The chain Rule

91

a.) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{d}{du}\left[\frac{u}{5}+7\right] \cdot \frac{d}{dx}\left[5x-35\right]$

$= \frac{1}{5} \cdot 5$

$= 1$

b.) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{d}{du}\left[1+\frac{1}{u}\right] \cdot \frac{d}{dx}\left[\frac{1}{x-1}\right]$

$= -\frac{1}{u^2} \cdot -\frac{1}{(x-1)^2}$

$= \frac{-1}{\left(\frac{1}{x-1}\right)^2} \cdot \frac{-1}{(x-1)^2}$

$= 1$

92 a) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{d\,u^3}{du} \cdot \frac{d\,\sqrt{x}}{dx}$

$= 3u^2 \cdot \frac{1}{2\sqrt{x}}$

$= 3(\sqrt{x})^2 \cdot \frac{1}{2\sqrt{x}} = \frac{3}{2}\sqrt{x}$

b.) $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d\sqrt{u}}{du} \cdot \frac{d}{dx} x^3$$

$$= \frac{1}{2\sqrt{u}} \cdot 3x^2$$

$$= \frac{1}{2\sqrt{x^3}} \cdot 3x^2$$

$$= \frac{3}{2}\sqrt{x}$$

$\boxed{93}$ $$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x-1}{x+1}\right]^2$$

$$= 2\left(\frac{x-1}{x+1}\right) \cdot \frac{d}{dx}\left(\frac{x-1}{x+1}\right)$$

$$= 2\left(\frac{x-1}{x+1}\right) \cdot \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2}$$

$$= 2\left(\frac{x-1}{x+1}\right) \cdot \frac{2}{(x+1)^2}$$

$$= \frac{4(x-1)}{(x+1)^3}$$

at $x=0$; $$\left.\frac{dy}{dx}\right|_{x=0} = \frac{4(0-1)}{(0+1)^3} = -4$$

62

(a) $\frac{dv}{dt}$ = the rate at which the ~~ta~~ volume is increasing

$\frac{dr}{dt}$ = the rate at which the radius is increasing

(b) We know $V = \frac{4}{3}\pi r^3$

Treat V and r as functions of t

$$V(t) = \frac{4}{3}\pi [r(t)]^3$$

Differentiate both sides with respect to t,

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3[r(t)]^2 \frac{dr}{dt}$$

Simplify to get $\boxed{\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}}$

(99) $s(t) = \sqrt{1+4t}$

Volocity $v(\cancel{t})$ is given by $v(t) = s'(t) = \dfrac{1}{2\sqrt{1+4t}}(4)$

$$v(t) = \frac{2}{\sqrt{1+4t}}$$

Acceleration $a(t) = v'(t) = \left[2(1+4t)^{-\frac{1}{2}}\right]'$

$$= 2\left(-\tfrac{1}{2}\right)(1+4t)^{-\frac{3}{2}}(4)$$

$$= 4(1+4t)^{-3/2} = \frac{4}{(1+4t)^{3/2}}$$

At $t = 6$ sec, we have

volocity $V(6) = \dfrac{2}{\sqrt{1+4\cdot 6}} = \dfrac{2}{5}$ m/s

Acceleration $a(6) = \dfrac{4}{(1+4\cdot 6)^{3/2}} = \dfrac{4}{125}$ m/s^2

(100) $s(t) =$ distance

$v(t) =$ velocity

We have $v(t) = \dfrac{k}{\sqrt{s(t)}}$, so $v'(t) = \left[k(s(t))^{-\frac{1}{2}}\right]'$

Acceleration $\Rightarrow a(t) = v'(t) = k\left(-\tfrac{1}{2}\right)[s(t)]^{-\frac{3}{2}} \cdot s'(t)$

But $s'(t) = v(t)$, we have $a(t) = k\left(-\tfrac{1}{2}\right) s(t)^{-3/2} \cdot v(t)$

$$= -\frac{k}{2} \cdot s^{-3/2} \cdot \frac{k}{\sqrt{s}}$$

$$= \frac{-k^2}{2s^2}$$

$v = ks$

$v' = \frac{1}{2\sqrt{s}} s'$

(100) $s(t)$ = distance

$v(t)$ = velocity = $s'(t)$

$a(t)$ = acceleration = $v'(t)$.

We know that $v(t) = k\sqrt{s} = k[s(t)]^{\frac{1}{2}}$

So, $a(t) = v'(t) = \left(k[s(t)]^{\frac{1}{2}}\right)' = k \cdot \frac{1}{2}[s(t)]^{-\frac{1}{2}} \cdot s'(t)$

$= \frac{k}{2} s^{-\frac{1}{2}} \cdot v(t)$ (because $s'(t) = v(t)$)

$= \frac{k}{2} s^{-\frac{1}{2}} \cdot k\sqrt{s} = \frac{k^2}{2}$ = constant

(101) We know that $v(t)$ is inversely proportional to $\sqrt{s}$.

Therefore, we have that $v \propto \frac{1}{\sqrt{s}}$

$\Rightarrow v = \frac{k}{\sqrt{s}} \Rightarrow v(t) = \frac{k}{\sqrt{s(t)}} = k[s(t)]^{-\frac{1}{2}}$

So, $a(t) = v'(t) = k\left(-\frac{1}{2}\right)[s(t)]^{-\frac{3}{2}} \cdot s'(t) = -\frac{k}{2} s^{-\frac{3}{2}} \cdot v(t)$

$= -\frac{k}{2} s^{-3/2}\left(\frac{k}{\sqrt{s}}\right) = -\frac{k^2}{2} \cdot \frac{1}{s^2}$

Therefore, we have that $a = \left(-\frac{k^2}{2}\right) \cdot \frac{1}{s^2}$

i.e. $a \propto \frac{1}{s^2}$ (a is inversely proportional to s^2.)

Implicit differentiations

21. $x^2 + y^2 = 1$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx} \cdot 1$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \cdot \left(-\frac{x}{y}\right)$$

$$= \frac{y \cdot (-1) - (-x)\frac{dy}{dx}}{y^2}$$

$$= \frac{1}{y^2}\left[-y + x\frac{dy}{dx}\right]$$

$$= \frac{1}{y^2}\left[-y + x \cdot \left(\frac{-x}{y}\right)\right]$$

$$= \frac{1}{y^2}\left[-y - \frac{x^2}{y}\right]$$

$$= \frac{-1}{y} - \frac{x^2}{y^3}$$

22. $x^{2/3} + y^{2/3} = 1$

Differentiate both sides to get

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0 \Rightarrow x^{-\frac{1}{3}} = -y^{-\frac{1}{3}}\frac{dy}{dx}$$

so, $$\frac{dy}{dx} = \frac{x^{-\frac{1}{3}}}{-y^{-\frac{1}{3}}} = -\frac{y^{1/3}}{x^{1/3}} = -\sqrt[3]{\frac{y}{x}}.$$

23. $y^2 = e^{x^2} + 2x$

Diff. both sides :

$$2yy' = (e^{x^2})\cdot(2x) + 2$$

$$y' = \frac{2xe^{x^2}+2}{2y} = \frac{xe^{x^2}+1}{y}$$

24. $y^2 - 2x = 1 - 2y$

Diff. both sides :

$$2yy' - 2 = 0 - 2y'$$

$$2yy' + 2y' = 2$$

$$(2y+2)y' = 2$$

$$y' = \frac{2}{2y+2} = \frac{1}{y+1}$$

25. $2\sqrt{y} = x - y$

Diff. both sides :

$$2\cdot\frac{1}{2\sqrt{y}}y' = 1 - y'$$

$$\frac{1}{\sqrt{y}}y' + y' = 1$$

$$\left(\frac{1}{\sqrt{y}}+1\right)y' = 1 \Rightarrow y' = \frac{1}{\frac{1}{\sqrt{y}}+1}$$

or $$y' = \frac{\sqrt{y}}{1+\sqrt{y}}$$

26. $xy + y^2 = 1$

Diff. both sides : $xy' + y \cdot 1 + 2yy' = 0$

$$xy' + 2yy' = -y$$

$$(x+2y)y' = -y, \text{ so } y' = \frac{-y}{x+2y}$$

27. $x^3 + y^3 = 16$

Diff. both sides : $3x^2 + 3y^2 \frac{dy}{dx} = 0$

$$x^2 + y^2 \frac{dy}{dx} = 0 \quad \text{— (I)}$$

Diff. both sides : $2x + 2y \frac{dy}{dx} \cdot \frac{dy}{dx} + y^2 \frac{d^2y}{dx^2} = 0$

$$2x + 2y \left(\frac{dy}{dx}\right)^2 + y^2 \frac{d^2y}{dx^2} = 0 \quad -(\text{II})$$

Plug-in $(x,y) = (2,2)$ into (I), we get $2^2 + 2^2 \left(\frac{dy}{dx}\right) = 0$

so, $\left.\frac{dy}{dx}\right|_{(x,y)=(2,2)} = \frac{-2^2}{2^2} = -1$

Plug-in $(x,y) = (2,2)$ and $\frac{dy}{dx} = -1$ into (II), we get

$$2(2) + 2(2)(-1)^2 + 2^2\left(\frac{d^2y}{dx^2}\right) = 0$$

$$4 + 4 + 4\left(\frac{d^2y}{dx^2}\right) = 0$$

$$4\left(\frac{d^2y}{dx^2}\right) = -8$$

so, $\left.\frac{d^2y}{dx^2}\right|_{(x,y)=(2,2)}$ is equal to $-\frac{8}{4} = -2$.

28. $xy + y^2 = 1$ Diff. $\Rightarrow$ $x\cancel{y}' + y \cdot 1 + 2yy' = 0$

$$xy' + y + 2yy' = 0 \qquad (\text{I})$$

Diff. one more time $\Rightarrow$ $xy'' + y' \cdot 1 + y' + 2yy'' + y'(2y') = 0$

$$xy'' + y' + y' + 2yy'' + 2(y')^2 = 0$$

$$xy'' + 2y' + 2yy'' + 2(y')^2 = 0$$

call this (II)

Plug-in $(x, y) = (0, -1)$ into (I),

$0(y') + (-1) + 2(-1)(y') = 0 \Rightarrow -1 - 2y' = 0$

$$-2y' = 1$$

$$y' = \frac{1}{-2}$$

so, $y' = -\frac{1}{2}$ at $(x, y) = (0, -1)$.

Plug-in $(x, y) = (0, -1)$ and $y' = -\frac{1}{2}$ into (II),

we get $0(y'') + 2(-\frac{1}{2}) + 2(-1)y'' + 2(-\frac{1}{2})^2 = 0$

$$0 - 1 - 2y'' + 2\left(\frac{1}{4}\right) = 0$$

$$-1 - 2y'' + \frac{1}{2} = 0$$

$$-2y'' = 1 - \frac{1}{2} = \frac{1}{2}$$

$$y'' = \frac{\frac{1}{2}}{-2} = -\frac{1}{4}$$

so, at $(x, y) = (0, -1)$,

we have $y'' = -\frac{1}{4}$.

29. $y^2 + x^2 = y^4 - 2x$ at $(-\cancel{2}, 1)$ and $(-2, -1)$

(~~done~~ We've done this in class.)

30. $(x^2+y^2)^2 = (x-y)^2$ at $(1,0)$ and $(1,-1)$

Diff both sides : $2(x^2+y^2)(2x+2yy') = 2(x-y)(1-y')$

$\Rightarrow (x^2+y^2)(2x+2yy') = (x-y)(1-y')$

Plug-in $(x,y) = (1,0) \Rightarrow (1+0)(2+0) = (1-0)(1-y')$

$2 = 1-y'$

$y' = -1$

so, slope at $(1,0)$ is -1

equation : $y - 0 = -1(x-1)$

$y = 1-x$

} Tangent line at ~~(-2,1)~~ $(1,0)$

Plug-in $(x,y) = (1,-1)$

$\Rightarrow (1+1)(2-2y') = (1+1)(1-y')$

$2(2-2y') = 2(1-y')$

$2-2y' = 1-y'$

$y' = 1$

so, slope at $(1,-1)$ is 1

equation : $y-(-1) = 1(x-1)$

$y+1 = x-1$

$y = x-2$

} Tangent line at $(1,-1)$.

$\boxed{15}$ $\quad y^2 = x^3(2-x)\,,\qquad (x,y) = (1,1)$

Diff. $\quad 2yy' = \cancel{3x^2}\, x^3(-1) + (2-x)\cdot 3x^2$

$$2yy' = -x^3 + (2-x)\cdot 3x^2$$

Plug-in $\quad (x,y) = (1,1)$

$$\Rightarrow\ 2\cdot 1\cdot y' = -1^3 + (2-1)\cdot 3\cdot 1^2$$

$$2y' = -1-3$$

$$y' = \frac{-4}{2} = -2$$

so, slope at $(1,1)$ is -2

equation: $\quad y-1 = -2(x-1)$

$$y-1 = -2x+2$$

$$y = -2x+3$$

16. $\quad x^{2/3} + y^{2/3} = 4 \ \Rightarrow\ \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0$

$$\Rightarrow\ x^{-\frac{1}{3}} + y^{-\frac{1}{3}}y' = 0$$

Plug-in $(x,y) = (-3\sqrt{3},1) \ \Rightarrow\ (-3\sqrt{3})^{-\frac{1}{3}} + 1^{-\frac{1}{3}}y' = 0$

$$-\frac{1}{\sqrt{3}} + y' = 0$$

$$y' = \frac{1}{\sqrt{3}}$$

equation: $\quad y-1 = \frac{1}{\sqrt{3}}(x-(-3\sqrt{3}))$

$$y-1 = \frac{1}{\sqrt{3}}(x+3\sqrt{3}) = \frac{1}{\sqrt{3}}x+3$$

$$y = \frac{1}{\sqrt{3}}x+4$$

Higher order derivatives

~~Example 3~~

Example 4 $s(t) = 160t - 16t^2$

From this, we get that $v(t) = s'(t) = 160 - 32t$

(a) The rock goes up until $v = 0$,

Solve $v = 0 \Rightarrow$

$$160 - 32t = 0$$
$$160 = 32t$$
$$t = \frac{160}{32} = 5 \text{ sec}$$

$$s(5) = 160(5) - 16(5)^2 = 400 \text{ ft}$$

(b) Solve for t in $s(t) = 256$

$\Rightarrow 256 = 160t - 16t^2 \Rightarrow 16t^2 - 160t + 256 = 0$

$\Rightarrow t^2 - 10t + 16 = 0 \Rightarrow (t-2)(t-8) = 0, \; t = 2, 8$

On the way up $(t=2)$: velocity $= v(2) = 160 - 32(2) = 96$ ft/s

speed $= |v(2)| = |96| = 96$ ft/s

On the way down $(t=8)$: velocity $= v(8) = 160 - 32(8) = -96$ ft/s

speed $= |v(8)| = |-96| = 96$ ft/s

(c) $a(t) = v'(t) = (160 - 32t)' = -32 \text{ ft/s}^2$

(d) The rock hits the ground when $s = 0$

$$0 = 160t - 16t^2$$
$$0 = t(160 - 16t)$$
$$t = 0 \text{ or } t = 10$$

So, the rock hits the ground after 10 seconds.

(21) B is derivative of A because $B < 0$ when A is decreasing and $B > 0$ when A is increasing.

On the other hand, A is the derivative of C because of the similar reason.

Therefore,

$$C = \text{position}$$
$$A = \text{velocity}$$
$$B = \text{acceleration}$$

(22) Using the similar reason as in (21), we get

$$A = B' , \quad B = C'$$

so,

$$C = \text{position}$$
$$B = \text{velocity}$$
$$A = \text{acceleration}$$

(35) $b = a'$, $c = b'$, so

$$f = a$$
$$f' = b$$
$$f'' = c$$

(36) $a = b'$, $b = c'$, $c = d'$

so,

$$f = d$$
$$f' = c$$
$$f'' = b$$
$$f''' = a$$