Extreme values   Solutions to selected problems.

① $f'(x) = x(x-1)$

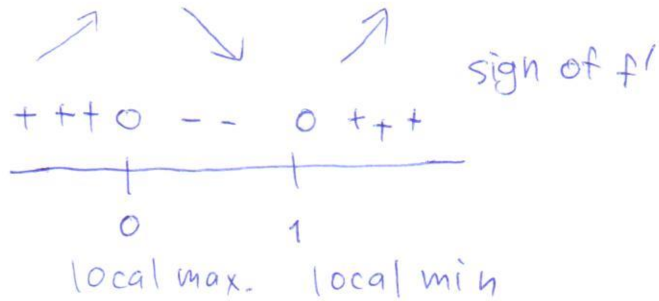
a) critical points : $x = 0, 1$

b) increase : $(-\infty, 0]$, $[1, \infty)$

decrease : $[0, 1]$

c) local maximum at $x = 0$

local minimum at $x = 1$



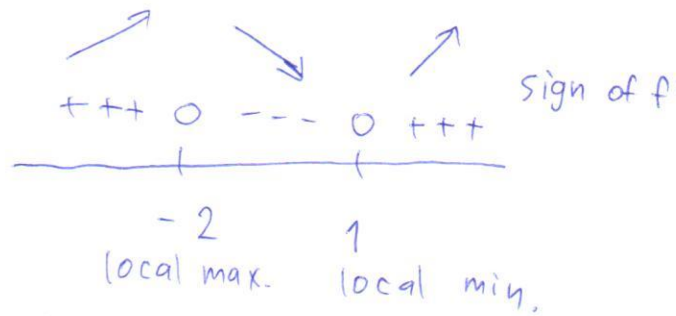
② $f'(x) = (x-1)(x+2)$

a) $x = 1, -2$

b) increase : $(-\infty, -2]$, $[1, \infty)$

decrease : $[-2, 1]$

c) local maximum at $x = -2$, local min at $x = 1$



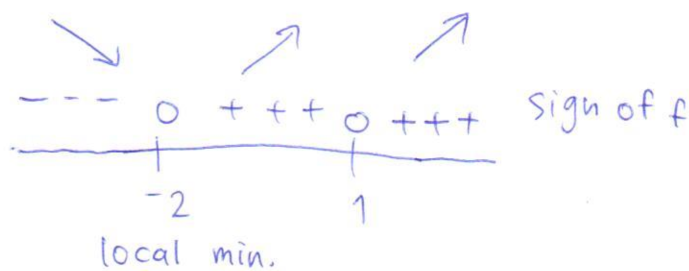
③ $f'(x) = (x-1)^2(x+2)$

a) $x = 1, -2$

b) increase : $[-2, \infty)$

decrease : $(-\infty, -2]$

c) local minimum at $x = -2$, no local maximum.

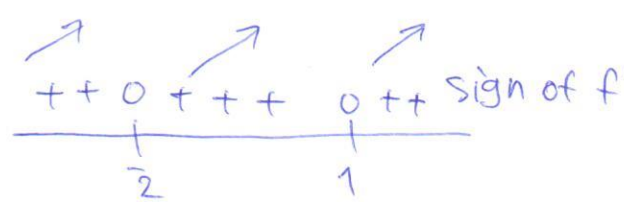


④ $f'(x) = (x-1)^2(x+2)^2$

a) $x = 1, -2$

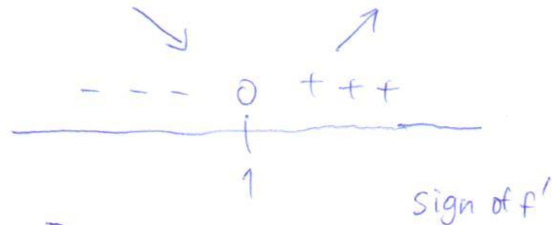
b) increase : $(-\infty, \infty)$

decrease : none



c) no local minimum or local maximum

(5) $f'(x) = (x-1)e^{-x} = \dfrac{x-1}{e^{x}}$

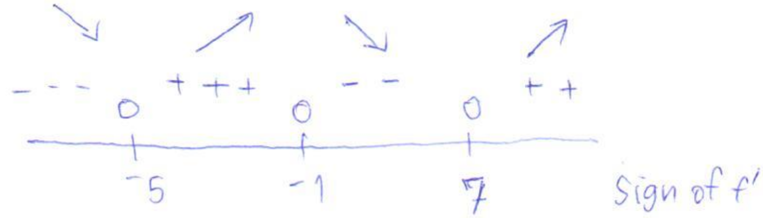


a) $x = 1$

b) inc. $[1, \infty)$ dec. $(-\infty, 1]$

c) local min. at $x = 1$

(6) $f'(x) = (x-7)(x+1)(x+5)$



a) $x = 7, -1, -5$

b) inc. $[-5, -1]$, $[7, \infty)$

dec. $(-\infty, -5]$, $[-1, 7]$

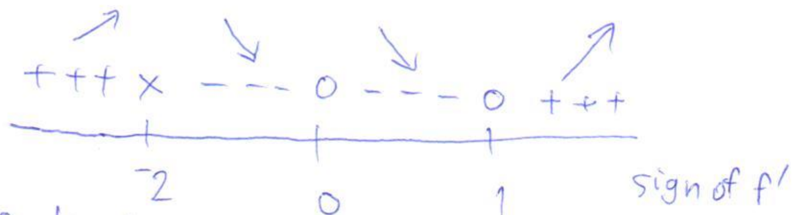
c) local min. at $x = -5, 7$, local max. at $x = -1$

(7) $f'(x) = \dfrac{x^2(x-1)}{x+2}$, $x \neq -2$

a) $x = 0, 1$ Note that $x = -2$ is not critical point because it's not in domain

b) inc. $(-\infty, -2)$, $(1, \infty)$

dec. $(-2, 1]$



c) local min. at $x = 1$, no local max.

Note that it's not local max. at $x = -2$ because -2 is not in the domain of f.

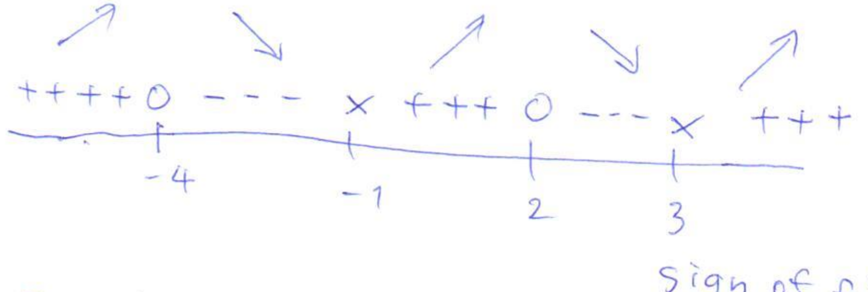
(8) $f'(x) = \dfrac{(x-2)(x+4)}{(x+1)(x-3)}$, $x \neq -1, 3$.

a) $x = 2, -4$

b) inc. $(-\infty, -4]$,

$(-1, 2]$,

$[3, \infty)$

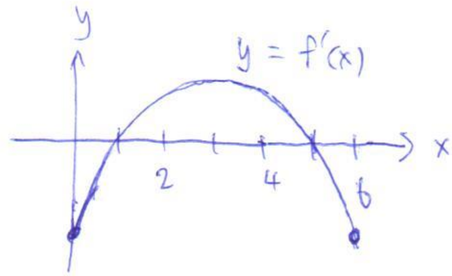


dec. $[-4, -1)$, $[2, 3)$.

c) local max. at $x = -4$ and $x = 2$

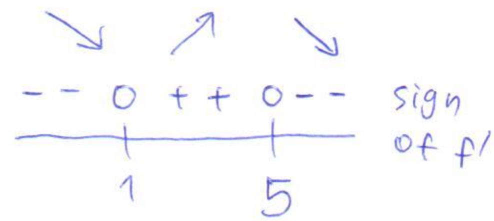
No local min.

5



(a) f is increasing on $[1, 5]$ because $f' > 0$ on $(1, 5)$.
f is decreasing on $[0, 1]$ and $[5, 6]$ because $f' < 0$ on $[0, 2)$ and $(5, 6]$.

(b) f has local minimum at $x = 1$ and has local ~~min~~ maximum at $x = 5$. Reason ⟶



6 (a) increasing : $[0, 1]$, $[3, 5]$
decreasing : $[1, 3]$, $[5, 6]$

(b) local min. at $x = 3$
local max. at $x = 1, 5$

15 (a) increasing : $(-2, 0)$, $(2, 4)$
decreasing : $(-4, -2)$, $(0, 2)$

(b) local min. $= 0$ at $x = -2$
$= -3$ at $x = 2$
local max. $= 1$ at $x = 0$

16 (a) increasing : $(-4, -3)$, $(-1, 1)$, $(2, 4)$
decreasing : $(-3, -1)$, $(1, 2)$

(b) local min. $= -1$ at $x = -1$
$= 0$ at $x = 2$

local max. $= 1$ at $x = -3$
$= 1$ at $x = 1$

(17) (a) increasing: $(-4, -1)$, $(\frac{1}{2}, 2)$, $(2, 4)$

decreasing: $(-1, \frac{1}{2})$

(b) local min. $= -1$ at $x = \frac{1}{2}$

local max. $= 2$ at $x = -1$

(18) (a) increasing: $(-4, -\frac{5}{2})$, $(-1, 1)$, $(3, 4)$

decreasing: $(-\frac{5}{2}, -1)$, $(1, 3)$

(b) local min. $= 1$ at $x = 3$

local max. $= 1$ at $x = -\frac{5}{2}$

(19) $g(t) = -t^2 - 3t + 3$

$g'(t) = -2t - 3$

$\nearrow \quad \searrow$

$+ + + \quad 0 \quad - - -$ sign of g'

$-\frac{3}{2}$ local max.

(a) increasing: $(-\infty, -\frac{3}{2})$

decreasing: $(-\frac{3}{2}, \infty)$

(b) local max. $= g(-\frac{3}{2}) = \frac{39}{4}$

no local min.

(20) $g(t) = -3t^2 + 9t + 5$

$g'(t) = -6t + 9 = -3(2t - 3)$

$\nearrow \quad \searrow$

$+ + + \quad 0 \quad - - -$ sign of g'

$\frac{3}{2}$

local max.

(a) inc. $(-\infty, \frac{3}{2})$

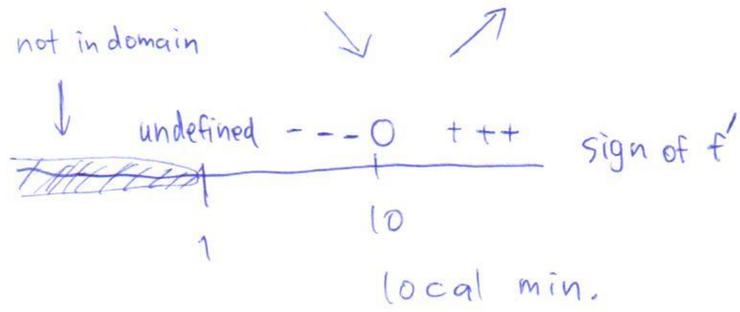
dec. $(\frac{3}{2}, \infty)$

(b) local max. $= g(\frac{3}{2}) = \frac{65}{4}$

(31) $f(x) = x - 6\sqrt{x-1}$

$f'(x) = 1 - 6 \cdot \dfrac{1}{2\sqrt{x-1}} \cdot 1$

$f'(x) = 1 - \dfrac{3}{\sqrt{x-1}}$



Note that $f'(x)$ is undefined at $x=1$

moreover, $f'(x)=0$ when $x=10$

$f(10) = 10 - 6\sqrt{10-1} = 10-18 = -8$

(a) increasing $(10, \infty)$

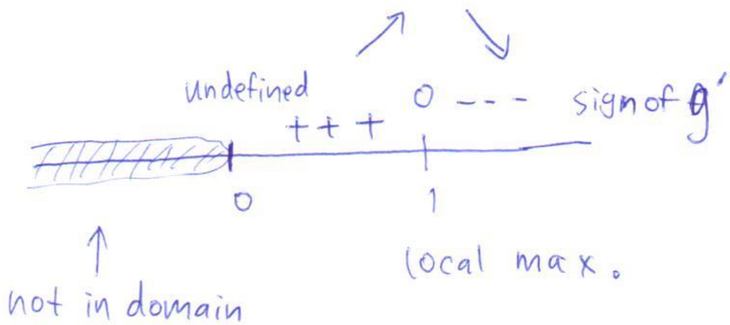
decreasing $(1, 10)$

(b) local min. $= -8$ at $x = 10$

(32) $g(x) = 4\sqrt{x} - x^2 + 3$

$g'(x) = \dfrac{2}{\sqrt{x}} - 2x$

- $f'(x)$ is undefined at $x=0$
- $f'(x) = 0$ when $x = 1$



(a) inc. $(0,1)$, dec. $(1, \infty)$

(b) local max. $= g(1) = 4\sqrt{1} - 1^2 + 3 = 0$

(33) $g(x) = x\sqrt{8-x^2}$

$g'(x) = x \cdot \dfrac{1}{2\sqrt{8-x^2}} \cdot (-2x) + \sqrt{8-x^2} \cdot 1$

$g'(x) = \dfrac{-x^2}{\sqrt{8-x^2}} + \sqrt{8-x^2}$

Mathematical Applications

① Let x = height, y = width

Then Area $= xy = 16 \Rightarrow y = \frac{16}{x}$

Perimeter $= 2x + 2y = 2x + 2\left(\frac{16}{x}\right) = 2x + \frac{32}{x}$

Let $f(x) = 2x + \frac{32}{x}$

$f'(x) = 2 - \frac{32}{x^2}$

Find critical point, set $f'(x) = 0$

$$2 - \frac{32}{x^2} = 0$$

$$x = \pm 4$$

But x = height, so $x = 4$.

$\boxed{\text{smallest perimeter} = f(4) = 16 \text{ in}}$

Sign of $f'(x)$ ↘ ↗

- - - - 0 + + +

4

so, local minimum occurs at $x = 4$ in.

② Let x = height, y = width

Then Perimeter $= 2x + 2y = 8 \Rightarrow x + y = 4 \Rightarrow y = 4 - x$

Area $= xy = x(4-x) = 4x - x^2$

Let $f(x) = 4x - x^2 \Rightarrow f'(x) = 4 - 2x$

Find critical point, set $f'(x) = 0 \Rightarrow 4 - 2x = 0$

$x = 2$

sign of $f'(x)$ ↗ ↘

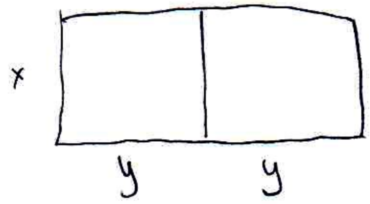
+ + + + 0 - - - -

2

so, local maximum occurs at $x = 2$

which means that $y = 4 - 2 = x$.

⑧ The shortest fence



Let x = height

Let y = half of the width

Area $= x(2y) = 216 \Rightarrow y = \frac{108}{x}$

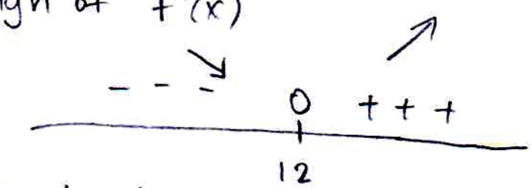
Length of fence $= 3x + 4y = 3x + 4\left(\frac{108}{x}\right)$

$f(x) = 3x + \frac{432}{x} \Rightarrow f'(x) = 3 - \frac{432}{x^2} \overset{\text{set}}{=} 0$

so, critical point is $x = 12$

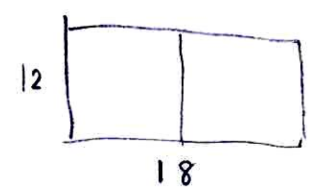
$$3 = \frac{432}{x^2}$$
$$x^2 = 144$$
$$x = \pm 12$$

Sign of $f'(x)$

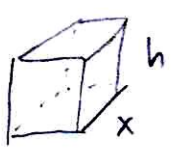


so, local minimum occurs at $x = 12$ m, $y = \frac{108}{12} = 9$

The dimension which require smallest length of fence is 12×18 m and it requires 72 meters of fence.



⑨



Let x = base, h = height

Then, Volume $= x^2 h = 500 \Rightarrow h = \frac{500}{x^2}$

To minimize the weight, we have to minimize surface area.

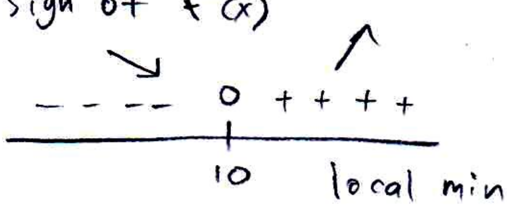
Surface area $= x^2 + 4xh = x^2 + 4x\left(\frac{500}{x^2}\right) = x^2 + \frac{2000}{x}$

$f(x) = x^2 + \frac{2000}{x} \Rightarrow f'(x) = 2x - \frac{2000}{x^2}$

set $f'(x) = 0 \Rightarrow 2x - \frac{2000}{x^2} = 0 \Rightarrow 2x = \frac{2000}{x^2} \Rightarrow x = 10$

so, critical point is $x = 10$.

sign of $f'(x)$



so, local minimum occurs at $x = 10$ ft.

Dimension of tank $\Rightarrow$