Calculus I

FINAL



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Part III

Final

5 Integration

5.1 Indefinite Integral

5.1.1 Antiderivative

Definition 1 (antiderivative) A function F is called an ***antiderivative*** of a function f on a given open interval if $F'(x) = f(x)$ for all x in the interval.

The process of finding antiderivatives is called ***antidifferentiation*** or ***integration*** . Thus, if

$$\frac{d}{dx}[F(x)] = f(x) \tag{5.1}$$

then ***integrating*** (or ***antidifferentiating***) the function $f(x)$ produces an antiderivative of the form $F(x) + C$ as in the following Theorem.

Theorem 5.1 If $F(x)$ is any antiderivative of $f(x)$ on an open interval, then for any constant C the function $F(x) + C$ is also an antiderivative on that interval. Moreover, each antiderivative of $f(x)$ on the interval can be expressed in the form $F(x) + C$ by choosing the constant C appropriately.

To emphasize this process, Equation (5.1) is recast using integral notation,

$$\int f(x)dx = F(x) + C \tag{5.2}$$

where C is understood to represent an arbitrary constant. Notice that the values of C result to the shifting of the function $F(x)$ up or down.
For example, $\frac{1}{3}x^3, \frac{1}{3}x^3 + 1, \frac{1}{3}x^3 - 3, \frac{1}{3}x^3 - \sqrt{2}$ are all antiderivatives of $f(x) = \ldots\ldots\ldots$

5.1.2 Integration Formulas

Differentiation Formula	Integration Formula
1. $\frac{d}{dx}(C) = 0$	1. $\int 0dx = C$
2. $\frac{d}{dx}[kx] = k$	2. $\int kdx = kx + C$
3. $\frac{d}{dx}[kf(x)] = kf'(x)$	3. $\int [kf(x)]dx = k\int f(x)dx$
4. $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	4. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
5. $\frac{d}{dx}[x^n] = nx^{n-1}$	5. $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
6. $\frac{d}{dx}[\ln\|x\|] = \frac{1}{x}$	6. $\int \frac{1}{x}dx = \ln\|x\| + C$
7. $\frac{d}{dx}[e^x] = e^x$	7. $\int e^x dx = e^x + C$
8. $\frac{d}{dx}[a^x] = a^x \ln a,\ a > 0$ and $a \neq 1$	8. $\int a^x dx = \frac{a^x}{\ln a} + C,\ a > 0$ and $a \neq 1$
9. $\frac{d}{dx}[\sin x] = \cos x$	9. $\int \cos x dx = \sin x + C$
10. $\frac{d}{dx}[\cos x] = -\sin x$	10. $\int \sin x dx = -\cos x + C$
11. $\frac{d}{dx}[\tan x] = \sec^2 x$	11. $\int \sec^2 x dx = \tan x + C$
12. $\frac{d}{dx}[\cot x] = -\csc^2 x$	12. $\int \csc^2 x dx = -\cot x + C$
13. $\frac{d}{dx}[\sec x] = \sec x \tan x$	13. $\int \sec x \tan x dx = \sec x + C$
14. $\frac{d}{dx}[\csc\ x] = -\csc\ x \cot x$	14. $\int \csc\ x \cot x dx = -\csc\ x + C$
15. $\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$	15. $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$
16. $\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$	16. $\int \frac{1}{1+x^2}dx = \arctan x + C$
17. $\frac{d}{dx}[\ln\|\sec x\|] = \tan x$	17. $\int \tan x dx = \ln\|\sec x\| + C$
18. $\frac{d}{dx}[\ln\|\sin x\|] = \cot x$	18. $\int \cot x dx = \ln\|\sin x\| + C$

Example 5.1

(a) $\int (x^6 - 7x + 4)\, dx =$

(b) $\int \dfrac{x^5 + 2x^3 - 1}{x^4}\, dx =$

(c) $\int (\sqrt{x} + \dfrac{1}{\sqrt[3]{x}})\, dx =$

(d) $\int (e^x + 2^x)\, dx =$

(e) $\int (4\sin x + 2\cos x)\, dx =$

(f) $\int (\dfrac{3}{\sqrt{1-x^2}} - \dfrac{2}{1+x^2})\, dx =$

5.2 Integration by Substitution

Theorem 5.2 Let u be a function of x and f be a function of u. Then

$$\int [f(u)]du = \int [f(u(x))u'(x)]dx.$$

Example 5.2 Evaluate $\int (2x+1)^{100}dx$

Example 5.3 Evaluate $\int \frac{x^2}{(3x^3-2)^9}dx$

Example 5.4 Evaluate $\int \cos(5x)dx$

Example 5.5 Evaluate $\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$

Example 5.6 Evaluate $\int e^x \sec^2(e^x + 1) dx$

Example 5.7 Evaluate $\int \frac{1}{\sqrt{2 - x^2}} dx$

Exercise 5a

Evaluate the integrals.

1. $\int x^{1/8} dx$
2. $\int \frac{1}{x^6} dx$
3. $\int 3x(2x-7) dx$
4. $\int x + \sqrt{7} dx$
5. $\int (\frac{x}{3} + 3x) dx$
6. $\int (x^3+1)\sqrt{x} dx$
7. $\int \frac{x^4 + 7x^3 - 5x^2 + 1}{x^2} dx$
8. $\int \frac{x^5 - \sqrt{x}}{x^3} dx$
9. $\int (6e^x - \ln x) dx$
10. $\int (1 + \sin x) dx$
11. $\int (5\sec^2 x + \csc^2 x) dx$
12. $\int \frac{\sin 2x}{\cos x} dx$
13. $\int \frac{\sec x}{\sec^2 x - 1} dx$
14. $\int (1 + \sin x + 8\cos x) dx$
15. $\int (\frac{15}{\sqrt{1-x^2}} - \frac{21}{1+x^2}) dx$
16. $\int x\sqrt{4-x^2} dx$
17. $\int \frac{x^4}{\sqrt{x^5 - 9}} dx$
18. $\int \frac{e^x}{1+e^{2x}} dx$
19. $\int \frac{1}{\sqrt{16-x^2}} dx$
20. $\int \sec^2(3x) dx$
21. $\int \frac{\cos x}{2 - \sin x} dx$
22. $\int \frac{\tan^2 x + 1}{\cot x} dx$
23. $\int \frac{7^{\ln x}}{x} dx$

6 Techniques of Integration

6.1 Overview of Integration Methods

A review of familiar integration formulas

1. $\int du = u + C$
2. $\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$
3. $\int \frac{1}{u}\, du = \ln|u| + C$
4. $\int a^u\, du = \frac{a^u}{\ln a} + C,\ a > 0, a \neq 1$
5. $\int e^u\, du = e^u + C$
6. $\int \sin u\, du = -\cos u + C$
7. $\int \cos u\, du = \sin u + C$
8. $\int \sec^2 u\, du = \tan u + C$
9. $\int \csc^2 u\, du = -\cot u + C$
10. $\int \sec u \tan u\, du = \sec u + C$
11. $\int \csc u \cot u\, du = -\csc u + C$
12. $\int \tan u\, du = \ln|\sec u| + C$
13. $\int \cot u\, du = \ln|\sin u| + C$
14. $\int \sec u\, du =$ _____ $+ C$
15. $\int \csc u\, du =$ _____ $+ C$
16. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin(\frac{u}{a}) + C$
17. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan(\frac{u}{a}) + C$

6.2 Integration by Parts

6.2.1 The Product Rule vs Integration by Parts

Let $G(x)$ be any antiderivative of $g(x)$; $G'(x) = g(x)$

$$\frac{d}{dx}[f(x)G(x)] = f(x)G'(x) + f'(x)G(x) = f(x)g(x) + f'(x)G(x)$$

$$\int [f(x)g(x) + f'(x)G(x)]\ dx = f(x)G(x)$$

$$\int f(x)g(x)\ dx = f(x)G(x) - \int f'(x)G(x)\ dx$$

Let $u = f(x), du = f'(x)dx$ and $v = G(x), dv = g(x)\ dx$

$$\int udv = uv - \int vdu$$

Example 6.1 Evaluate $\int x \sin x\ dx$.

Example 6.2 Evaluate $\int x^3 \ln x \, dx$.

6.2.2 Repeated Integration by Parts

Example 6.3 Evaluate $\int x^2 e^x \, dx$.

Example 6.4 Evaluate $\int e^x \cos x \; dx$.

Example 6.5 Evaluate $\int \arctan x \; dx$.

Exercise 6a

Evaluate the following integrals.

1. $\int x \sin \frac{x}{2}\ dx$
2. $\int \sqrt{x} \ln x\ dx$
3. $\int x\sec^2 x\ dx$
4. $\int (\ln x)^2\ dx$
5. $\int x^2 \sin x\ dx$
6. $\int x\cos^2 x\ dx$
7. $\int e^{\sqrt{3x+9}}\ dx$
8. $\int \sin(\ln x)\ dx$
9. $\int xe^{3x}\ dx$
10. $\int xe^{-2x}\ dx$
11. $\int x^2 e^x\ dx$
12. $\int x^2 e^{-2x}\ dx$
13. $\int x \sin 3x\ dx$
14. $\int x \cos 2x\ dx$
15. $\int x^2 \cos x\ dx$
16. $\int x \ln x\ dx$
17. $\int \ln(3x-2)\ dx$
18. $\int \ln(x^2+4)\ dx$
19. $\int \arcsin x\ dx$
20. $\int \arccos(2x)\ dx$
21. $\int \arctan(3x)\ dx$
22. $\int x \arctan x\ dx$
23. $\int e^x \sin x\ dx$
24. $\int e^{3x} \cos(2x)\ dx$
25. $\int \cos(\ln x)\ dx$
26. $\int x \tan^2 x\ dx$
27. $\int x^3 e^{x^2}\ dx$
28. $\int \frac{\ln x}{\sqrt{x}}\ dx$
29. $\int \frac{xe^x}{(x+1)^2}\ dx$
30. $\int_0^{\pi} (x + x\cos x)\ dx$
31. $\int_0^2 xe^{2x}\ dx$
32. $\int_0^1 xe^{-5x}\ dx$
33. $\int_1^e x^2 \ln x\ dx$
34. $\int_{\sqrt{e}}^e \frac{\ln x}{x^2}\ dx$
35. $\int_{-1}^1 \ln(x+2)\ dx$
36. $\int_0^{\sqrt{3}/2} \arcsin x\ dx$
37. $\int_2^4 \sec^{-1}\sqrt{x}\ dx$
38. $\int_1^2 x \sec^{-1} x\ dx$
39. $\int_0^{\pi} x \sin 2x\ dx$
40. $\int_1^3 \sqrt{x} \arctan\sqrt{x}\ dx$
41. $\int_0^2 \ln(x^2+1)\ dx$

6.3 Integrating Trigonometric Functions

We start the section by reviewing important trigonometric identities as following:

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x = \sec^2 x - 1$$

$$\sin 2x = 2\sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

6.3.1 Integrating Products of Sines and Cosines

If m and n are positive integers, the integral $\int \sin^m x \cos^n x \, dx$ can be evaluated by one of the following procedures, depending on whether m and n are even or odd.

$\int \sin^m x \cos^n x \, dx$	Procedure	relevant identity
m odd	set $\sin^m x = \sin^{m-1} x \sin x$	$\sin^2 x = 1 - \cos^2 x$
n odd	set $\cos^n x = \cos^{n-1} x \cos x$	$\cos^2 x = 1 - \sin^2 x$
m and n even	set $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ or set $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Example 6.6 Evaluate $\int \sin^3 x \, dx$.

Example 6.7 Evaluate $\int \cos^5 x \, dx$.

Example 6.8 Evaluate $\int \sin^2 x \cos^3 x \, dx$.

Example 6.9 Evaluate $\int \cos^{1/3} x \sin^3 x \, dx$.

Example 6.10 Evaluate $\int (1 + \sin x)^2 \, dx$.

6.3.2 Integrating Products of Sines and Cosines with Different Angles

Integrals of the form

$$\int \sin mx \cos nx \; dx, \int \sin mx \sin nx \; dx, \int \cos mx \cos nx \; dx$$

can be evaluated using the following identities:

$$\sin(mx)\cos(nx) = \tfrac{1}{2}\{\sin(m+n)x + \sin(m-n)x\}$$
$$\sin(mx)\sin(nx) = \tfrac{1}{2}\{\cos(m-n)x - \cos(m+n)x\}$$
$$\cos(mx)\cos(nx) = \tfrac{1}{2}\{\cos(m+n)x + \cos(m-n)x\}$$

Example 6.11 Evaluate $\int \sin 3x \;\; \cos 5x \; dx$.

Example 6.12 Evaluate $\int \sin 3x \;\; \sin \dfrac{5x}{2} \; dx$.

Exercise 6b

Evaluate the following integrals.

1. $\int \sin^3 x \, \cos^2 x \, dx$
2. $\int \frac{\cos^3 x}{\sin x} \, dx$
3. $\int \cos^{5/3} x \sin x \, dx$
4. $\int \sin^4 x \, dx$
5. $\int \cos^4 x \, \sin^4 x \, dx$
6. $\int \cos^3 x \sin x \, dx$
7. $\int \sin^5 3x \cos 3x \, dx$
8. $\int \sin^2 5x \, dx$
9. $\int \cos^2 3x \, dx$
10. $\int \sin^3 ax \, dx$
11. $\int \cos^3 ax \, dx$
12. $\int \sin x \cos^3 x \, dx$
13. $\int \sin^2 x \cos^2 x \, dx$
14. $\int \sin^2 x \cos^4 x \, dx$
15. $\int \sin 2x \cos 3x \, dx$
16. $\int \sin 3x \cos 2x \, dx$
17. $\int \sin x \cos(x/2) \, dx$
18. $\int \sin 7x \, \sin 2x \, dx$
19. $\int \cos 4x \, \cos 9x \, dx$

6.4 Trigonometric Substitution

We will be concerned with integrals that contain expressions of the form $\sqrt{a^2-u^2}, \sqrt{u^2 \pm a^2}$.

$\sqrt{a^2-u^2}$	$u = a\sin\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{a^2+u^2}$	$u = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{u^2-a^2}$	$u = a\sec\theta$	$0 \le \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \le \pi$

Example 6.13 Evaluate $\int \frac{dx}{\sqrt{9+x^2}}$.

Example 6.14 Evaluate $\int \frac{x^3}{\sqrt{9-x^2}}dx$.

Example 6.15 Evaluate $\int \frac{dx}{x^2\sqrt{4x^2-3}}$.

Exercise 6c

Evaluate the following integrals.

1. $\int \sqrt{4-x^2}\,dx$
2. $\int \sqrt{25-x^2}\,dx$
3. $\int \dfrac{dx}{\sqrt{4x^2-49}}$
4. $\int \dfrac{x^3}{\sqrt{x^2+4}}\,dx$
5. $\int \dfrac{dx}{(x^2-1)^{3/2}}$
6. $\int \dfrac{\sqrt{1-x^2}}{x^2}\,dx$
7. $\int \dfrac{8}{(4x^2+1)^2}\,dx$
8. $\int \dfrac{dx}{\sqrt{x^2+2x-3}}$
9. $\int \dfrac{dx}{(x^2-2x+10)^{\frac{3}{2}}}$
10. $\int \dfrac{x+6}{\sqrt{4x-x^2}}\,dx$
11. $\int \dfrac{dx}{x^2\sqrt{x^2-16}}$
12. $\int \dfrac{3x^3}{\sqrt{1-x^2}}\,dx$
13. $\int \dfrac{x^2}{\sqrt{16-x^2}}\,dx$
14. $\int \dfrac{\sqrt{x^2-9}}{x}\,dx$
15. $\int \dfrac{3x^3}{\sqrt{x^2-25}}\,dx$
16. $\int \dfrac{\cos x}{\sqrt{2-\sin^2 x}}\,dx$
17. $\int \dfrac{\sqrt{2x^2-4}}{x}\,dx$
18. $\int \dfrac{x^3}{(3+x^2)^{5/2}}\,dx$
19. $\int \dfrac{x^2}{\sqrt{5+x^2}}\,dx$
20. $\int \dfrac{dx}{x^2\sqrt{9-x^2}}$
21. $\int \dfrac{dx}{(4+x^2)^2}$
22. $\int \dfrac{dx}{x^2\sqrt{9x^2-4}}$
23. $\int \dfrac{dx}{(1-x^2)^{3/2}}$
24. $\int \dfrac{dx}{x^2\sqrt{x^2+25}}$
25. $\int \dfrac{dx}{\sqrt{x^2-9}}$
26. $\int \dfrac{dx}{1+2x^2+x^4}$
27. $\int \dfrac{dx}{(4x^2-9)^{3/2}}$
28. $\int \dfrac{dx}{(1-x^2)^2}$
29. $\int \dfrac{dx}{x^2\sqrt{x^2-1}}$
30. $\int \dfrac{dx}{x^4\sqrt{x^2+3}}$
31. $\int \dfrac{e^x}{\sqrt{e^{2x}+e^x+1}}\,dx$
32. $\int \sqrt{1-4x^2}\,dx$
33. $\int e^x\sqrt{1-e^{2x}}\,dx$

6.5 Integrating Rational Functions by Partial Fractions

Recall that a rational function is a function that can be written as a quotient of two polynomials. Assume that $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function, where $P(x)$ and $Q(x)$ are polynomials. If $\deg P(x) < \deg Q(x)$, then $f(x)$ is said to be ***proper.*** If $\deg P(x) \geq \deg Q(x)$, then $f(x)$ is said to be ***improper*** .

We now find the form of partial fraction decomposition of a proper rational function $f(x) = \dfrac{P(x)}{Q(x)}$. Elementary algebra tells us that $Q(x)$ has only two types of irreducible factors which are of degree 1 or degree 2. Therefore, the partial fraction decomposition of $f(x)$ can be determined by using the following rules, **Linear factor and Quadratic factor rules**.

Linear factor rule: For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:
$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \ \ldots + \dfrac{A_m}{(ax+b)^m}$, where $A_i \quad (i = 1, 2, \ldots, m)$ are constants.

Example 6.16 Evaluate $\displaystyle\int \frac{dx}{x^2+x-2}$.

Example 6.17 Evaluate $\int \dfrac{2x^2 - 3x + 4}{(x+1)(x-2)^2}\, dx$.

Quadratic factor rule : For each factor of the form $(ax^2 + bx + c)^m$ with $b^2 - 4ac < 0$, the partial fraction decomposition contains the following sum of m partial fractions:
$\dfrac{A_1 x + B_1}{ax^2 + bx + c} + \dfrac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \ldots + \dfrac{A_m x + B_m}{(ax^2 + bx + c)^m}$,
where $A_i, B_i \quad (i = 1, 2, \ldots, m)$ are constants.

Example 6.18 Evaluate $\int \dfrac{3x^2 + x - 2}{(x-1)(x^2+1)}\, dx$.

Example 6.19 Evaluate $\int \dfrac{x+4}{x^2(x^2+4)}\,dx$.

Example 6.20 Evaluate $\int \dfrac{x^3-4x}{(x^2+1)^2}\,dx$.

6.5.1 Integrating Improper Rational Functions

Example 6.21 Evaluate $\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$.

The integrand can be expressed as

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = (3x^2+1) + \frac{1}{x^2+x-2}$$

and hence

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = \int (3x^2+1)\,dx + \int \frac{1}{x^2+x-2}\,dx = x^3+x+\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C.$$

Exercise 6d

Write out the form of the partial fraction decomposition.

1. $\dfrac{3x-1}{(x-3)(x+4)}$
2. $\dfrac{5}{x(x^2-4)}$
3. $\dfrac{2x-3}{x^3-x^2}$
4. $\dfrac{x^2}{(x+2)^3}$
5. $\dfrac{1-x^2}{x^3(x^2+2)}$
6. $\dfrac{3x}{(x-1)(x^2+6)}$
7. $\dfrac{4x^3-x}{(x^2+5)^2}$
8. $\dfrac{1-3x^4}{(x-2)(x^2+1)^2}$
9. $\dfrac{1}{x^2}$

Evaluate the following integrals.

10. $\displaystyle\int \frac{dx}{x^2-3x-4}$
11. $\displaystyle\int \frac{dx}{x^2-6x-7}$
12. $\displaystyle\int \frac{11x+17}{2x^2+7x-4}\,dx$
13. $\displaystyle\int \frac{5x-5}{3x^2-8x-3}\,dx$
14. $\displaystyle\int \frac{2x^2-9x-9}{x^3-9x}\,dx$
15. $\displaystyle\int \frac{dx}{x(x^2-1)}$
16. $\displaystyle\int \frac{x^2-8}{x+3}\,dx$
17. $\displaystyle\int \frac{x^2+1}{x-1}\,dx$
18. $\displaystyle\int \frac{3x^2-10}{x^2-4x+4}\,dx$
19. $\displaystyle\int \frac{x^2}{x^2-3x+2}\,dx$
20. $\displaystyle\int \frac{2x-3}{x^2-3x-10}\,dx$
21. $\displaystyle\int \frac{3x+1}{3x^2+2x-1}\,dx$
22. $\displaystyle\int \frac{x^5+x^2+2}{x^3-x}\,dx$
23. $\displaystyle\int \frac{x^5-4x^3+1}{x^3-4x}\,dx$
24. $\displaystyle\int \frac{2x^2+3}{x(x-1)^2}\,dx$
25. $\displaystyle\int \frac{3x^2-x+1}{x^3-x^2}\,dx$
26. $\displaystyle\int \frac{2x^2-10x+4}{(x+1)(x-3)^2}\,dx$
27. $\displaystyle\int \frac{2x^2-2x-1}{x^3-x^2}\,dx$
28. $\displaystyle\int \frac{x^2}{(x+1)^3}\,dx$
29. $\displaystyle\int \frac{2x^2+3x+3}{(x+1)^3}\,dx$
30. $\displaystyle\int \frac{2x^2-1}{(4x-1)(x^2+1)}\,dx$
31. $\displaystyle\int \frac{dx}{x^3+2x}$
32. $\displaystyle\int \frac{x^3+3x^2+x+9}{(x^2+1)(x^2+3)}\,dx$
33. $\displaystyle\int \frac{x^4+6x^3+10x^2+x}{x^2+6x+10}\,dx$

7 Definite Integration and its Applications

7.1 An Overview of Area Problem

Problem 2 Given a function f that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x-axis (Figure 7.1).

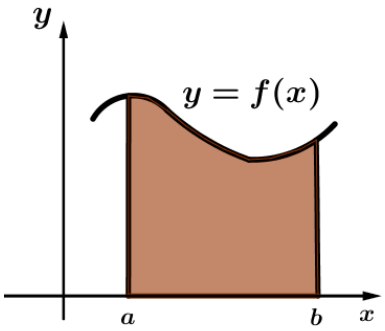


Figure 7.1: Area problem

7.2 The Definition of Area as a Limit; Sigma Notation

7.2.1 Sigma Notation

To simplify our computations, we will begin by discussing a useful notation for expressing lengthy sums in a compact form. This notation is called sigma notation or summation notation because it uses the uppercase Greek letter $\sum$ to denote various kinds of sums.

If $f(k)$ is a function of k, and if m and n are integers such that $m \leq n$, then

$$\sum_{k=m}^{n} f(k)$$

denotes the sum of the terms that result when we substitute successive integers for k, starting with $k = m$ and ending with $k = n$.

Example 7.1

$\sum_{k=4}^{8} k^3 =$

$\sum_{k=0}^{5} (-1)^k (2k-1) =$

7.2.2 Properties of Sums

Theorem 7.1 (a) $\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k$

(b) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(c) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

7.2.3 The Rectangle Method for Finding Areas

One approach to the area problem is to use Archimedes' method of exhaustion in the following way:

Divide the interval $[a, b]$ into n equal subintervals, and over each subinterval construct a rectangle that extends from the x-axis to any point on the curve $y = f(x)$ that is above the subinterval; the particular point does not matter – it can be above the center, above an endpoint, or above any other point in the subinterval.

For each n, the total area of the rectangles can be viewed as an approximation to the exact area under the curve over the interval $[a, b]$. Moreover, it is evident intuitively that as n increases these approximations will get better and better and will approach the exact area as a limit (Figure 7.2). That is, if A denotes the exact area under the curve and A_n denotes the approximation to A using n rectangles, then

$$A = \lim_{n \to \infty} A_n$$

We will call this the rectangle method for computing A.

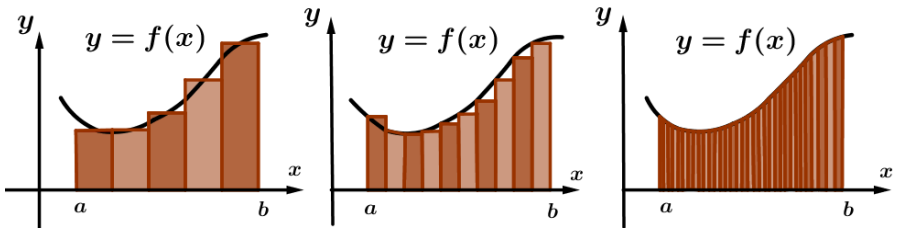
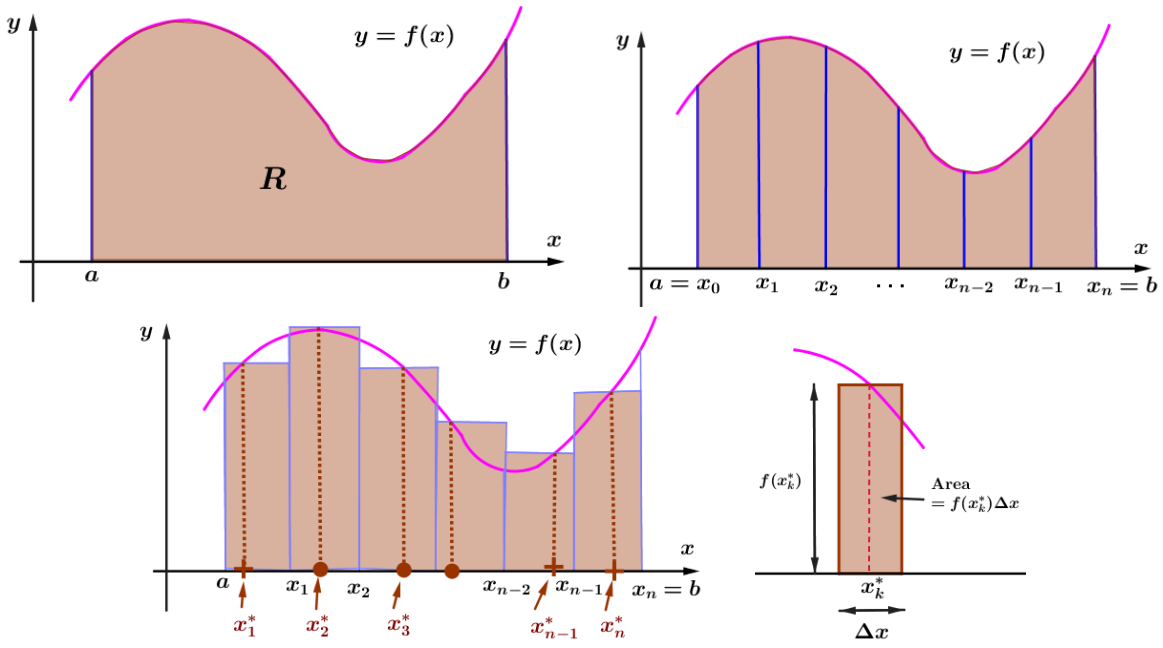


Figure 7.2: Finding Area

7.2.4 A Definition of Area

Definition 2 (Area Under a Curve) If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the area A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$



It is probably easiest to see how we do this with an example. So let's determine the area between $f(x) = x^2 + 2$ on $[-1, 1]$. In other words, we want to determine the area of the shaded region below.

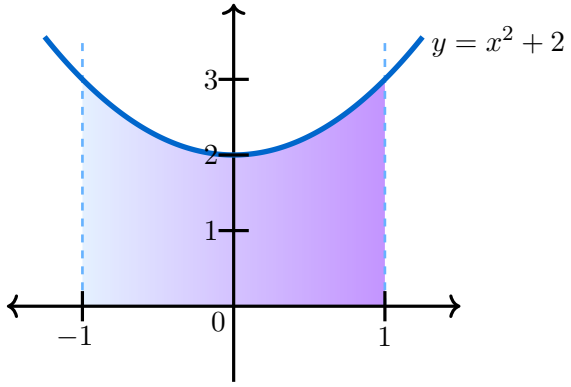


Figure 7.3: $y = x^2 + 2$

So, let's divide up the interval into 6 subintervals and use the function value on the left of each interval to define the height of the rectangle.

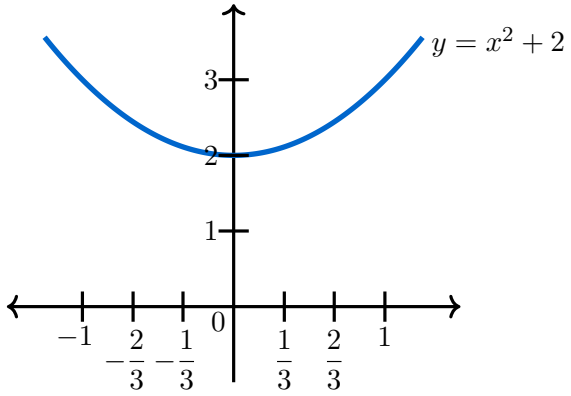


Figure 7.4: $y = x^2 + 2$

First, the width of each of the rectangles is $\Delta x = \dfrac{1-(-1)}{6} = \ldots\ldots\ldots$

The height of each rectangle is determined by the function value on the left. Here is the estimated area.

$$
\begin{aligned}
A_6 &= f(-1)\Delta x + f(-\tfrac{2}{3})\Delta x + f(-\tfrac{1}{3})\Delta x + f(0)\Delta x + f(\tfrac{1}{3})\Delta x + f(\tfrac{2}{3})\Delta x \\
&= \left[f(-1) + f(-\tfrac{2}{3}) + f(-\tfrac{1}{3}) + f(0) + f(\tfrac{1}{3}) + f(\tfrac{2}{3})\right]\Delta x \\
&= \left[\left((-1)^2+2\right) + \left((-2/3)^2+2\right) + \left((-1/3)^2+2\right) + \left((0)^2+2\right) + \left((1/3)^2+2\right) + \left((2/3)^2+2\right)\right]\Delta x \\
&= \left[3 + \frac{22}{9} + \frac{19}{9} + 2 + \frac{19}{9} + \frac{22}{9}\right]\Delta x \\
&= \left[\frac{87}{9}\right] \cdot \Delta x \\
&= \left[\frac{87}{9}\right] \cdot\ldots\ldots\ldots = \ldots\ldots\ldots
\end{aligned}
$$

Now, let's move on to the general case. We'll divide the interval into n subintervals, the width of each of the rectangles is $\Delta x = \dfrac{1-(-1)}{n} = \ldots\ldots\ldots$

The total area A_n of the n rectangles will be

$$\begin{aligned}
A_n &= f(-1)\Delta x + f(-1+\Delta x)\Delta x + f(-1+2\Delta x)\Delta x + \cdots + f(-1+(n-1)\Delta x)\Delta x \\
&= f(-1)\cdot\frac{2}{n} + f\left(-\frac{(n-2)}{n}\right)\cdot\frac{2}{n} + f\left(-\frac{(n-4)}{n}\right)\cdot\frac{2}{n} + \cdots + f\left(\frac{(n-2)}{n}\right)\cdot\frac{2}{n} \\
&= \left[f(-1) + f\left(-\frac{(n-2)}{n}\right) + f\left(-\frac{(n-4)}{n}\right) + \cdots + f\left(\frac{(n-2)}{n}\right)\right]\cdot\frac{2}{n} \\
&= \left[(-1)^2 + 2 + \left(-\frac{(n-2)}{n}\right)^2 + 2 + \left(-\frac{(n-4)}{n}\right)^2 + 2 + \cdots + \left(\frac{(n-2)}{n}\right)^2 + 2\right]\cdot\frac{2}{n} \\
&= \left[2n + \frac{n^2+2}{3n}\right]\cdot\frac{2}{n} \\
&= 4 + \frac{2n^2+4}{3n^2} \\
&= 4 + \frac{2}{3} + \frac{4}{3n^2} \\
&= 4\frac{2}{3} + \frac{4}{3n^2}
\end{aligned}$$

$$A_n = 4\frac{2}{3} + \frac{4}{3n^2} \tag{7.1}$$

Table 7.1 below shows the result of evaluating (7.1) on a computer for some increasingly large values of n. These computations suggest that the exact area is close to ……..

n	6	10	100	1,000	10,000
A_n	4.7	4.68	4.6668	4.666668	4.66666668

Table 7.1: estimation of area

So, increasing the number of rectangles improves the accuracy of the estimation as we would guess.

Later in this chapter we will show that

$$\lim_{n\to\infty} A_n = 4\frac{2}{3}.$$

7.2.5 Net Signed Area

If f is continuous and attains both positive and negative values on $[a, b]$, then the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

no longer represents the area between the curve $y = f(x)$ and the interval $[a, b]$ on the x-axis; rather, it represents a difference of areas -— the area of the region that is above the interval $[a, b]$ and below the curve $y = f(x)$ minus the area of the region that is below the interval $[a, b]$ and above the curve $y = f(x)$. We call this the **net signed area**.

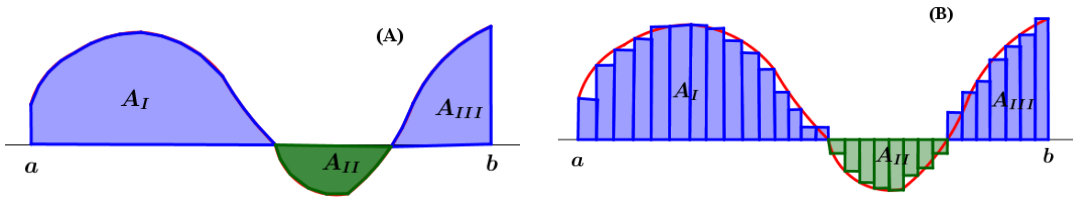


Figure 7.5: net sign area

For example, in Figure 7.5, the net signed area between the curve $y = f(x)$ and the interval $[a, b]$ is

$$(A_I + A_{III}) - A_{II} = [\text{ area above } [a, b]] - [\text{ area below } [a, b]]$$

Definition 3 (Net Signed Area) If the function f is continuous on $[a, b]$, then the net signed area A between $y = f(x)$ and the interval $[a, b]$ is defined by

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

7.3 Definite Integral

7.3.1 Riemann Sums and the Definite Integral

In previous section, we assumed that for each positive number n, the interval $[a, b]$ was subdivided into n subintervals of equal length to create bases for the approximating rectangles. For some functions it may be more convenient to use rectangles with different widths; however, if we are to exhaust an area with rectangles of different widths, then it is important that successive subdivisions are constructed in such a way that the widths of all the rectangles approach zero as n increases (Figure 7.6-left). Thus, we must preclude the kind of situation that occurs in Figure 7.6-right in which the right half of the interval is never subdivided. If this kind of subdivision were allowed, the error in the approximation would not approach zero as n increased.

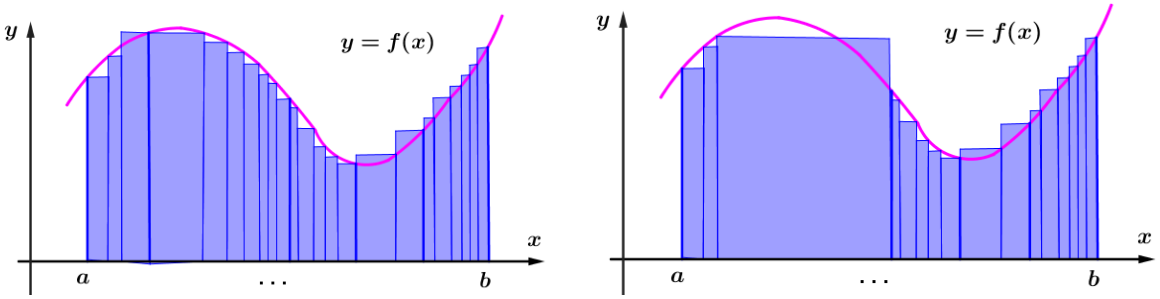


Figure 7.6: Definite integral

A partition of the interval $[a, b]$ is a collection of points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n1} < x_n = b$$

that divides $[a, b]$ into n subintervals of lengths

$$\Delta x_1 = x_1 - x_0, \Delta x_2 = x_2 - x_1, \Delta x_3 = x_3 - x_2, \ldots, \Delta x_n = x_n - x_{n-1}$$

The partition is said to be regular provided the subintervals all have the same length

$$\Delta x_k = \Delta x = \frac{b-a}{n}.$$

For a regular partition, the widths of the approximating rectangles approach zero as n is made large. Since this need not be the case for a general partition, we need some way to measure the size of these widths. One approach is to let $\max \Delta x_k$ denote the largest of the subinterval widths. The magnitude $\max \Delta x_k$ is called the **mesh size** of the partition. For example, Figure 7.7 shows a partition of the

interval $[0, 6]$ into four subintervals with

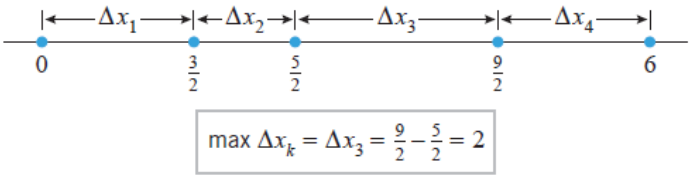


Figure 7.7: partition of [0,6]

where $\Delta x_1 = \ldots\ldots\ldots, \Delta x_2 = \ldots\ldots\ldots, \Delta x_3 = \ldots\ldots\ldots, \ldots, \Delta x_n = \ldots\ldots\ldots$

If we are to generalize Definition 2 so that it allows for unequal subinterval widths, we must replace the constant length Δx by the variable length Δx_k. When this is done the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x \text{ is replaced by } \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

We also need to replace the expression $n \to \infty$ by an expression that guarantees us that the lengths of all subintervals approach zero. We will use the expression $\max \Delta x_k \to 0$ for this purpose.

Definition 4 A function f is said to be integrable on a finite closed interval $[a, b]$ if the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

which is called the ***definite integral*** of f from a to b. The numbers a and b are called the ***lower limit of integration*** and the ***upper limit of integration*** , respectively, and $f(x)$ is called the ***integrand*** .

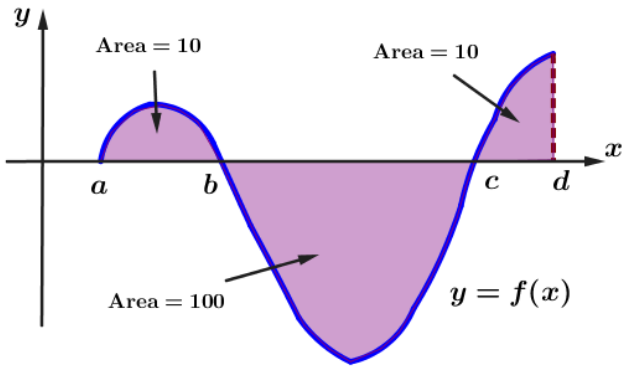
Theorem 7.2 If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$, and the net signed area A between the graph of f and the interval $[a, b]$ is

$$A = \int_a^b f(x)dx.$$

Example 7.2 Use the areas shown in the figure to find

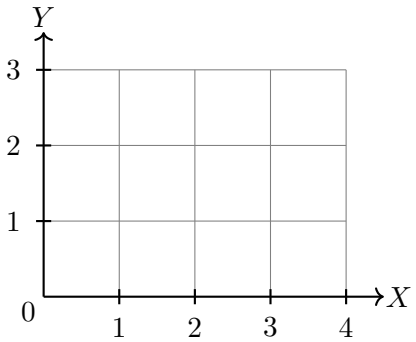
(a) $\int_a^b f(x)dx$ (b) $\int_b^c f(x)dx$ (c) $\int_a^c f(x)dx$ (d) $\int_a^d f(x)dx$

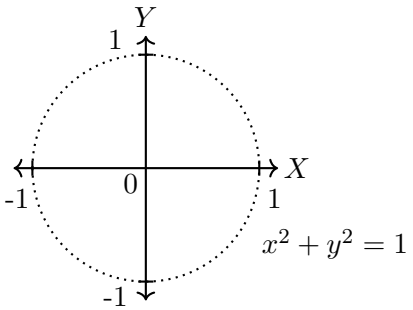
Solution



Example 7.3 Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

(a) $\int_1^4 2\ dx$ (b) $\int_0^1 \sqrt{1-x^2}\ dx$





7.3.2 Properties of the Definite Integral

Theorem 7.3 (a) If a is in the domain of f , we define

$$\int_a^a f(x)dx = 0$$

(b) If f is integrable on $[a, b]$, then we define

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Example 7.4

(a) $\int_1^1 (\sin 1 - x^2)dx =$

(b) $\int_1^0 (\sqrt{1-x^2})dx =$

Theorem 7.4 If f and g are integrable on $[a, b]$ and if c is a constant, then cf, $f+g$, and $f-g$ are integrable on $[a, b]$ and

(a) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

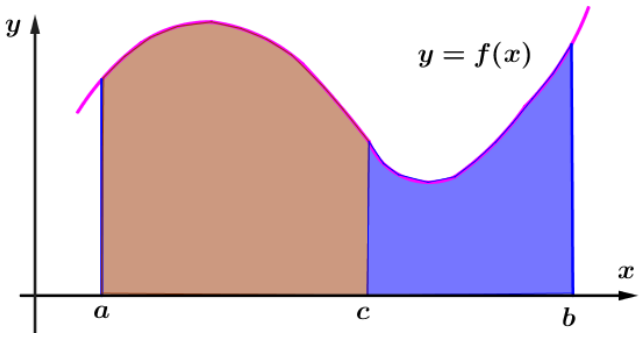
(b) $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(c) $\int_a^b f(x) - g(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

Example 7.5 Evaluate $\int_0^1 (5 - 3\sqrt{1-x^2})dx$

Theorem 7.5 If f is integrable on a closed interval containing the three points a, b, and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

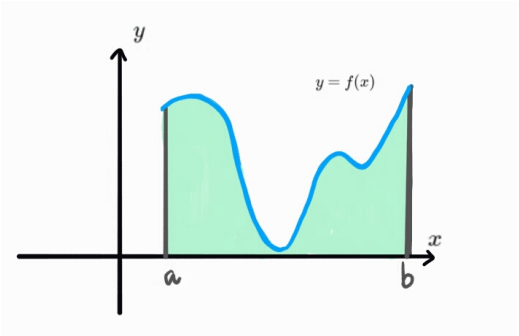


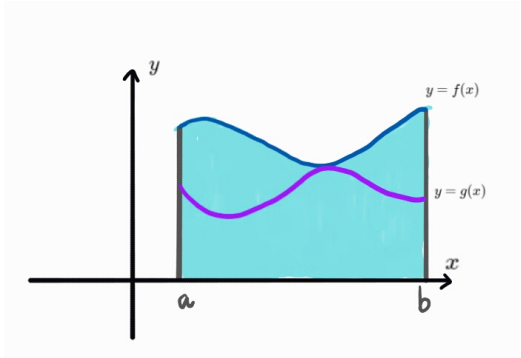
Theorem 7.6 (a) If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \geq 0$$

(b) If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$





7.4 Fundamental Theorem of Calculus

7.4.1 Part I of the Fundamental Theorem of Calculus

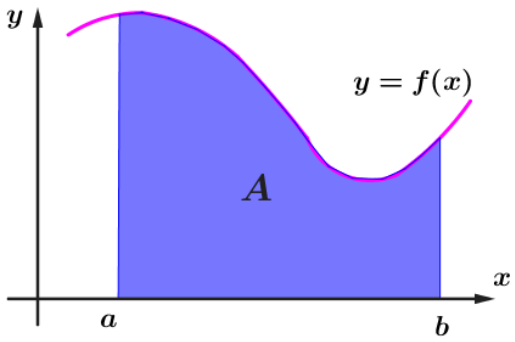


Figure 7.8: area under the graph

Assume that f is a non-negative continuous function on the interval $[a, b]$, the area A under the graph of f over the interval $[a, b]$ is represented by the definite integral

$$A = \int_a^b f(x)dx$$

(Figure 7.8). If $A(x)$ denotes the area under the graph of f over the interval $[a, x]$, where x is any point in the interval $[a, b]$ (Figure 7.9), then

$$A'(x) = f(x) \tag{7.2}$$

The following example confirms Formula (7.2) in some cases where a formula for $A(x)$ can be found using elementary geometry.

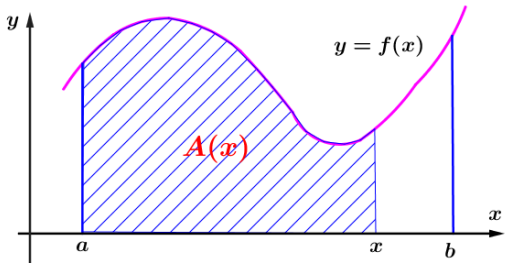
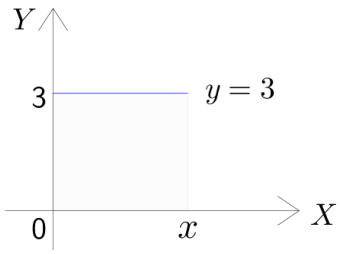


Figure 7.9: area under the graph from a to x

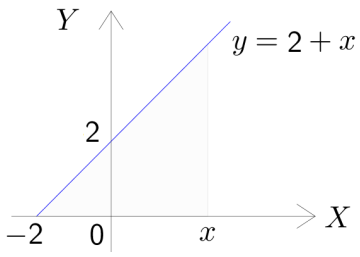
Example 7.6 For each of the functions f, find the area $A(x)$ between the graph of f and the interval $[a, x]$, and find the derivative $A'(x)$ of this area function.

Solution

(a) $f(x) = 3; \quad a = 0$



(b) $f(x) = 2 + x; \quad a = -2$



The procedure for finding areas via antidifferentiation is called the ***antiderivative method*** .

Recap that if $A(x)$ is the area under the graph of f from a to x (Figure 7.9), then

- $A'(x) = f(x)$
- $A(a) = 0$ (The area under the curve from a to a is the area above the single point a, and hence is zero.)
- $A(b) = A$ (The area under the curve from a to b is A.)

The formula $A'(x) = f(x)$ states that $A(x)$ is an antiderivative of $f(x)$, which implies that every other antiderivative of $f(x)$ on $[a, b]$ can be obtained by adding a constant to $A(x)$. Accordingly, let

$$F(x) = A(x) + C$$

be any antiderivative of $f(x)$, and consider what happens when we subtract $F(a)$ from $F(b)$:

$$F(b) - F(a) = [A(b) + C] - [A(a) + C] = A(b) - A(a) = A - 0 = A = \int_a^b f(x)dx$$

Theorem 7.7 (The Fundamental Theorem of Calculus, Part 1) If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example 7.7 Evaluate $\int_1^2 xdx$.

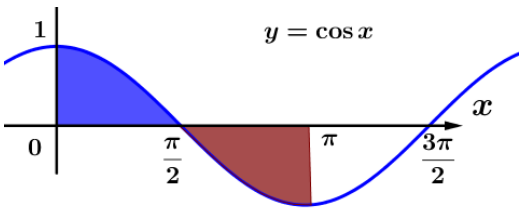
Example 7.8

(a) Find the area under the curve $y = cosx$ over the interval $[0, \pi/2]$.

(b) Make a conjecture about the value of the integral

$$\int_0^\pi \cos x dx$$

and confirm your conjecture using the Fundamental Theorem of Calculus.



7.4.2 Relationship between Definite and Indefinite Integrals

Let F be any antiderivative of the integrand on $[a, b]$, and let C be any constant, then

$$\int_a^b f(x)dx = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

Thus, for purposes of evaluating a definite integral we can omit the constant of integration in

$$\int_a^b f(x)dx = [F(x) + C]_a^b = \left[\int f(x)dx\right]_a^b$$

which relates the definite and indefinite integrals.

Example 7.9

(a) $\int_4^9 x^2\sqrt{x}\ dx =$

(b) $\int_0^{\ln 3} 5e^x\ dx =$

(c) $\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\ dx =$

7.4.3 Part 2 of the Fundamental Theorem of Calculus

Theorem 7.8 If f is continuous on an interval, then f has an antiderivative on that interval. In particular, if a is any point in the interval, then the function F defined by

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of f ; that is, $F'(x) = f(x)$ for each x in the interval, or in an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Example 7.10 Find $\frac{d}{dx}\left[\int_1^x t^3 dt\right]$

7.4.4 Evaluating Definite Integrals by Substitution

Two Methods for Making Substitutions in Definite Integrals

A definite integral of the form

$$\int_a^b [f(u(x))u'(x)]dx,$$

we need to account for the effect that the substitution has on the x-limits of integration. There are two ways of doing this.

Method 1. First evaluate the indefinite integral

$$\int [f(u(x))u'(x)]dx$$

by substitution, and then use the relationship

$$\int_a^b [f(u(x))u'(x)]dx = \left[\int [f(u(x))u'(x)]dx\right]_a^b,$$

Method 2. Make the substitution directly in the definite integral, and then replace the x-limits, $x = a$ and $x = b$, by corresponding u-limits, $u(a)$ and $u(b)$. This produces a new definite integral

$$\int_a^b [f(u(x))u'(x)]dx = \int_{u(a)}^{u(b)} [f(u)]du$$

that is expressed entirely in terms of u.

Example 7.11 Use the two methods above to evaluate $\int_0^2 x(x^2+5)^3 dx$

Solution by Method 1.

Solution by Method 2.

Example 7.12 Evaluate

(a) $\displaystyle\int_0^{3/4} \frac{1}{1-x}dx$

(b) $\displaystyle\int_0^{\ln 3} e^x(1+e^x)^{1/2}dx$

7.4.5 Integration by Parts for Definite Integrals

Example 7.13 Evaluate $\displaystyle\int_0^1 \arctan x \; dx$.

Exercise 7

Compute the following integrations.

1. $\displaystyle\int_0^2 [3x^2 + x - 5]\,dx$
2. $\displaystyle\int_\pi^{2\pi} [\sin x \cos x]\,dx$
3. $\displaystyle\int_0^2 [g(t)]\,dt$ where $g(t) = \begin{cases} t, & 0 \le t < 1 \\ \sin \pi t, & 1 \le t \le 2 \end{cases}$
4. Let $\displaystyle\int_1^2 f(x)\,dx = -4$, $\displaystyle\int_1^5 f(x)\,dx = 6$, $\displaystyle\int_1^5 g(x)\,dx = 8$. Find
   (a) $\displaystyle\int_1^2 5f(x)\,dx$
   (b) $\displaystyle\int_2^5 f(x)\,dx$
   (c) $\displaystyle\int_1^5 [3f(x) - g(x)]\,dx$
5. Define $F(x)$ by
$$\int_1^x [t^3 + 1]dt$$
   (a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.
   (b) Check the result in part (a) by first integrating and then differentiating.
6. Define $F(x)$ by
$$\int_4^x \left[\frac{1}{\sqrt{t}}\right]dt$$
   (a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.
   (b) Check the result in part (a) by first integrating and then differentiating.

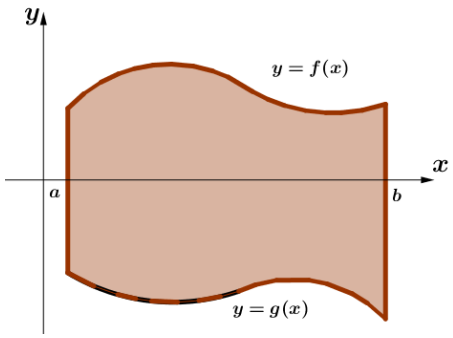
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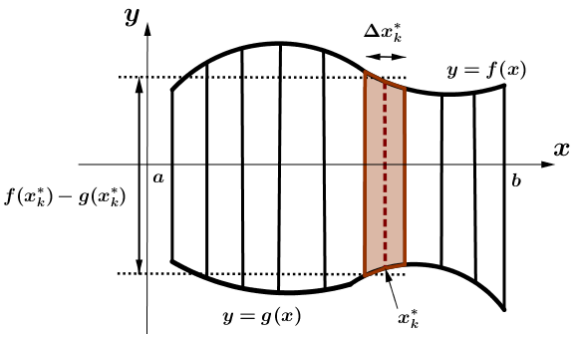
Applications of the Definite Integral in Geometry

8.1 Area between Two Curves

Theorem 8.1 If f and g are continuous functions on the interval $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$. Then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

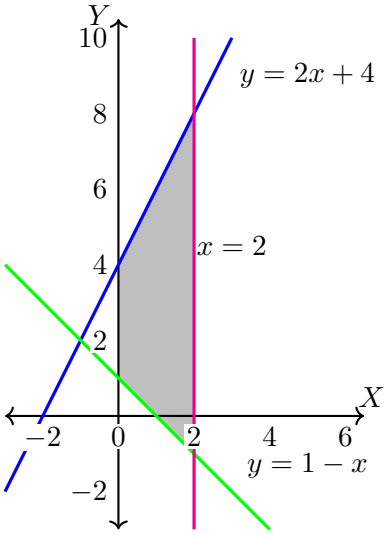
$$A = \int_a^b [f(x) - g(x)]dx. \tag{8.1}$$



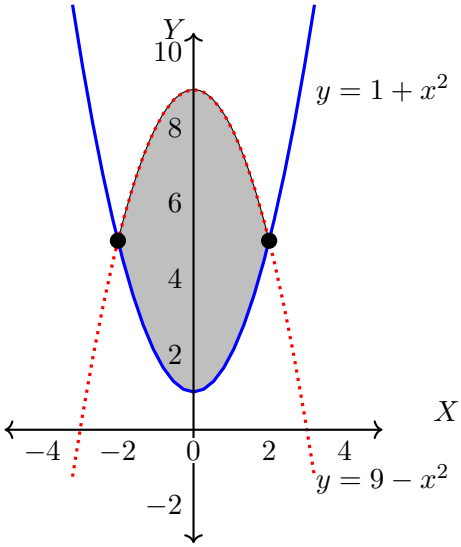


$$A = \lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\,\Delta x_k = \int_a^b [f(x) - g(x)]dx$$

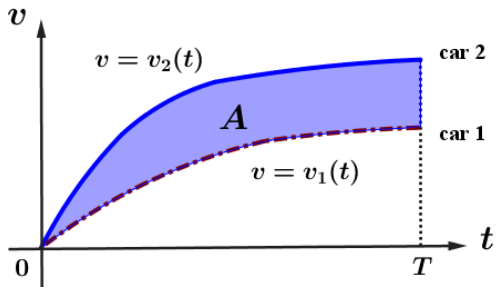
Example 8.1 Find the area of the region bounded above by $y = 2x + 4$, bounded below by $y = 1 - x$, and bounded on the sides by the line $x = 0$ and $x = 2$.



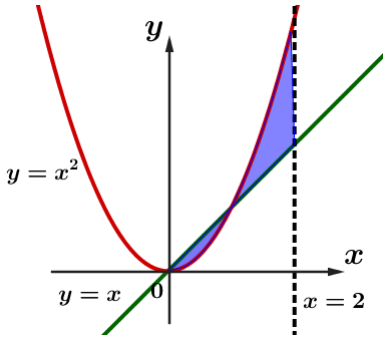
Example 8.2 Find the area of the region enclosed by $y = 9 - x^2$ and $y = 1 + x^2$.



Example 8.3 Figure shows velocity versus time curves for two race cars that move along a straight track, starting from rest at the same time. Give a physical interpretation of the area A between the curves over the interval $0 \leq t \leq T$.



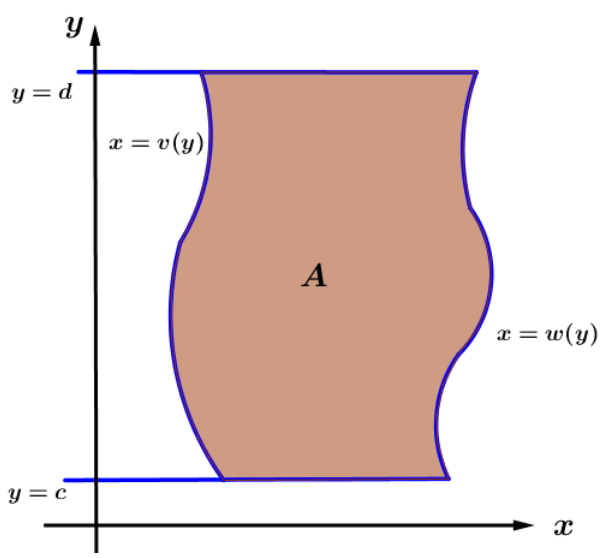
Example 8.4 Find the area of the region enclosed by $y = x$, $y = x^2$, $x = 0$ and $x = 2$.

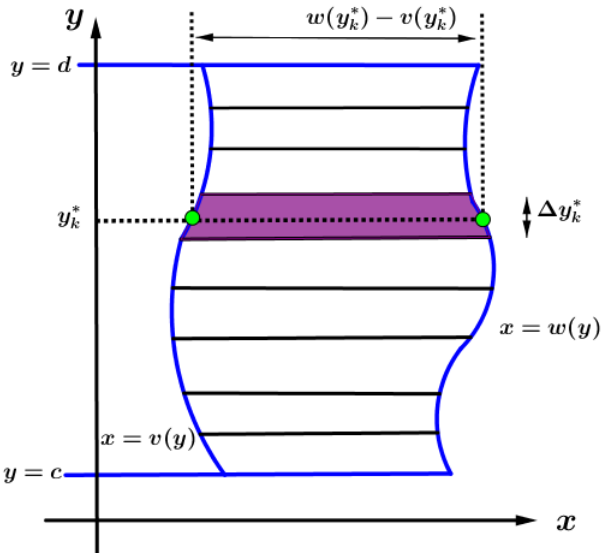


Reversing the Roles of x and y

Theorem 8.2 If w and v are continuous functions on the interval [c, d] and $w(y) \geq v(y)$ for all y in [c, d]. Then the area of the region bounded on the right by $x = w(y)$, on the left by $x = v(y)$, below by the line $y = c$, and above by the line $y = d$ is

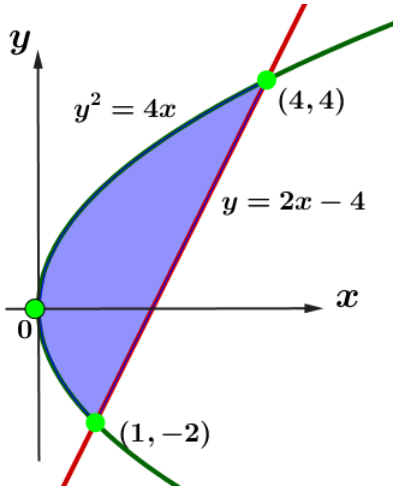
$$A = \int_c^d [w(y) - v(y)]dy \tag{8.2}$$





$$A = \lim_{max\Delta y_k \to 0} \sum_{k=1}^{n} [w(y_k^*) - v(y_k^*)]\, \Delta y_k = \int_c^d [w(y) - v(y)]dy$$

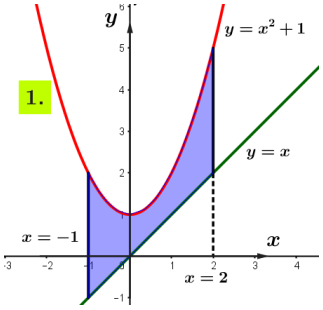
Example 8.5 Find the area of the region enclosed by $y^2 = 4x$ and $y = 2x - 4$.



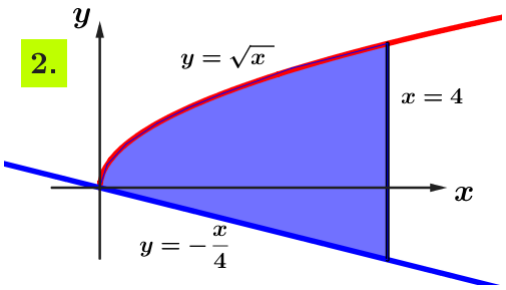
Exercise 8a

1-4 Find the area of the shaded regions.

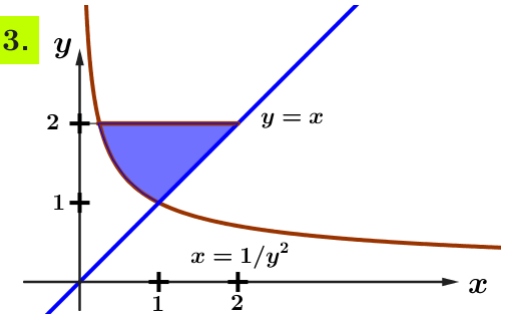
1.



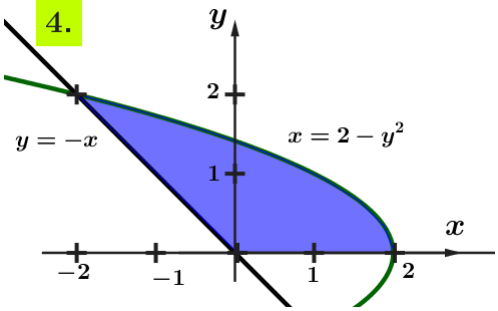
2.



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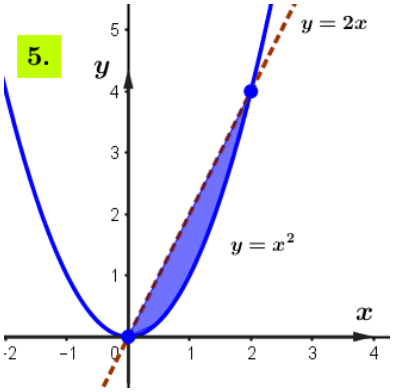
4.



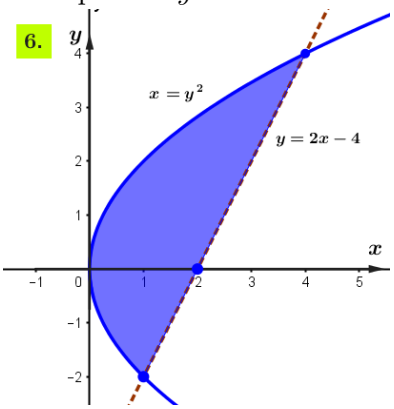
5-6 Find the area of the shaded region by

(a) integrating with respect to x (b) integrating with respect to y.

5.



6.



7. $y = x^2, \quad y = \sqrt{x}, \quad x = \dfrac{1}{4}, \quad x = 1.$

8. $y = x^3 - 4x, \quad y = 0, \quad x = 0, \quad x = 2.$

9. $y = \cos 2x, \quad y = 0, \quad x = \pi/4, \quad x = \pi/2.$

10. $y = \sec^2 x, \quad y = 2, \quad x = -\pi/4, \quad x = \pi/4.$

11. $y = \sin y, \quad x = 0, \quad y = \pi/4, \quad y = 3\pi/4.$

12. $x^2 = y, \quad x = y - 2.$

13. $y = e^x, \quad y = e^{2x}, \quad x = 0, \quad x = \ln 2.$

14. $x = 1/y, \quad x = 0, \quad y = 1, \quad y = e.$

15. $y = 2/(1 + x^2), \quad y = |x|.$

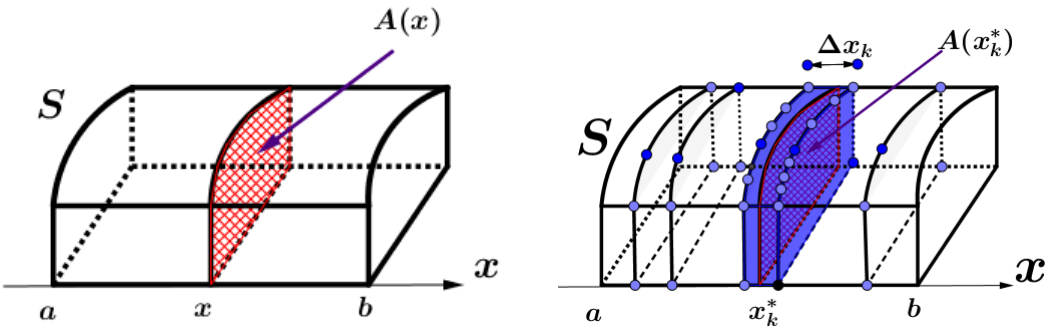
16. $y = 1/\sqrt{1 - x^2}\,, \quad y = 2.$

17. $y = x, \quad y = 4x, \quad y = -x + 2.$

8.2 Volumes by Slicing; Disks and Washers

Theorem 8.3 (**Volume formula**) Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in [a, b], the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)dx. \tag{8.3}$$



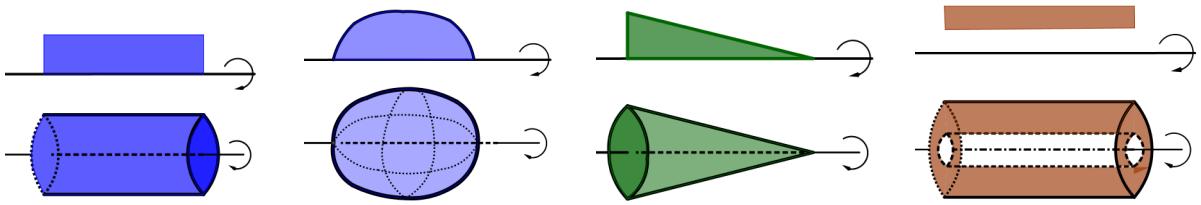
$$V = \lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} A(x_k^*)\Delta x_k = \int_a^b A(x)dx$$

There is a similar result for cross sections perpendicular to the y-axis.

Theorem 8.4 (**Volume formula**) Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in [c, d], the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

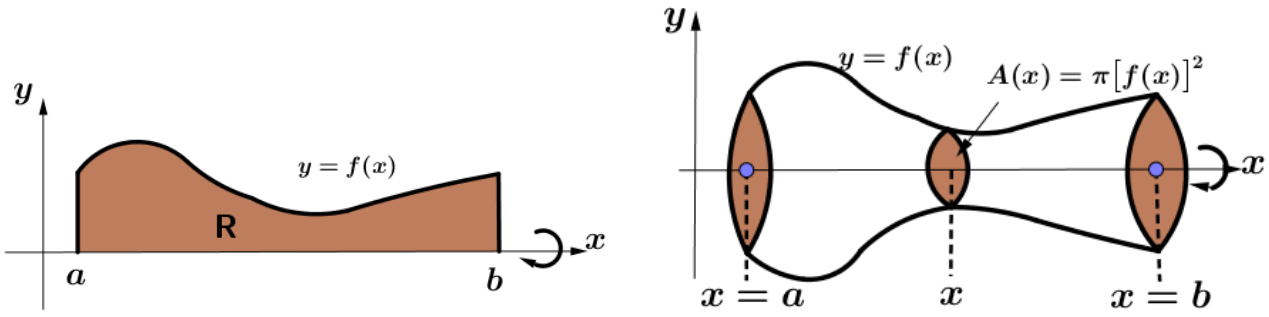
$$V = \int_c^d A(y)dy. \tag{8.4}$$

Solid of Revolution



Volume by Disks perpendicular to the X-axis

Problem 3 Let f be continuous and nonnegative on [a, b], and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. Find the volume of the solid of revolution that is generated by revolving the region R about the X-axis.



We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the X-axis at the point x is a circular disk of radius $f(x)$. The area of this region is

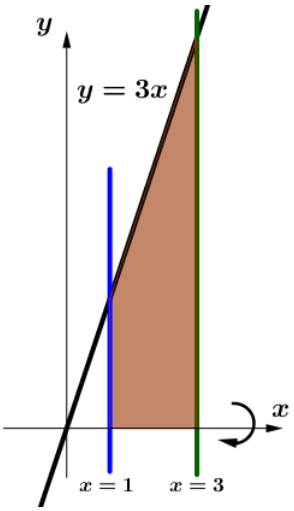
$$A(x) = \pi[f(x)]^2.$$

Thus, from (8.3) the volume of the solid is

$$V = \int_a^b \pi[f(x)]^2 dx. \tag{8.5}$$

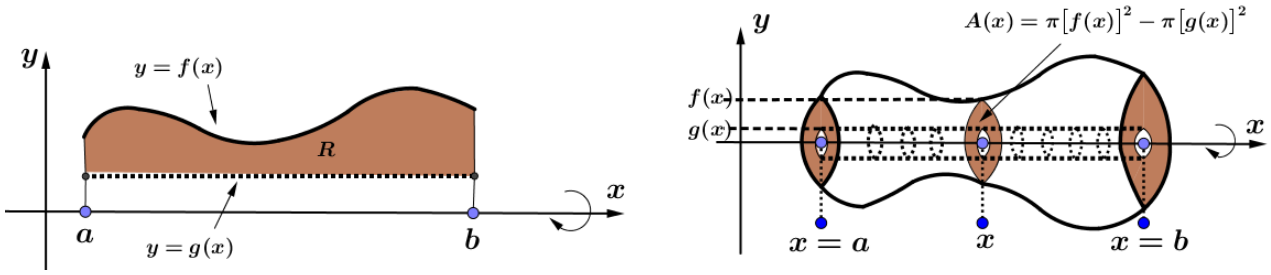
Because the cross sections are disk shaped, the application of this formula is called the ***method of disks.***

Example 8.6 Find the volume of the solid that is obtained when the region under the curve $y = 3x$ over the interval $[1, 3]$ is revolved about the X-axis.



Volume by Washers Perpendicular to the X-axis

Problem: Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$. Find the volume of the solid of revolution that is generated by revolving the region R about the X-axis.



We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the X-axis at the point x is the annular or "washer-shaped", region with inner radius $g(x)$ and outer radius $f(x)$. The area of this region is

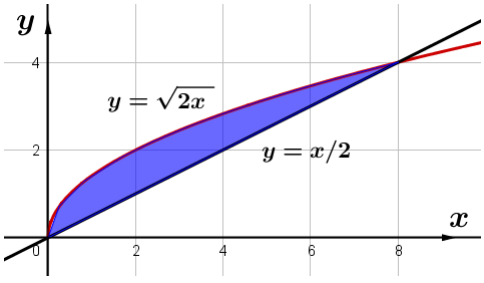
$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 = \pi\left([f(x)]^2 - [g(x)]^2\right)$$

Thus, from (8.3) the volume of the solid is

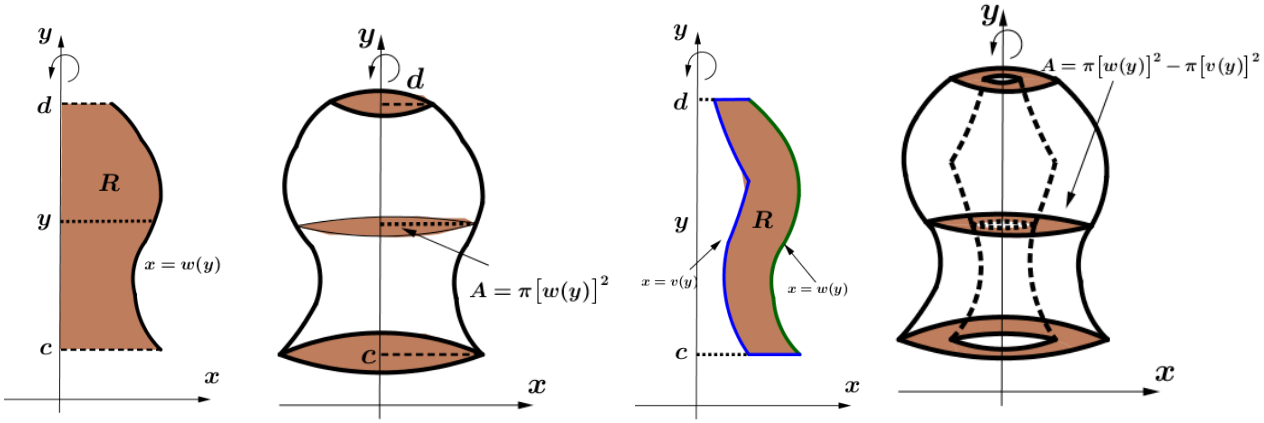
$$V = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right) dx \tag{8.6}$$

Because the cross sections are washer shaped, the application of this formula is called the ***method of washers.***

Example 8.7 Find the volume of the solid that is obtained when the region between the graphs of the equations $y = \sqrt{2x}$ and $y = \frac{x}{2}$ over the interval $[0, 8]$ is revolved about the X-axis.



Volume by Disks and Washers perpendicular to the Y-axis

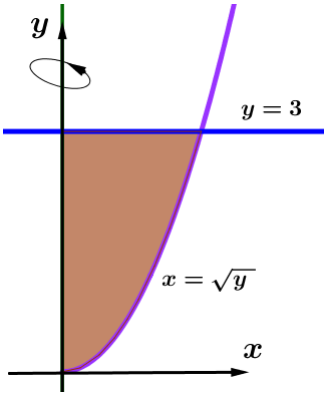


The methods of disks and washers have analogs for regions that are revolved about the Y-axis. Using the method of slicing and Formula (8.4), the following formulas for the volumes of the solid are

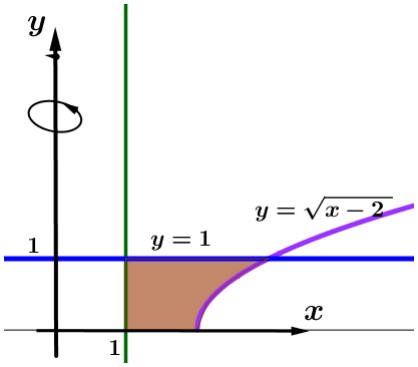
$$V = \int_c^d \pi[w(y)]^2 dy \qquad (disks), \tag{8.7}$$

$$V = \int_c^d \pi\left([w(y)]^2 - [v(y)]^2\right) dy \qquad (washers). \tag{8.8}$$

Example 8.8 Find the volume of the solid generated when the region enclosed by $x = \sqrt{y}$, $x = 0$, and $y = 3$ is revolved about the Y-axis.



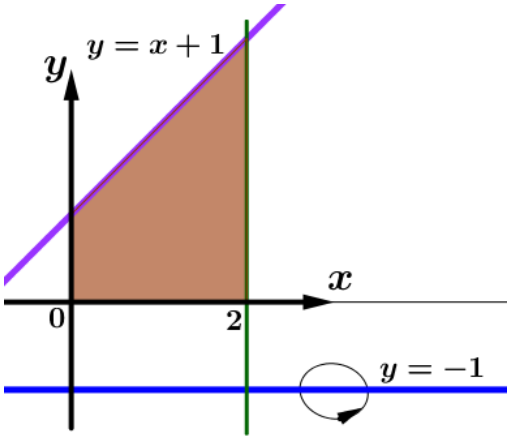
Example 8.9 Find the volume of the solid generated when the region enclosed by $x = 1$, $y = \sqrt{x-2}$, $y = 0$, and $y = 1$ is revolved about the Y-axis.



Other axes of revolution

It is possible to use the method of disks and the method of washers to find the volume of a solid of revolution whose axis of revolution is a line other than one of the coordinate axes. Instead of developing a new formula for each situation, we will appeal to Formulas (8.3) and (8.4) and integrate an appropriate cross-sectional area to find the volume.

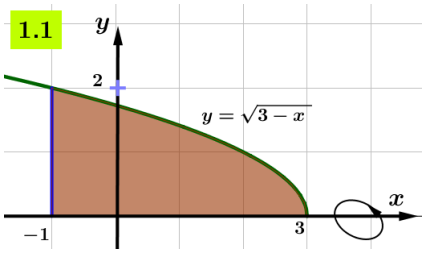
Example 8.10 Find the volume of the solid that is obtained when the region between the curve $y = x + 1$ and $y = 0$ over the interval $[0, 2]$ is rotated about the line $y = -1$.



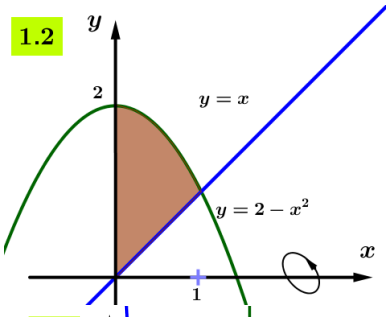
Exercise 8b

1. Find the volume of the solid that results when the shaded region is revolved about the indicated axis.

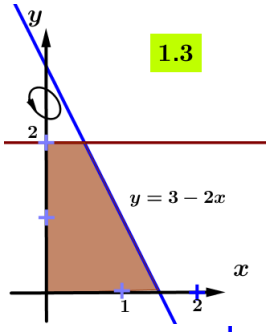
1.1



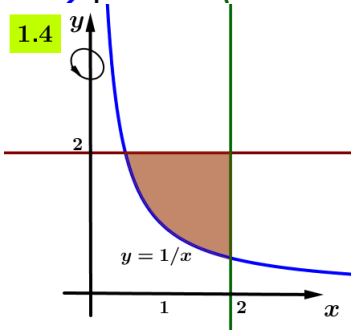
1.2



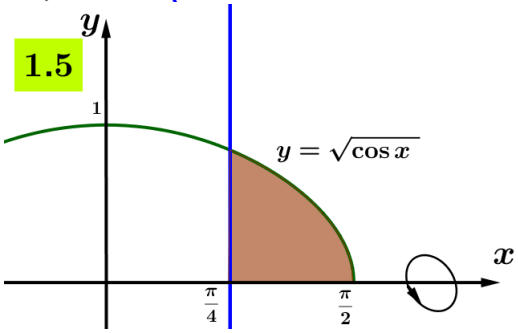
1.3



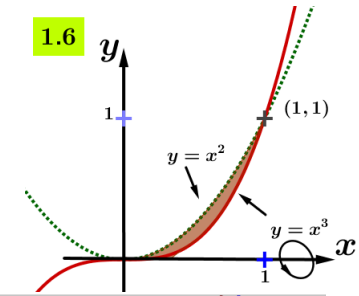
1.4



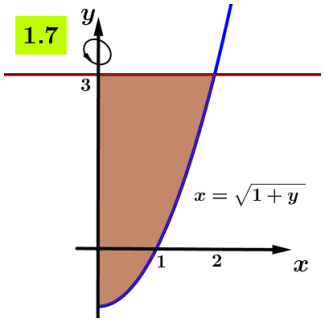
1.5



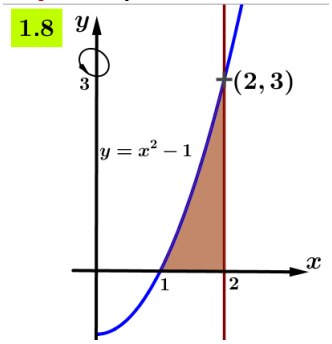
1.6



1.7



1.8



2. Find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis.

   (a) $y = \sqrt{25 - x^2}, y = 3$

   (b) $y = 9 - x^2, y = 0$

   (c) $x = \sqrt{y}, x = y/4$

   (d) $y = e^x, y = 0, x = 0, x = \ln 3$

   (e) $y = e^{-2x}, y = 0, x = 0, x = 1$

3. Find the volume of the solid that results when the region enclosed by the given curves is revolved about the y-axis.

   (a) $y = \csc y, y = \pi/4, y = 3\pi/4, x = 0$

   (b) $y = x^2, x = y^2$

   (c) $x = y^2, x = y + 2$

   (d) $x = 1 - y^2, x = 2 + y^2, y = -1, y = 1$

   (e) $y = \ln x, x = 0, y = 0, y = 1$

4. Find the volume of the solid that results when the region enclosed by $y = \sqrt{x}, y = 0$, and $x = 9$ is revolved about the line $x = 9$.

5. Find the volume of the solid that results when the region in Problem 4 is revolved about the line $x = 9$.

6. Find the volume of the solid that results when the region enclosed by $x = y^2$ and $x = y$ is revolved about the line $y = -1$.

7. Find the volume of the solid that results when the region in Problem 6 is revolved about the line $x = -1$.

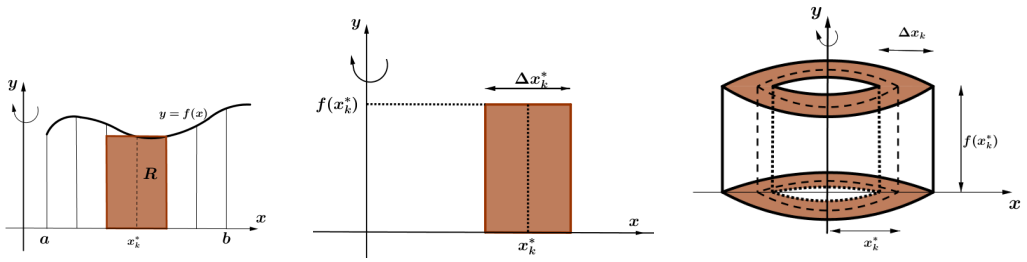
8. Find the volume of the solid that results when the region enclosed by $y = x^2$ and $y = x^3$ is revolved about the line $x = 1$.

9. Find the volume of the solid that results when the region in the problem of item 8 is revolved about the line $y = -1$.

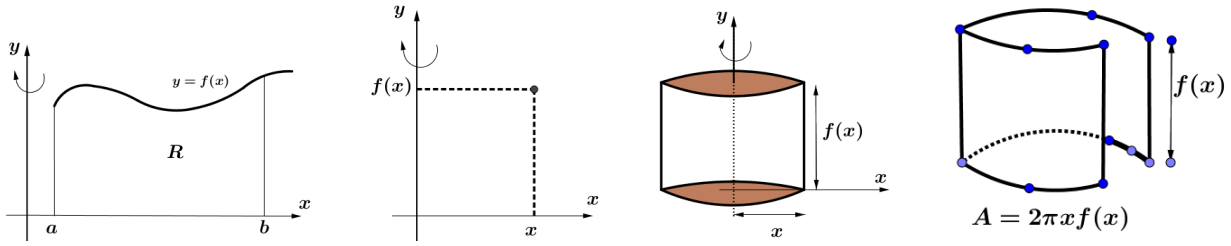
8.3 Volumes by Cylindrical Shells

Theorem 8.5 (Volume by cylindrical shells about the Y-axis) Let f be continuous and non-negative on [a, b] and let R be the region that is bounded above by $y = f(x)$, below by the X-axis, and on the sides by the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution that is generated by revolving the region R about the Y-axis is given by

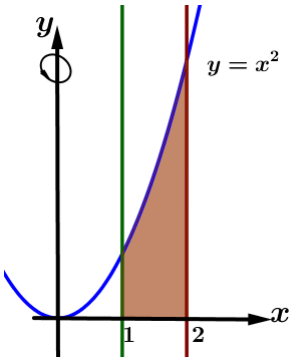
$$V = \int_a^b 2\pi x f(x)dx. \tag{8.9}$$



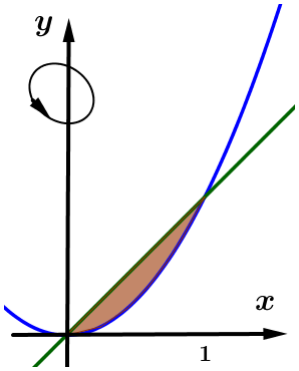
$$V = \lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*)\Delta x_k = \int_a^b 2\pi x f(x)dx.$$



Example 8.11 Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y = x^2$, $x = 1$, $x = 2$ and the X-axis is revolved about the Y-axis.

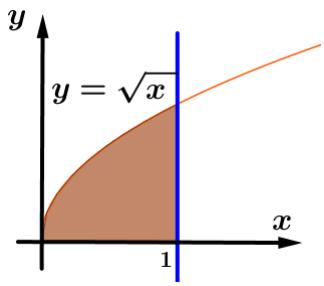


Example 8.12 Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed between $y = x$ and $y = x^2$ is revolved about the Y-axis.



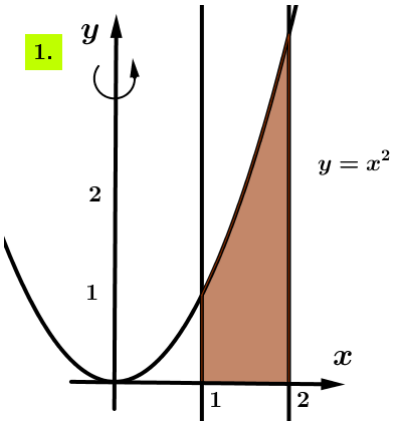
Example 8.13 Use cylindrical shells to find the volume of the solid generated when the region R under $y = \sqrt{x}$ over the interval $[0, 1]$ is revolved about

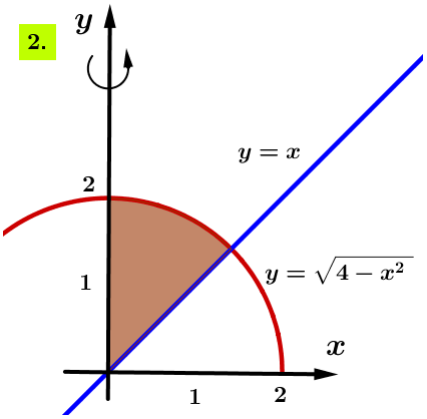
1. line $y = -1$.
2. x-axis.
3. y-axis.

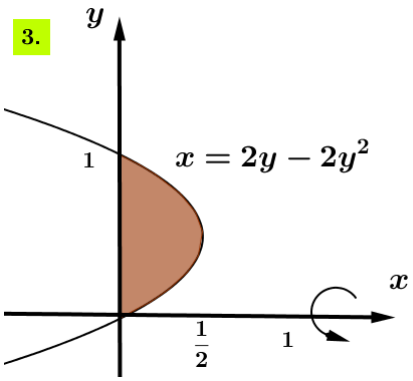


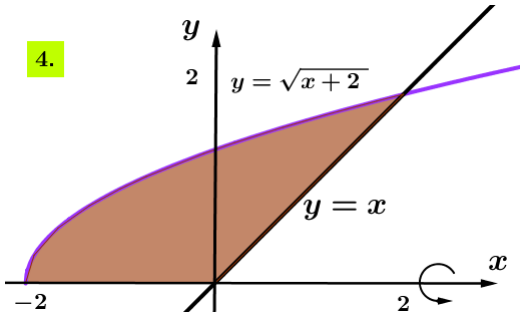
Exercise 8c

1. Use cylindrical shells to find the volume of the solid generated when the shaded region is revolved about the indicated axis.









2. Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the y-axis.

   (a) $y = x^3, x = 1, y = 0$

   (b) $y = \sqrt{x}, x = 4, x = 9, y = 0$

   (c) $y = 1/x, y = 0, x = 1, x = 3$

   (d) $y = \cos(x^2), x = 0, x = \frac{1}{2}\sqrt{\pi}, y = 0$

   (e) $y = 2x - 1, y = -2x + 3, x = 2$

   (f) $y = 2x - x^2, y = 0$

3. Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the x-axis.

   (a) $y^2 = x, y = 1, x = 0$

   (b) $x = 2y, y = 2, y = 3, x = 0$

   (c) $y = x^2, x = 1, y = 0$

   (d) $xy = 4, x + y = 5$

4. Using the method of cylindrical shells, set up but do not evaluate an integral for the volume of the solid generated when the region R is revolved about (A) the line $x = 1$ and (B) the line $y = -1$.

   (a) R is the region bounded by the graphs of $y = x$, $y = 0$, and $x = 1$.

   (b) R is the region in the first quadrant bounded by the graphs of $y = \sqrt{1 - x^2}, y = 0$ and $x = 0$.

5. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = 1/x^3, x = 1, x = 2, y = 0$ is evolved about the line $x = -1$.

6. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = x^3, y = 1, x = 0$ is evolved about the line $y = 1$.

8.4 Improper Integrals

It is assumed in the definition of the definite integral $\int_a^b f(x)\,dx$ that $[a,b]$ is a finite interval and that the limit that defines the integral exists; that is, the function f is integrable.

Our main objective is to extend the concept of definite integrals for infinite intervals of integration and for integrands with vertical asymptotes within the interval of integration. If a function f has a vertical asymptote, then f is said to have an ***infinite discontinuity*** .

An integral over an infinite interval of integration or an integral with an infinite discontinuity will be called an ***improper integral*** .

There are three types of improper integrals:

1. Improper integrals with infinite intervals of integration.
2. Improper integrals with infinite discontinuities in the interval of integration.
3. Improper integrals with infinite discontinuities over infinite intervals of integration.

Example 8.14 Determine if each of the following integrals is improper. If so, specify its type.

1. $\displaystyle\int_0^{+\infty} \frac{1}{1-x^2}\,dx$

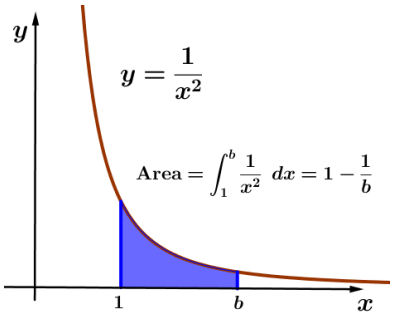
2. $\displaystyle\int_{-\infty}^{+\infty} x^3\,dx$

3. $\displaystyle\int_0^{\pi} \sec^2\theta\,d\theta$

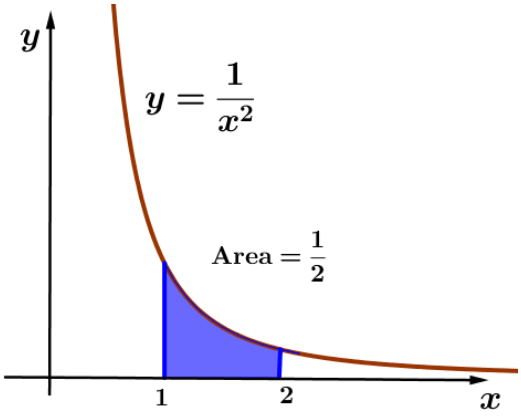
4. $\displaystyle\int_{-3}^{3} \frac{x}{x^2+x+1}\,dx$

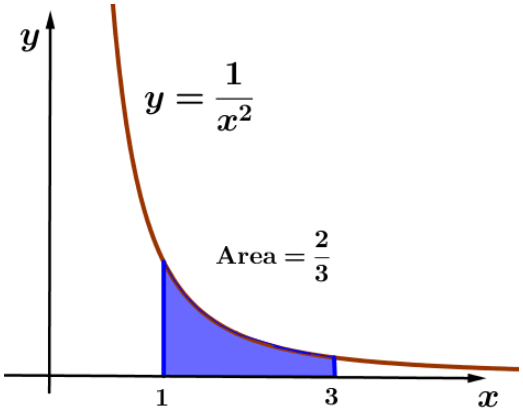
8.4.1 Integrals over Infinite Intervals :

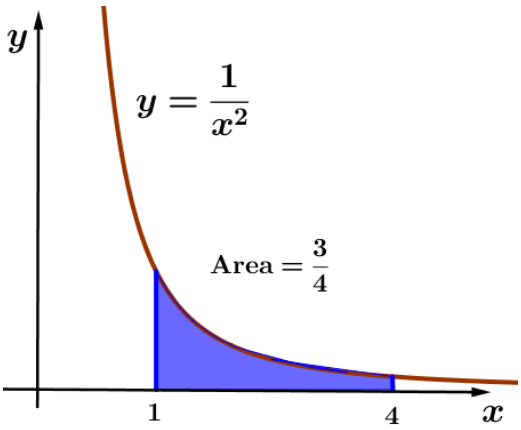
Suppose we are interested in the area A of the region that lies below the curve $y = 1/x^2$ and above the interval $[1, +\infty)$ on the x-axis. Let us begin by calculating the portion of the area that lies above a finite interval $[1, b]$, where $b > 1$ is arbitrary. That area is $\int_1^b \frac{dx}{x} = 1 - \frac{1}{b}$

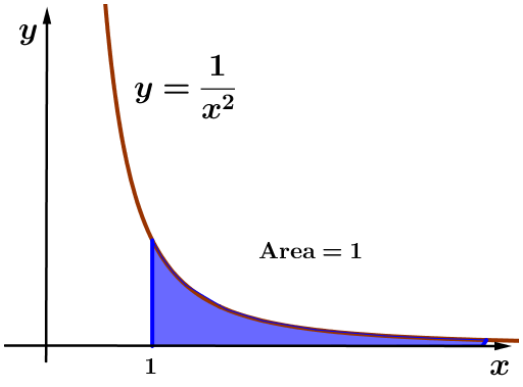


If we now allow b to increase so that $b \to +\infty$, then the portion of the area over the interval $[1, b]$ will begin to fill out the area over the entire interval $[1, +\infty)$, and hence we can reasonably define the area A under $y = 1/x^2$ over the interval $[1, +\infty)$ to be $A = \int_1^\infty \frac{dx}{x} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x} = \lim_{b\to+\infty}(1 - \frac{1}{b}) = 1$









Definition 5 The ***improper integral*** of f over the interval $[a, +\infty)$ is defined to be

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx,$$

The integral is said to converge if the limit exists and diverge if it does not.

The improper integral of f over the interval $(-\infty, b]$ is defined to be

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx,$$

The integral is said to converge if the limit exists and diverge if it does not.

The improper integral of f over the interval $(-\infty, +\infty)$ is defined as

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx$$

where c is any real number. The improper integral is said to converge if both terms converge and diverge if either term diverges.

Example 8.15 Compute $\displaystyle\int_1^{+\infty} \frac{1}{x^3}\, dx$.

Example 8.16 Compute $\int_0^{+\infty} \cos x \; dx$.

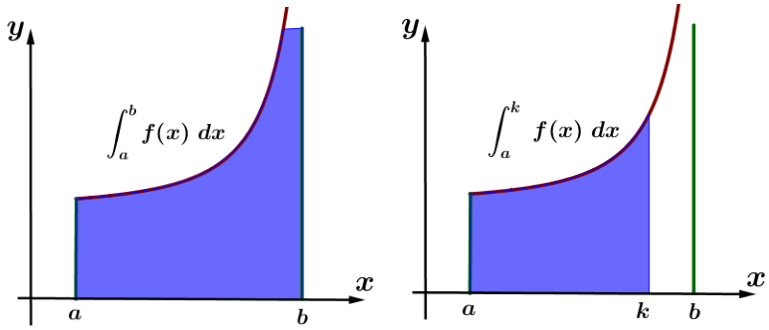
Example 8.17 Compute $\int_1^{\infty} \frac{\ln x}{x} dx$.

Example 8.18 Compute $\int_{-\infty}^{1} \frac{1}{3-2x} dx$.

Example 8.19 Compute $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1}dx$.

8.4.2 Integrals whose Integrands have Infinite Discontinuities:

Let us consider the case where f is nonnegative on $[a, b]$, so we can interpret the improper integral $\int_a^b f(x)\ dx$ as the area of the region. The problem of finding the area of this region is complicated by the fact that it extends indefinitely in the positive y-direction. However, instead of trying to find the entire area at once, we can proceed indirectly by calculating the portion of the area over the interval $[a, k]$, where $a \leq k < b$, and then letting k approach b to fill out the area of the entire region.



Definition 6 If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at b, then the improper integral of f over the interval $[a, b]$ is defined as

$$\int_a^b f(x)dx = \lim_{k\to b^-} \int_a^k f(x)dx,$$

The integral is said to converge if the indicated limit exists and diverge if it does not.

If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a, then the improper integral of f over the interval $[a, b]$ is defined as

$$\int_a^b f(x)dx = \lim_{k\to a^+} \int_k^b f(x)dx,$$

The integral is said to converge if the indicated limit exists and diverge if it does not.

If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a point c in (a, b), then the improper integral of f over the interval $[a, b]$ is defined as

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

where the two integrals on the right side are themselves improper. The improper integral on the left side is said to converge if both terms on the right side converge and diverge if either term on the right side diverges.

Example 8.20 Compute $\int_0^1 \frac{1}{1-x}\ dx$.

Example 8.21 Compute $\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

The following example is an improper integral with infinite discontinuity over infinite intervals of integration. Remark that it combines the first and second type of improper integral. With this type, we should separate the imiproper integral into the first and second types. Then, the convergence will be consiered using the mentioned methods.

Example 8.22 Compute $\int_{-\infty}^{\infty} \frac{dx}{(x-1)^{2/3}}$.

Exercise 8d

Evaluate the following integrals if they converge.

1. $\int_3^4 \frac{1}{(x-3)^2}dx$
2. $\int_1^{+\infty} \frac{dx}{x^{1.001}}$
3. $\int_{-\infty}^0 \theta e^{\theta} d\theta$
4. $\int_2^{+\infty} \frac{2}{v^2 - v}dv$
5. $\int_0^{+\infty} \sin \pi x \ dx$
6. $\int_{-\infty}^{+\infty} \frac{1}{1+4x^2}dx$
7. $\int_0^{+\infty} \frac{xdx}{\sqrt{x+1}}$
8. $\int_{-\infty}^0 \frac{dx}{x^2+4}$
9. $\int_1^{+\infty} \frac{dx}{x^4+x^2}$
10. $\int_{-\infty}^{+\infty} 3^x dx$
11. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$
12. $\int_0^2 \frac{x}{x^2-1}dx$
13. $\int_0^{+\infty} e^{2x}dx$
14. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$
15. $\int_0^2 \frac{dx}{(x-1)^{1/3}}$
16. $\int_1^2 \frac{dx}{(2-x)^{3/4}}$
17. $\int_0^{\pi/2} \frac{xdx}{\sin x^2}$
18. $\int_{-\infty}^{-2} \frac{2}{x^2-1}dx$
19. $\int_0^1 \frac{\ln x}{x}dx$
20. $\int_0^{+\infty} \frac{dx}{(1+x)\sqrt{x}}$
21. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}d\theta$
22. $\int_2^{\infty} \frac{1}{x^2+4}dx$
23. $\int_1^3 x(x^2-4)^{-3}dx$
24. $\int_0^2 \frac{2x+1}{x^2+x-6}dx$
25. $\int_0^4 \frac{\ln\sqrt{x}}{\sqrt{x}}dx$
26. $\int_2^4 \frac{x}{\sqrt[3]{x-2}}dx$
27. $\int_{-1}^3 \frac{1}{\sqrt[3]{x}}dx$
28. $\int_{-1}^1 \frac{1}{\sqrt{|x|}}dx$
29. $\int_0^3 \frac{1}{x^2+2x-3}dx$
30. $\int_{-1}^2 \frac{1}{x^2}\cos\frac{1}{x}dx$
31. $\int_{-1}^2 \frac{1}{x^2-x-2}dx$
32. $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$
33. $\int_0^{\infty} x\mathrm{e}^{-x}dx$
34. $\int_0^{\pi/2} \sec^2 x dx$
35. $\int_0^1 x\ln x dx$
36. $\int_0^4 \frac{1}{(4-x)^{3/2}}dx$
37. $\int_{-\infty}^{\infty} \frac{x}{(x^2+3)^2}dx$

38. $\displaystyle\int_{-\infty}^{\infty} \frac{|1+x|}{x^2+1}dx$

39. $\displaystyle\int_{-\infty}^{0} \frac{1}{(x-8)^{2/3}}dx$

40. $\displaystyle\int_{-\infty}^{0} \frac{1}{(1-x)^{5/2}}dx$

41. $\displaystyle\int_{-\infty}^{0} \mathrm{e}^{3x}dx$

42. $\displaystyle\int_{1}^{\infty} \frac{1}{\sqrt{x}(1+\mathrm{e}^{\sqrt{x}})^2}dx$

43. $\displaystyle\int_{0}^{\infty} \mathrm{e}^{-x}\cos x dx$

44. $\displaystyle\int_{-1}^{\infty} \frac{x}{1+x^2}dx$

45. $\displaystyle\int_{2}^{4} (x-3)^{-7}dx$

46. $\displaystyle\int_{0}^{\infty} \cos x dx$

47. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\mathrm{e}^{x}+\mathrm{e}^{-x}}dx$

48. $\displaystyle\int_{-\infty}^{0} \frac{1}{2x^2+2x+1}dx$

49. $\displaystyle\int_{-\infty}^{-1} \frac{x}{\sqrt{1+x^2}}dx$

50. $\displaystyle\int_{0}^{\infty} \frac{1}{\mathrm{e}^{2x}+\mathrm{e}^{x}}dx$

51. $\displaystyle\int_{-\infty}^{0} \frac{\mathrm{e}^{x}}{3-2\mathrm{e}^{x}}dx$

Applications and Concepts:

52. Find the area of the region between the x-axis and the curve $8/(x^2-4), x>4$.

53. Let R be the region to the right of $x=1$ that is bounded by the x-axis and the curve $y=1/x$. When this region is revolved about the x-axis, it generates a solid whose surface is known as Gabriel's Horn (for reasons that should be clear from the accompanying figure 8.1). Show that the solid has a finite volume but its surface has an infinite area. **Note:** It has been suggested that if one could saturate the interior of the solid with paint and allow it to seep through to the surface, then one could paint an infinite surface with a finite amount of paint. What do you think?

True or False: Determine whether the following statements are true or false. Explain your answer.

54. $\displaystyle\int_{1}^{\infty} x^{-4/3}\,dx$ converges to 3.

55. If f is continuous on $[a,+\infty)$ and $\lim\limits_{x\to+\infty} f(x)=1$, then $\displaystyle\int_{a}^{+\infty} f(x)dx$ converges.

56. $\displaystyle\int_{1}^{2} \frac{1}{x(x-3)}\,dx$ is an improper integral.

57. $\displaystyle\int_{-1}^{1} \frac{1}{x^3}\,dx = 0.$

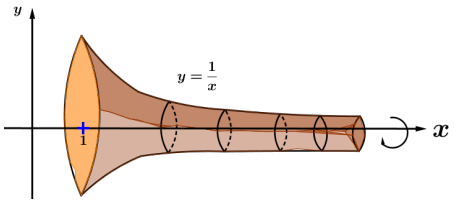


Figure 8.1: Gabriel's Horn

9 Differential Equations

In this chapter, we introduce two methods for solving some form of the first order of differential equations (ODEs). First, we introduce some basic definitions of ODEs. We, then, solve the particular ODEs in the forms of ***Separable equations*** and ***Linear first order ODEs*** . Lastly, some examples of linear first order ODEs.

9.1 Introduction to Ordinary Differential Equations

Consider the equation, $y = 2x^3 - 2x^2 + 5$. By differentiation, it can be shown that

$$\frac{dy}{dx} = 6x^2 - 4x. \tag{9.1}$$

Similarly, for a function $p(x) = 10000e^{-0.04x}$, we have

$$p'(x) = -400e^{-0.04x}. \tag{9.2}$$

These equations are example of ***differential equations*** .

In general, an equation is a **differential equation** if it involves an unknown function and one or more of its derivatives. Other examples of differential equations are

$$\frac{dy}{dx} = ky, \qquad y'' - xy' + x^2 = 5, \qquad \frac{dy}{dx} = 2xy$$

The first and third equations are called **first-order** equations because each involves a first derivative but no higher derivative. The second equation is called a **second-order** equation because it involves a second derivative and no higher derivatives. In general, the ***order*** of a differential equation is the order of the highest derivative that it contains.

9.2 General and Particular Solutions

A ***solution*** of differential equation is the function which matches the differential equation.

Example 9.1 Show that the function $y = e^x$ is a solution of

$$\frac{dy}{dx} - y = 0$$

Example 9.2 Show that, for any constant C, the function $y = e^x - x + C$ is a solution of

$$\frac{dy}{dx} = e^x - 1$$

Remark:

- The ***general solution*** of a differential equation is a solution that contains all possible solutions. The general solution always contains an arbitrary constant.
- The ***particular solution*** of a differential equation is a solution that satisfies the initial condition of the equation. A ***first-order initial value problem*** is a first-order differential equation $y' = f(x, y)$ whose solution must satisfy an initial condition $y(x_0) = y_0$.

Example 9.3 Find the particular solution of

$$\frac{dy}{dx} = e^x - 1, \quad y(0) = 1.$$

Example 9.4 Show that the function

$$y = (x + 1) - \frac{1}{3}e^x$$

is a solution to the first order initial-value problem

$$\frac{dy}{dx} = y - x, \quad y(0) = 2/3.$$

9.3 Separable Equations

We will now consider a method of solution that can often be applied to first-order equations that are expressible in the form

$$h(y)\frac{dy}{dx} = g(x). \tag{9.3}$$

Such first-order equations are said to be ***separable*** . The name separable arises from the fact that (9.3) can be rewritten in the differential form

$$h(y)dy = g(x)dx \tag{9.4}$$

in which the expressions involving x and y appear on opposite sides. To motivate a method for solving separable equations, assume that $h(y)$ and $g(x)$ are continuous functions of their respective variables, and let $H(y)$ and $G(x)$ denote antiderivatives of $h(y)$ and $g(x)$, respectively. Consider the results if we integrate both sides of (9.4), the left side with respect to y and the right side with respect to x. We then have

$$\int h(y)dy = \int g(x)dx, \tag{9.5}$$

or, equivalently,

$$H(y) = G(x) + C \tag{9.6}$$

where C denotes a constant. We claim that a differentiable function $y = y(x)$ is a solution to (9.3) if and only if y satisfies (9.6) for some choice of the constant C.

Example 9.5 Write these first-order differential equation in the separable form.

Equation	Form	$h(y)$	$g(x)$
$\frac{dy}{dx} = \frac{x}{y}$			
$\frac{dy}{dx} = x^2y^3$			
$\frac{dy}{dx} = y$			
$\frac{dy}{dx} = y - \frac{y}{x}$			

Example 9.6 Find the general solution of

$$\frac{dy}{dx} = \frac{x}{y}.$$

Example 9.7 Find the general solution of

$$\frac{dy}{dx} = ye^x.$$

Example 9.8 Find the general solution of

$$\frac{dy}{dx} = \sqrt{xy}\,.$$

Example 9.9 Find the general solution of

$$\frac{dy}{dx} = \frac{xy + y}{xy - x}$$

Example 9.10 Solve the initial value problem

$$\frac{dy}{dx} = -4xy^2, \quad y(0) = 1.$$

Example 9.11 Solve the initial value problem

$$yy' - (x^2 + 1) = 0, \quad y(4) = 2.$$

Example 9.12 Solve the initial value problem

$$(4y - \cos y)\frac{dy}{dx} - 3x^2 = 0, \quad y(0) = 0.$$

9.4 Linear Equations

A first-order differential equation is called **linear** if it is expressible in the form

$$\frac{dy}{dx} + p(x) \cdot y = q(x). \tag{9.7}$$

Some examples of first-order linear differential equations are

$$\begin{aligned}
&\frac{dy}{dx} = x^3 - xy,\\
&\frac{dy}{dx} + x^2 y = e^x,\\
&\frac{dy}{dx} + (\sin x)y + x^3 = 0,\\
&\frac{dy}{dx} + 5y + 2 = 0.
\end{aligned}$$

We will assume that the functions $p(x)$ and $q(x)$ in (9.7) are continuous and we will look for a general solution that is valid on that interval. One method for doing this is based on the observation that if we define the function $I = I(x)$ by

$$I = e^{\int p(x)dx}. \tag{9.8}$$

then

$$\frac{dI}{dx} = e^{\int p(x)dx} \cdot \frac{d}{dx}\int p(x)dx = I \cdot p(x).$$

Thus,

$$\frac{d}{dx}(Iy) = I\frac{dy}{dx} + \frac{dI}{dx}y = I\frac{dy}{dx} + Ip(x)y. \tag{9.9}$$

If (9.7) is multiplied through by I, it becomes

$$I\frac{dy}{dx} + Ip(x) \cdot y = Iq(x).$$

Combine this with (9.9), we have

$$\frac{d}{dx}(Iy) = Iq(x).$$

This equation can be solved for y by integrating both sides with respect to x and then dividing through by I to obtain

$$y = \frac{1}{I(x)} \int I(x)q(x)dx$$

which is a general solution of (9.7) on the interval. The function $I(x)$ in (9.8) is called **an integrating factor** for (9.7), and this method for finding a general solution of (9.7) is called **the method of integrating factors**.

The Method of Integrating Factors

Step 1 Calculate the integrating factor

$$I = e^{\int p(x)dx}.$$

Step 2 Multiply both sides of (9.7) by I and express the result as

$$\frac{d}{dx}(Iy) = Iq(x)$$

Step 3 Integrate both sides of the equation obtained in Step 2 and then solve for y. Be sure to include a constant of integration in this step.

Example 9.13 Find the general solution of

$$\frac{dy}{dx} - y = e^{2x}$$

Example 9.14 Solve the initial value problem

$$x\frac{dy}{dx} - y = x, \quad x > 0, \quad y(1) = 2.$$

Example 9.15 Find the general solution of

$$\frac{dy}{dx} = xe^x + y - 1$$

Example 9.16 Solve the initial value problem

$$\frac{dy}{dx} = \frac{x-1}{e^{2x}} - 2y, \quad y(0) = 1.$$

Example 9.17 Find the general solution of

$$\frac{dy}{dx} = \frac{cosx - y}{x}, \quad x > 0.$$

9.5 Applications of Differential Equations

9.5.1 Exponential Growth Law

In general, if the rate of change of a quantity Q with respect to time is proportional to the amount of Q present and $Q(0) = Q_0$, then, we obtain the following theorem:

EXPONENTIAL GROWTH LAW If $\frac{dQ}{dt} = rQ$ and $Q(0) = Q_0$ then $Q = Q_0 e^{rt}$ where

- Q_0 is amount of Q at $t = 0$
- r is relative growth rate
- t is time
- Q is quantity at time t

If r is positive, this becomes ***exponential growth*** . If r is negative, this becomes an ***exponential decay*** problem.

The constant r in the exponential growth law is called ***the relative growth rate*** . If the relative growth rate is $r = 0.02$, then the quantity Q is growing at a rate $dQ/dt = 0.02Q$ (that is 2% of the quantity Q per unit of time t). Note the distinction between the relative growth rate r and the rate of growth dQ/dt of the quantity Q. Relative growth rate is 0.02 and the rate of growth is $0.02Q$. Once we know that the rate of growth of something is proportional to the amount present, we know that it has exponential growth and we can use the exponential growth formula.

Example 9.18 The world population passed 1 billion in 1804, 2 billion in 1927, 3 billion in 1960, 4 billion in 1974, 5 billion in 1987, and 6 billion in 1999, as illustrated in Figure 9.1. Population growth over certain periods can be approximated by the exponential growth law.

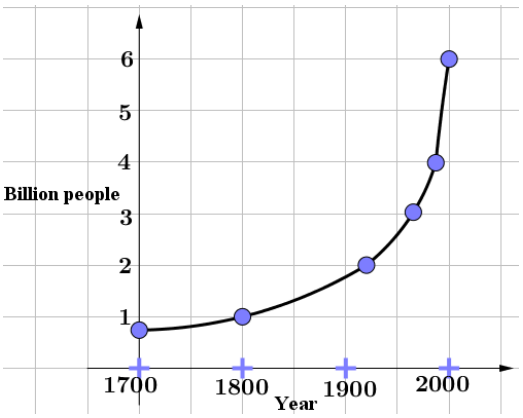


Figure 9.1: World population growth

Example 9.19 Population Growth India had a population of about 1.2 billion in 2010. Let P represent the population (in billions) t years after 2010, and assume a growth rate of 1.5% compounded continuously.

(A) Find an equation that represents India's population growth after 2010, assuming that the 1.5% growth rate continues.

(B) What is the estimated population (to the nearest tenth of a billion) of India. in the year 2030?

We now turn to another type of exponential growth: ***radioactive decay*** . In 1946,Willard Libby (who later received a Nobel Prize in chemistry) found that as long as a plant or animal is alive, radioactive carbon-14 is maintained at a constant level in its tissues. Once the plant or animal is dead, however, the radioactive carbon-14 diminishes by radioactive decay at a rate proportional to the amount present.

$$\frac{dQ}{dt} = rQ \qquad Q(0) = Q_0$$

This is another example of the exponential growth law. The continuous compound rate of decay for radioactive carbon-14 is 0.0001238, so $r = -0.0001238$, since decay implies a negative continuous compound growth rate.

Example 9.20 A human bone fragment was found at an archaeological site in Africa. If 10% of the original amount of radioactive carbon-14 was present, estimate the age of the bone.

Example 9.21 **Half-life** is the time required for a radioactive element to reduce its quantity by half.

Denote by T the half-life of a radioactive element. Use $y = y_0 e^{-kt}$ to write T in terms of the decay constant k. If the half-life of radium-226 is 1600 years, find its decay constant.

9.6 Comparison of Exponential Growth Phenomena

The graphs and equations given in Figure below compare several widely used growth models. These models are divided into two groups: unlimited growth and limited growth. Following each equation and graph is a short (and necessarily incomplete) list of areas in which the models are used.

Table 1 Exponential Growth

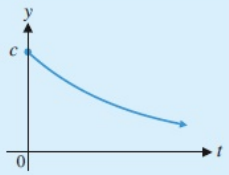
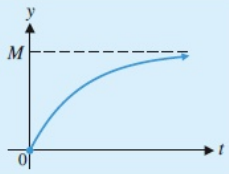
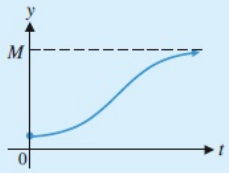
Description	Model	Solution	Graph	Uses
Unlimited growth: Rate of growth is proportional to the amount present	$\frac{dy}{dt} = ky$ $k, t > 0$ $y(0) = c$	$y = ce^{kt}$		• Short-term population growth (people, bacteria, etc.) • Growth of money at continuous compound interest • Price–supply curves
Exponential decay: Rate of growth is proportional to the amount present	$\frac{dy}{dt} = -ky$ $k, t > 0$ $y(0) = c$	$y = ce^{-kt}$		• Depletion of natural resources • Radioactive decay • Absorption of light in water • Price–demand curves • Atmospheric pressure (t is altitude)
Limited growth: Rate of growth is proportional to the difference between the amount present and a fixed limit	$\frac{dy}{dt} = k(M - y)$ $k, t > 0$ $y(0) = 0$	$y = M(1 - e^{-kt})$		• Sales fads (for example, skateboards) • Depreciation of equipment • Company growth • Learning
Logistic growth: Rate of growth is proportional to the amount present and to the difference between the amount present and a fixed limit	$\frac{dy}{dt} = ky(M - y)$ $k, t > 0$ $y(0) = \frac{M}{1 + c}$	$y = \frac{M}{1 + ce^{-kMt}}$		• Long-term population growth • Epidemics • Sales of new products • Spread of a rumor • Company growth

Figure 9.2: Exponential growth

Barnett et al. *Calculus for Business, Economics, Life Sciences and Social Sciences (12 ed.)* , Pearson (2011).

Exercise 9

1. Verify that both $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of the differential equation $y'' = y$. How about their linear combination $y = c_1 e^x + c_2 e^{-x}$?

2. Solve the initial-value problem

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

for $k > 0$.

3. China had a population of 1.32 billion in 2007 (t = 0). Let P represent the population (in billions) t years after 2007, and assume a continuous growth rate of 0.6%. Find the estimated population for China in the year 2025.

4. A bone from an ancient tomb was discovered and was found to have 5% of the original radioactive carbon present. Estimate the age of the bone.

5. Give an example of a first-order differential equation with unique solution.

6. Give an example of a first-order differential equation which has $y = e^{-2x}$ as a solution.

7. Classify the following first-order differential equations as separable, linear, both, or neither.

   (a) $\dfrac{dy}{dx} - 3y = \sin x$

   (b) $\dfrac{dy}{dx} + xy = x$

   (c) $y\dfrac{dy}{dx} - x = 1$

   (d) $\dfrac{dy}{dx} + xy^2 = \sin(xy)$

8. Solve the following differential equations.

   (a) $\dfrac{dy}{dx} + 2xy = 3x$

   (b) $\dfrac{\sqrt{1+x^2}}{1+y}\dfrac{dy}{dx} = -x$

   (c) $(1+x^4)\dfrac{dy}{dx} = \dfrac{x^3}{y}$

   (d) $y' + y = \sin(e^x)$

(e) $e^{-y}\sin x - y'\cos^2 x = 0$

(f) $\dfrac{dy}{dx} + y + \dfrac{1}{1-e^x} = 0$

(g) $\dfrac{dy}{dx} - \dfrac{y^2 - y}{\sin x} = 0$

(h) $\dfrac{dy}{dx} + 5y = e^{-3x}$

(i) $(1+x^2)\dfrac{dy}{dx} + xy = 0$

(j) $y' - (1+x)(1+y^2) = 0$

9. Solve the following initial-value problems.

(a) $x\dfrac{dy}{dx} + y = x, \quad y(1) = 3$

(b) $y' = \dfrac{3x^2}{2y + \cos y}, \quad y(0) = \pi$

(c) $\dfrac{dy}{dx} = \dfrac{2x+1}{2y-2}, \quad y(0) = 1$

(d) $x\dfrac{dy}{dx} - y = x^2, \quad y(1) = 1$

(e) $2\dfrac{dy}{dx} - y = 4\sin(3x), \quad y(0) = 0$

(f) $y' = -4xy^2, \quad y(0) = 1$

10. Find a curve that satisfies $y' = -\dfrac{x}{y}$ and passes through $(3, 1)$.

11. Polonium-210 is a radioactive element with a half-life of 140 days. Assume that 20 milligrams of the element are placed in a lead container and that $y(t)$ is the number of milligrams present t days later.

(a) Find the initial-value problem whose solution is $y(t)$.

(b) Find a formula for $y(t)$.

(c) How many milligrams will be present after 10 weeks?

(d) How long will it take for 70% of the original sample to decay?

12. Suppose that 40% of a certain radioactive element decays in 5 years.

(a) What is the half-life of this element in years?

(b) Suppose that a certain quantity of this substance is stored in a cave. What percentage of it will remain after t years?