrating first with respect to x.





From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1045

EXERCISES 4.3

1. The iterated integral $\int_1^5 \int_2^4 \int_3^6 f(x,y,z)\,dx\,dz\,dy$ integrates f over the rectangular box defined by

$$\ldots\ldots\ldots \le x \le \ldots\ldots\ldots, \quad \ldots\ldots\ldots \le y \le \ldots\ldots\ldots, \quad \ldots\ldots\ldots \le z \le \ldots\ldots\ldots$$

2. Let G be the solid in the first octant bounded below by the surface $z = y + x^2$ and bounded above by the plane $z = 4$. Supply the missing limits of integration.

$$(a)\ \iiint\limits_G f(x,y,z)\,dV = \int_\square^\square \int_\square^\square \int_{y+x^2}^4 f(x,y,z)\,dz\,dx\,dy$$

$$(b)\ \iiint\limits_G f(x,y,z)\,dV = \int_\square^\square \int_\square^\square \int_{y+x^2}^4 f(x,y,z)\,dz\,dy\,dx$$

$$(c)\ \iiint\limits_G f(x,y,z)\,dV = \int_\square^\square \int_\square^\square \int_\square^\square f(x,y,z)\,dy\,dz\,dx$$

3-10 Evaluate the iterated integral.

3. $\int_{-1}^1 \int_0^2 \int_0^1 (x^2 + y^2 + z^2)\,dx\,dy\,dz$

4. $\int_{1/3}^{1/2} \int_0^\pi \int_0^1 zx \sin xy\,dz\,dy\,dx$

5. $\int_0^2 \int_{-1}^{y^2} \int_{-1}^z yz\,dx\,dz\,dy$

6. $\displaystyle\int_0^{\pi/4}\int_0^1\int_0^{x^2} x\cos y\,dz\,dx\,dy$

7. $\displaystyle\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^x xy\,dy\,dx\,dz$

8. $\displaystyle\int_1^3\int_x^{x^2}\int_0^{\ln z} xe^y\,dy\,dz\,dx$

9. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_{-5+x^2+y^2}^{3-x^2-y^2} x\,dz\,dy\,dx$

10. $\displaystyle\int_1^2\int_z^2\int_0^{\sqrt{3}y} \frac{y}{x^2+y^2}\,dx\,dy\,dz$

11-14 Evaluate the iterated integral.

11. $\displaystyle\iiint_G xy\sin yz\,dV$, where G is the rectangular box dened by the inequalities $0 \leq x \leq \pi$, $0 \leq y \leq 1$, $0 \leq z \leq \pi/6$.

12. $\displaystyle\iiint_G y\,dV$, where G is the solid enclosed by the plane $z = y$, the xy-plane, and the parabolic cylinder $y = 1 - x^2$.

13. $\displaystyle\iiint_G xyz\,dV$, where G is the solid in the first octant that bounded by the parabolic cylinder $z = 2 - x^2$ and the planes $z = 0$, $y = x$, and $y = 0$.

14. $\displaystyle\iiint_G \cos(z/y)\,dV$, where G is the solid defined by the inequalities $\pi/6 \leq y \leq \pi/2$, $y \leq x \leq \pi/2$, $0 \leq z \leq xy$.

15-18 Use a triple integral to nd the volume of the solid.

15. The solid in the first octant bounded by the coordinate planes and the plane $3x+6y+4z = 12$.

16. The solid bounded by the surface $z = \sqrt{y}$ and the plane $x + y = 1,\ x = 0$, and $z = 0$.

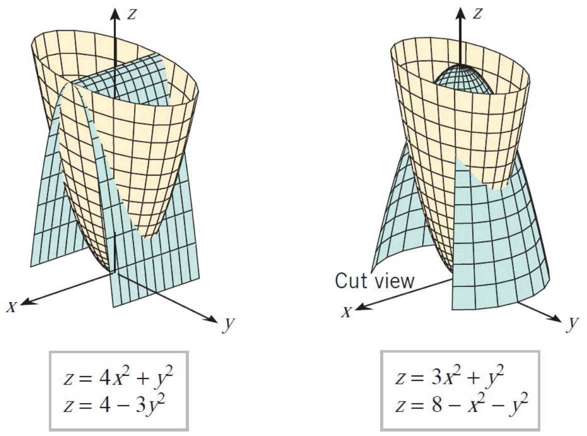
17. The solid bounded by the surface $y = x^2$ and the planes $y + z = 4$ and $z = 0$.

18. The wedge in the first octant that is cut from the solid cylinder $y^2 + z^2 1$ by the planes $y = x$ and $x = 0$.

19. Let G be the solid enclosed by the surfaces in the accompanying figure. Fill in the missing limits of integration.

$$\iiint\limits_G f(x,y,z)\,dV = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} f(x,y,z)\,dz\,dx\,dy$$



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1046

20-22 Set up (but do not evaluate) an iterated triple integral for the volume of the solid enclosed between the given surfaces.

20. The surfaces in Exercise 19

21. The elliptic cylinder $x^2 + 9y^2 = 9$ and the planes $z = 0$ and $z = x + 3$.

22. The cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

23-28 Sketch the solid whose volume is given by the integral.

23. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{y+1} dz\,dy\,dx$

24. $\displaystyle\int_{0}^{9}\int_{0}^{y/3}\int_{0}^{\sqrt{y^2-9x^2}} dz\,dx\,dy$

25. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{2} dy\,dz\,dx$

26. $\displaystyle\int_{0}^{3}\int_{x^2}^{9}\int_{0}^{2} dz\,dy\,dx$

27. $\displaystyle\int_{0}^{2}\int_{0}^{2-y}\int_{0}^{2-x-y} dz\,dx\,dy$

28. $\int_{-2}^{2}\int_{0}^{4-y^2}\int_{0}^{2} dx\,dz\,dy$

29-34 Express each integral as an equivalent integral in which the z-integration is performed rst, the y-integration second, and the x-integration last.

23. $\int_{0}^{5}\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} f(x,y,z)\,dx\,dy\,dz$

24. $\int_{0}^{9}\int_{0}^{3-\sqrt{x}}\int_{0}^{z} f(x,y,z)\,dy\,dz\,dx$

25. $\int_{0}^{4}\int_{y}^{8-y}\int_{0}^{\sqrt{4-y}} f(x,y,z)\,dx\,dz\,dy$

26. $\int_{0}^{3}\int_{0}^{\sqrt{9-z^2}}\int_{0}^{\sqrt{9-y^2-z^2}} f(x,y,z)\,dx\,dy\,dz$

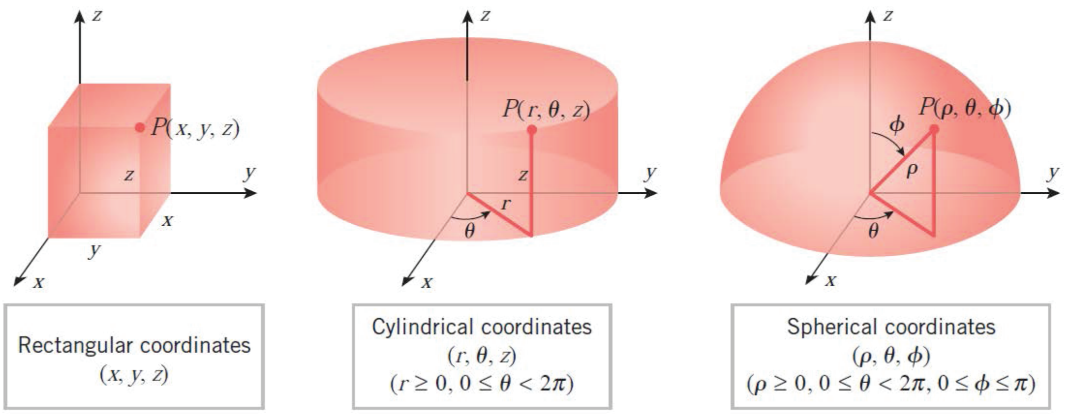
27. $\int_{0}^{4}\int_{0}^{2}\int_{0}^{x/2} f(x,y,z)\,dy\,dz\,dx$

28. $\int_{0}^{4}\int_{0}^{4-y}\int_{0}^{\sqrt{z}} f(x,y,z)\,dx\,dz\,dy$

See more exercises from : Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1045-1047

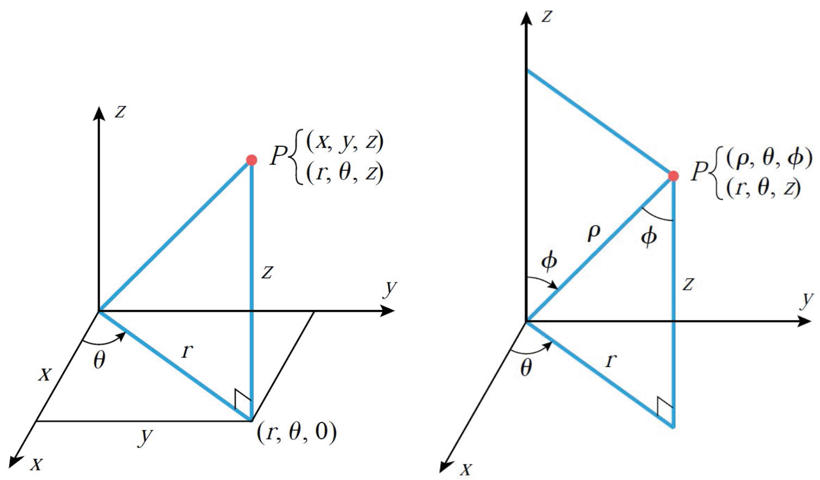
4.4 Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical and Spherical Coordinate Systems



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 832

Converting Coordinates



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 834

Cylindrical to Rectangular

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Rectangular to Cylindrical

$$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}, \quad z = z$$

Spherical to Rectangular

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

Rectangular to Spherical

$$\rho = \sqrt{x^2+y^2+z^2}, \quad \tan\theta = \frac{y}{x}, \quad \cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

Spherical to Cylindrical

$$r = \rho\sin\phi, \quad \theta = \theta, \quad z = \rho\cos\phi$$

Cylindrical to Spherical

$$\rho = \sqrt{r^2+z^2}, \quad \theta = \theta, \quad \tan\phi = \frac{r}{z}$$

Constant surfaces

In **rectangular coordinates**, the surfaces represented by equations of the form

$$x = x_0, \quad y = y_0, \ Vz = z_0$$

where x_0, y_0, and z_0 are constants, are planes parallel to the yz-plane, xz-plane, and xy-plane, respectively.

In **cylindrical coordinates**, the surfaces represented by equations of the form

$$r = r_0, \quad \theta = \theta_0, \quad z = z_0$$

where r_0, θ_0, and z_0 are constants,

- The surface $r = r_0$ is a right circular cylinder of radius r_0 centered on the z-axis.

- The surface $\theta = \theta_0$ is a half-plane attached along the z-axis and making an angle θ_0 with the positive x-axis.

- The surface $z = z_0$ is a horizontal plane.

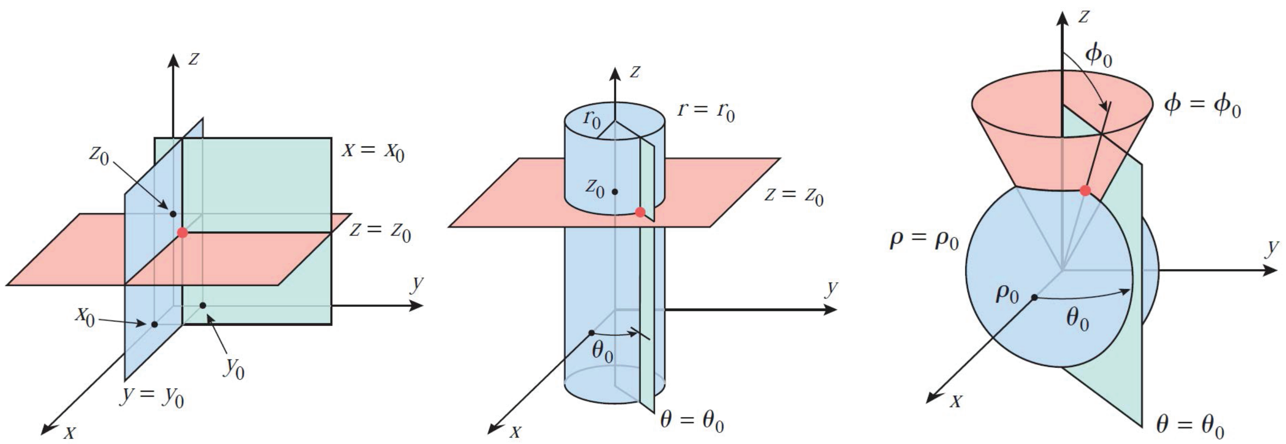
In **spherical coordinates**, the surfaces represented by equations of the form

$$\rho = \rho_0, \quad \theta = \theta_0, \quad \phi = \phi_0$$

- The surface $\rho = \rho_0$ consists of all points whose distance ρ from the origin is ρ_0. Assuming ρ to be nonnegative, this is a sphere of radius ρ centered at the origin.

- As in cylindrical coordinates, the surface $\theta = \theta_0$ is a half-plane attached along the z-axis, making an angle of θ_0 with the positive x-axis.

- The surface $\phi = \phi_0$ consists of all points from which a line segment to the origin makes an angle of ϕ_0 with the positive z-axis. If $0 < \phi_0 < \pi/2$, this will be the nappe of a cone opening up, while if $\pi/2 < \phi_0 < \phi$, this will be the nappe of a cone opening down. (If $\phi_0 = \pi/2$, then the cone is flat, and the surface is the xy-plane.)

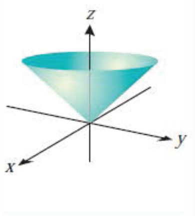
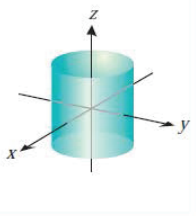
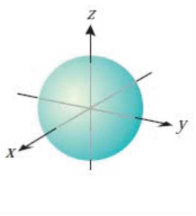
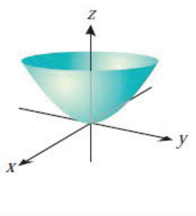
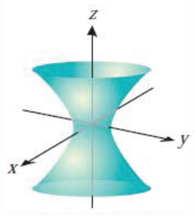


From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 832-833

Equations of Surfaces in Cylindrical and Spherical Coordinates

Surfaces of revolution about the z-axis of a rectangular coordinate system usually have simpler equations in cylindrical coordinates than in rectangular coordinates.

Example 6. Find equations of the paraboloid $z = \sqrt{x^2 + y^2}$ in cylindrical and spherical coordinates.

	CONE	CYLINDER	SPHERE	PARABOLOID	HYPERBOLOID
					
RECTANGULAR	$z = \sqrt{x^2 + y^2}$	$x^2 + y^2 = 1$	$x^2 + y^2 + z^2 = 1$	$z = x^2 + y^2$	$x^2 + y^2 - z^2 = 1$
CYLINDRICAL	$z = r$	$r = 1$	$z^2 = 1 - r^2$	$z = r^2$	$z^2 = r^2 - 1$
SPHERICAL	$\phi = \pi/4$	$\rho = \csc\phi$	$\rho = 1$	$\rho = \cos\phi \csc^2\phi$	$\rho^2 = -\sec 2\phi$

From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 835

Triple Integrals in Cylindrical Coordinates

Recall that in rectangular coordinates the triple integral of a continuous function f over a solid region G is defined as

$$\iiint_G f(x, y, z)\, dV \;=\; \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*) \Delta V_k$$

where ΔV_k denotes the volume of a rectangular parallelepiped interior to G and (x_k^*, y_k^*, z_k^*) is a point in this parallelepiped. Triple integrals in cylindrical and spherical coordinates are defined similarly, except that the region G is divided not into rectangular parallelepipeds but into regions more appropriate to these coordinate systems.

In cylindrical coordinates, the simplest equations are of the form

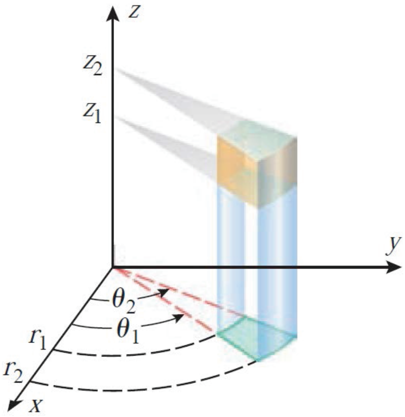
$$r = \text{constant}, \quad \theta = \text{constant}, \quad z = \text{constant}$$

These surfaces can be paired up to determine solids called *cylindrical wedges* or *cylindrical elements of volume*. To be precise, a cylindrical wedge is a solid enclosed between six surfaces of the following form:

two cylinders $r = r_1,\ r = r_2 \quad (r_1 < r_2)$

two vertical half-planes $\theta = \theta_1,\ \theta = \theta_2 \quad (\theta_1 < \theta_2)$

two horizontal planes $z = z_1,\ z = z_2 \quad (z_1 < z_2)$



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1048

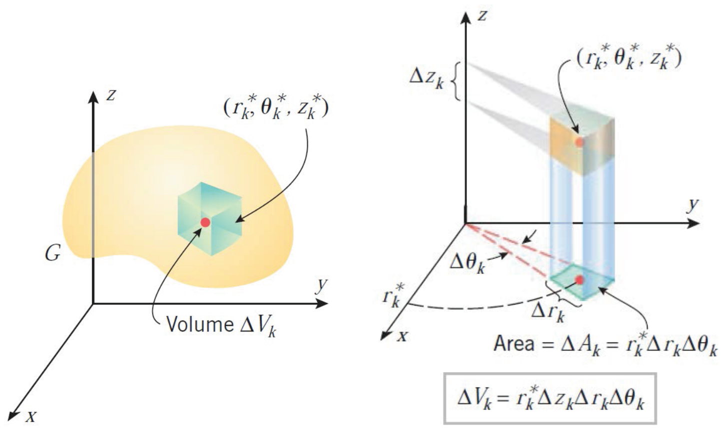
To define the triple integral over G of a function $f(r, \theta, z)$ in cylindrical coordinates we proceed as follows:

• Subdivide G into pieces by a three-dimensional grid consisting of concentric circular cylinders centered on the z-axis, half-planes hinged on the z-axis, and horizontal planes. Exclude from consideration all pieces that contain any points outside of G, thereby leaving only cylindrical wedges that are subsets of G.

• Assume that there are n such cylindrical wedges, and denote the volume of the kth cylindrical wedge by ΔV_k. Let $(r_k^*, \theta_k^*, z_k^*)$ be any point in the kth cylindrical wedge.

• Repeat this process with more and more subdivisions so that as n increases, the height, thickness, and central angle of the cylindrical wedges approach zero. Define

$$\iiint_G f(r, \theta, z)\, dV = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*) \Delta V_k = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*) r_k^* \Delta z_k \Delta r_k \Delta \theta_k.$$



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1048-1049

which suggests that a triple integral in cylindrical coordinates can be evaluated as an iterated integral of the form

$$\iiint_G f(r,\theta,z)\,dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(r,\theta,z) r\,dz\,dr\,d\theta.$$

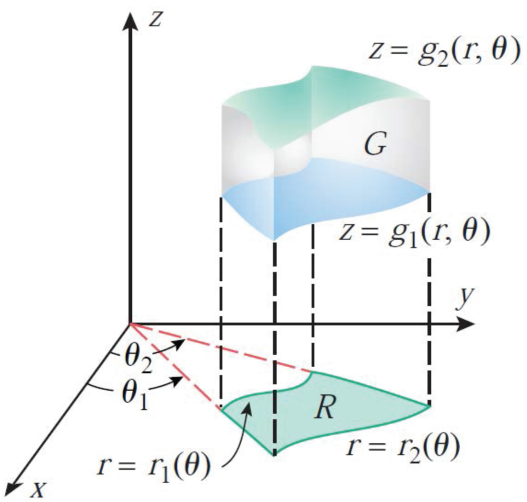
Theorem. Let G be a solid region whose upper surface has the equation $z = g_2(r,\theta)$ and whose lower surface has the equation $z = g_1(r,\theta)$ in cylindrical coordinates.

If the projection of the solid on the xy-plane is a simple polar region R, and if $f(r,\theta,z)$ is continuous on G, then

$$\iiint_G f(r,\theta,z)\,dV = \iint_R \left[\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\,dz\right] dA$$

where the double integral over R is evaluated in polar coordinates. In particular, if the projection R is as shown in below figure, then the integral can be written as

$$\iiint_G f(r,\theta,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z) r\,dz\,dr\,d\theta.$$



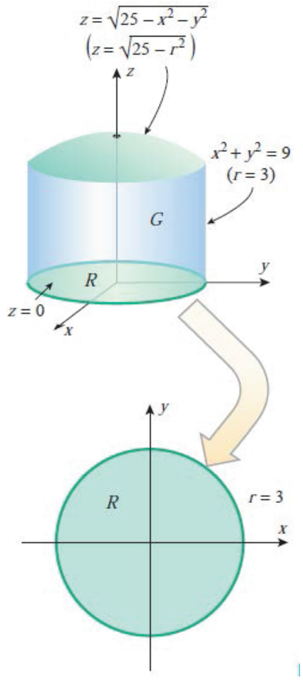
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1049

Determining Limits of Integration: Cylindrical Coordinates

Step 1. Identify the upper surface $z = g_2(r,\theta)$ and the lower surface $z = g_1(r,\theta)$ of the solid. The functions $g_1(r,\theta)$ and $g_2(r,\theta)$ determine the z-limits of integration. (If the upper and lower surfaces are given in rectangular coordinates, convert them to cylindrical coordinates.)

Step 2. Make a two-dimensional sketch of the projection R of the solid on the xy-plane. From this sketch the r- and θ-limits of integration may be obtained exactly as with double integrals in polar coordinates.

Example 7. Use triple integration in cylindrical coordinates to find the volume of the solid G that is bounded above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane, and laterally by the cylinder $x^2 + y^2 = 9$.



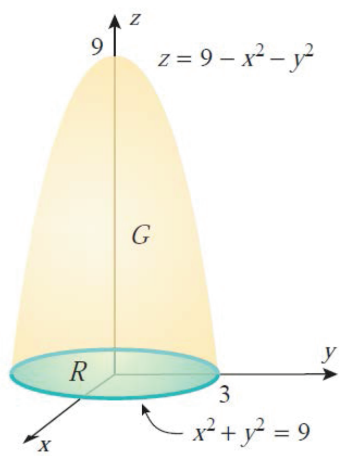
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1050

Converting Triple Integrals from Rectangular to Cylindrical Coordinates

Sometimes a triple integral that is difficult to integrate in rectangular coordinates can be evaluated more easily by making the substitution $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ to convert it to an integral in cylindrical coordinates. Under such a substitution, a rectangular triple integral can be expressed as an iterated integral in cylindrical coordinates as

$$\iiint_G f(x, y, z)\, dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta.$$

Example 8. Use cylindrical coordinates to evaluate $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} x^2\, dz\, dy\, dx.$



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1051

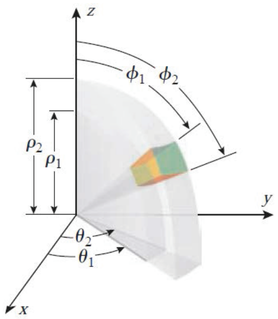
Triple Integrals in Spherical Coordinates

In spherical coordinates, the simplest equations are of the form

$$\rho = \text{constant}, \quad \theta = \text{constant}, \quad \phi = \text{constant}$$

These surfaces can be paired up to determine solids called *spherical wedges* or *spherical elements of volume*. To be precise, a spherical wedge is a solid enclosed between six surfaces of the following form:

two spheres	$\rho = \rho_1,\ \rho = \rho_2 \quad (\rho_1 < \rho_2)$
two vertical half-planes	$\theta = \theta_1,\ \theta = \theta_2 \quad (\theta_1 < \theta_2)$
nappes of two circular cones	$\phi = \phi_1,\ \phi = \phi_2 \quad (\phi_1 < \phi_2)$



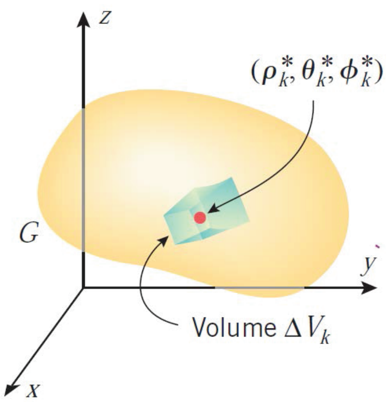
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1051

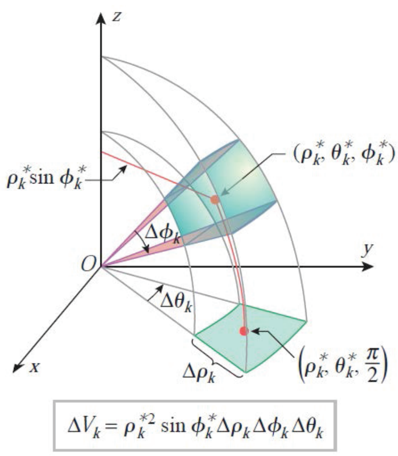
If G is a solid region in three-dimensional space, then the triple integral over G of a continuous function $f(\rho, \theta, \phi)$ in spherical coordinates is similar in definition to the triple integral in cylindrical coordinates, except that the solid G is partitioned into spherical wedges by a three-dimensional grid consisting of spheres centered at the origin, half-planes hinged on the z-axis, and nappes of right circular cones with vertices at the origin and lines of symmetry along the z-axis.

The defining equation of a triple integral in spherical coordinates is

$$\iiint_G f(\rho, \theta, \phi)\, dV \;=\; \lim_{n\to+\infty} \sum_{k=1}^{n} f(\rho_k^*, \theta_k^*, \phi_k^*)\Delta V_k$$

where ΔV_k is the volume of the kth spherical wedge that is interior to G, $(\rho_k^*, \theta_k^*, \phi_k^*)$ is an arbitrary point in this wedge, and n increases in such a way that the dimensions of each interior spherical wedge tend to zero.





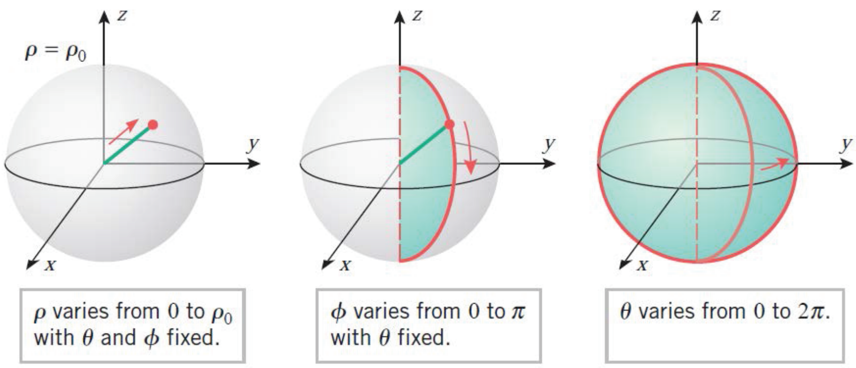
$$\iiint\limits_G f(\rho,\theta,\phi)\,dV \;=\; \lim_{n\to+\infty}\sum_{k=1}^{n} f(\rho_k^*,\theta_k^*,\phi_k^*)(\rho_k^*)^2\sin\phi_k^*\Delta\rho_k\Delta\phi_k\Delta\theta_k.$$

which suggests that a triple integral in cylindrical coordinates can be evaluated as an iterated integral of the form

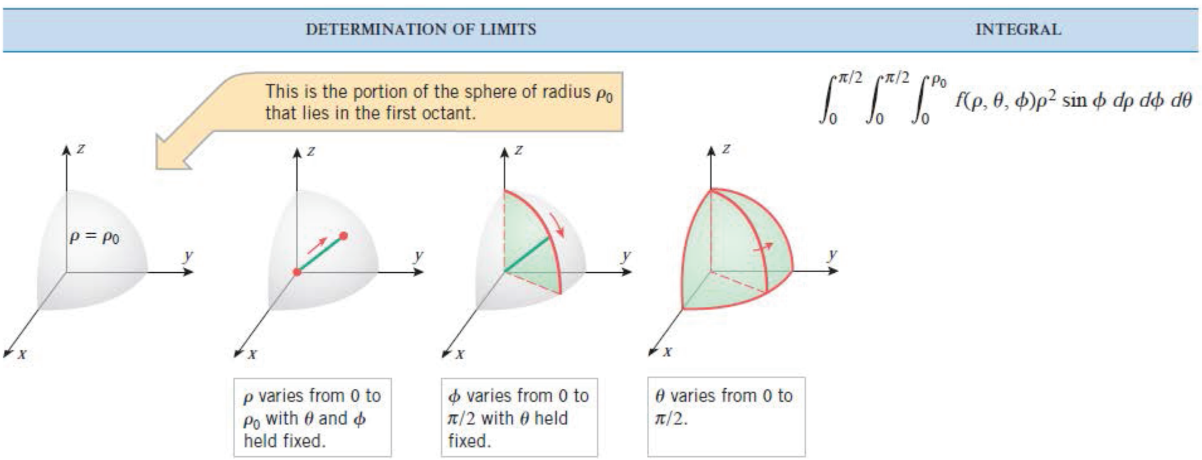
$$\iiint\limits_G f(\rho,\theta,\phi)\,dV \;=\; \iiint\limits_{\substack{\text{appropriate}\\ \text{limits}}} f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Suppose that we want to integrate $f(\rho,\theta,\phi)$ over the spherical solid G enclosed by the sphere $\rho=\rho_0$. The basic idea is to choose the limits of integration so that every point of the solid is accounted for in the integration process. The below figure illustrates one way of doing this. Holding θ and ϕ fixed for the first integration, we let ρ vary from 0 to ρ_0. This covers a radial line from the origin to the surface of the sphere. Next, keeping θ fixed, we let ϕ vary from 0 to π so that the radial line sweeps out a fan-shaped region. Finally, we let θ vary from 0 to 2π so that the fan-shaped region makes a complete revolution, thereby sweeping out the entire sphere. Thus, the triple integral of $f(\rho,\theta,\phi)$ over the spherical solid G can be evaluated by writing

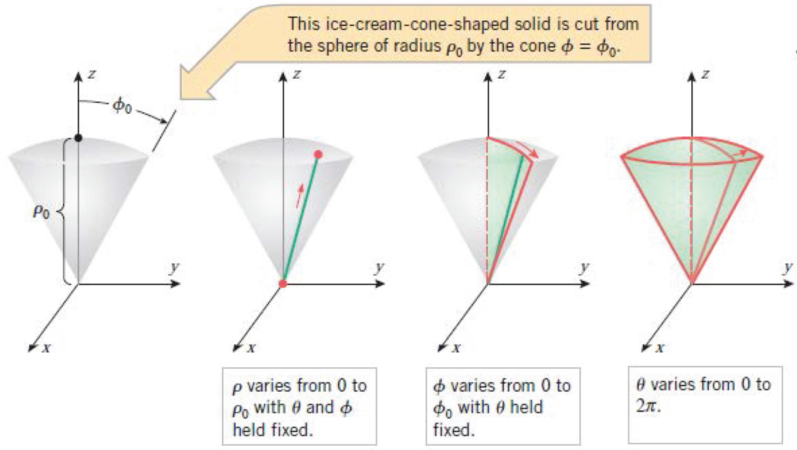
$$\iiint\limits_G f(\rho,\theta,\phi)\,dV \;=\; \int_0^{2\pi}\int_0^{\pi}\int_0^{\rho_0} f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$



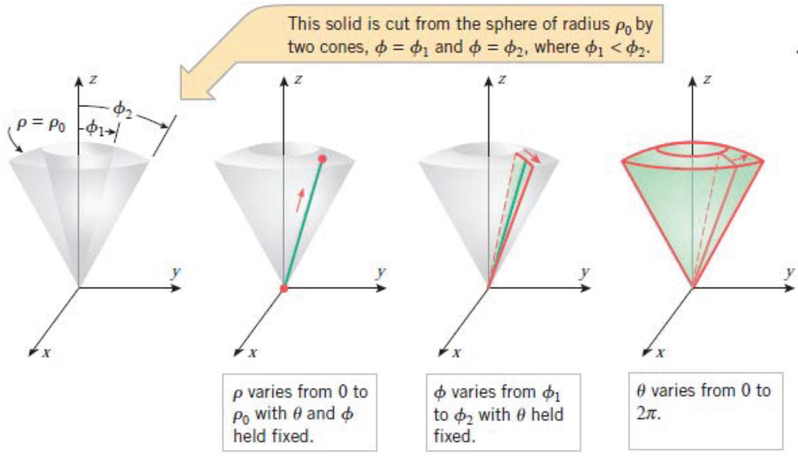
From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1052



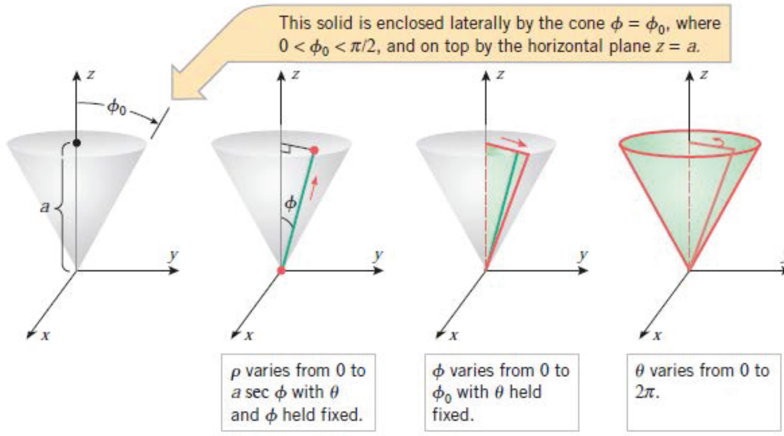
$$\int_0^{2\pi}\int_0^{\phi_0}\int_0^{\rho_0} f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



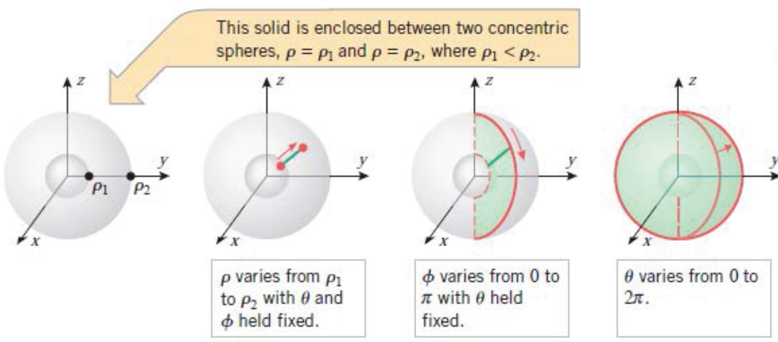
$$\int_0^{2\pi}\int_{\phi_1}^{\phi_2}\int_0^{\rho_0} f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



$$\int_0^{2\pi}\int_0^{\phi_0}\int_0^{a\sec\phi} f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

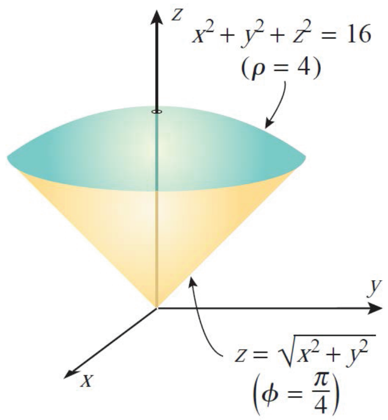


$$\int_0^{2\pi}\int_0^{\pi}\int_{\rho_1}^{\rho_2} f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



From:Calculus Early Transcendentals,10th edition,Howard Anton,Irl C. Beven,Stephen Deavis, page 1053-1054

Example 9. Use spherical coordinates to find the volume of the solid G bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the cone $z = \sqrt{x^2 + y^2}$.



From:Calculus Early Transcendentals,
10th edition,Howard Anton,Irl C. Beven,
Stephen Deavis, page 1054

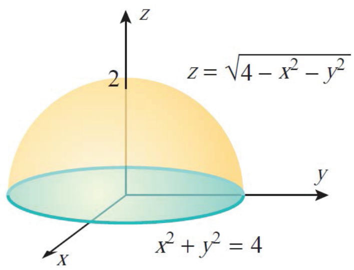
Converting Triple Integrals from Rectangular to Spherical Coordinates

Triple integrals can be converted from rectangular coordinates to spherical coordinates by making the substitution $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho\cos\phi$. The two integrals are related by the equation

$$\iiint_G f(x, y, z)\, dV \;=\; \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(\rho, \theta, \phi)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$

Example 10. Use spherical coordinates to evaluate

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$



From:Calculus Early Transcendentals,
10th edition,Howard Anton,Irl C. Beven,
Stephen Deavis, page 1055

EXERCISES 4.4

1. Let G be the solid region inside the sphere of radius 2 centered at the origin and above the plane $z = 1$. In each part, supply the missing integrand and limits of integration for the iterated integral in cylindrical and spherical coordinates.

$$\text{The volume of } G \text{ is } \iiint_G dV = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} \ldots\ldots\ldots\ldots\, dz\,dr\,d\theta$$

$$= \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} \ldots\ldots\ldots\ldots\, d\rho\,d\phi\,d\theta$$

$$\iiint_G \frac{z}{x^2+y^2+z^2}\,dV = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} \ldots\ldots\ldots\ldots\, dz\,dr\,d\theta$$

$$= \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} \ldots\ldots\ldots\ldots\, d\rho\,d\phi\,d\theta$$

2-5 Evaluate the iterated integral, sketch the region G and identify the function f.

2. $\displaystyle\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} zr\,dz\,dr\,d\theta$

3. $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{r^2} r\sin\theta\,dz\,dr\,d\theta$

4. $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1} \rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta$

5. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a\sec\theta} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

6-9 Use cylindrical coordinates to nd the volume of the solid.

6. The solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$.

7. The solid that is bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the cone $z = \sqrt{x^2 + y^2}$.

8. The solid that is inside the surface $r^2 + z^2 = 20$ but not above the surface $z = r^2$.

9. The solid enclosed between the cone $z = (hr)/a$ and the plane $z = h$.

10-13 Use spherical coordinates to nd the volume of the solid.

10. The solid bounded above by the sphere $\rho = 4$ and below by the cone $\phi = \pi/3$.

11. The solid within the cone $\phi = \pi/4$ and between the spheres $\rho = 1$ and $\rho = 2$.

12. The solid enclosed by the sphere $x^2 + y^2 + z^2 = 4a^2$ and the planes $z = 0$ and $z = a$.

13. The solid within the sphere $x^2 + y^2 + z^2 = 9$, outside the cone $z = \sqrt{x^2 + y^2}$, and above the xy-plane.

13-16 Use cylindrical or spherical coordinates to evaluate the integral.

13. $\displaystyle\int_0^a \int_a^{\sqrt{a^2-x^2}} \int_0^{a^2-x^2-y^2} x^2\, dz\, dy\, dx.$

14. $\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} e^{-(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx.$

15. $\displaystyle\int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2\, dz\, dx\, dy.$

16. $\displaystyle\int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy.$

17-20 True−False. Determine whether the statement is true or false. Explain your answer.

17. A rectangular triple integral can be expressed as an iterated integral in cylindrical coordinates as

$$\iiint_G f(x, y, z)\, dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta, z) r^2\, dz\, dr\, d\theta.$$

18. If $0 \le \rho_1 < \rho_2$, $0 \le \theta_1 < \theta_2$, and $0 \le \phi_1 < \phi_2 \le \pi$, then the volume of the spherical wedge bounded by the spheres $\rho = \rho_1$ and $\rho = \rho_2$, the half-planes $\theta = \theta$ and $\theta = \theta_2$, and the cones $\phi = \phi_1$ and $\phi = \phi_2$ is

$$\int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

19. Let G be the solid region in 3-space between the spheres of radius 1 and 3 centered at the origin and above the cone $z = \sqrt{x2 + y2}$. The volume of G equals

$$\int_0^{\pi/4} \int_0^{2\pi} \int_1^3 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi.$$

20. Let G be the solid region in 3-space between the spheres of radius 1 and 3 centered at the origin and above the cone $z = \sqrt{x2 + y2}$. If $f(x, y, z)$ is continuous on G, then

$$\iiint_G f(x, y, z)\, dV = \int_0^{\pi/4} \int_0^{2\pi} \int_1^3 F(\rho, \theta, \phi) \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi.$$

where $F(\rho, \theta, \phi) = f(\rho\sin\phi\sin\theta, \rho\sin\phi\cos\theta, \rho\cos\phi)$.

See more exercises from : Calculus Early Transcendentals, 10th edition, Howard Anton, Irl C. Beven, Stephen Deavis, page 1056-1057