Limit & Continuity of function

Limits from Graphs

1.
   - 1.1 Does not exist because $\lim\limits_{x \to 1^-} g(x) = 1$ but $\lim\limits_{x \to 1^+} g(x) = 0$
   - 1.2 1
   - 1.3 0
   - 1.4 0.5

2.
   - 2.1 True because $\lim\limits_{x \to 0} f(x) = 0$
   - 2.2 True
   - 2.3 False
   - 2.4 False because $\lim\limits_{x \to 1} f(x)$ does not exist since $\lim\limits_{x \to 1^-} f(x) = -1$, but $\lim\limits_{x \to 1^+} f(x) = 0$.
   - 2.5 False
   - 2.6 True because f is continuous on $(-1,1)$
   - 2.7 True.

Existence of limits

1.
   (a) $\lim\limits_{x \to 0} \frac{x}{|x|}$ does not exist because $\lim\limits_{x \to 0^-} \frac{x}{|x|} = \lim\limits_{x \to 0^-} \frac{x}{-x} = -1$

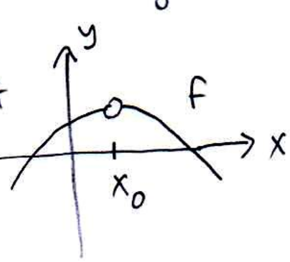
   but $\lim\limits_{x \to 0^+} \frac{x}{|x|} = \frac{x}{x} = 1$.

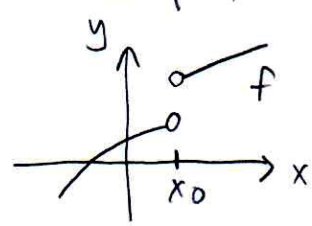
   (b) $\lim\limits_{x \to 1} \frac{1}{x-1}$ does not exist because $\lim\limits_{x \to 1^-} \frac{1}{x-1} = -\infty$

   but $\lim\limits_{x \to 1^+} \frac{1}{x-1} = +\infty$.

2. $\lim\limits_{x \to x_0} f(x)$ may or may not exist depending on the left limit and the right limit. For example,

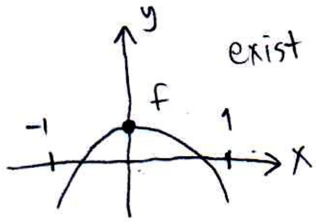
exist



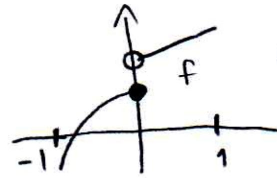


does not exist

(3) The answer is similar to the previos problem.

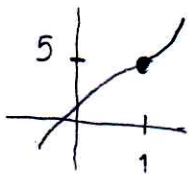


exist

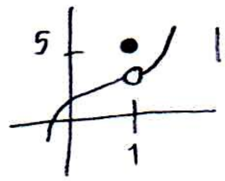


does not exist

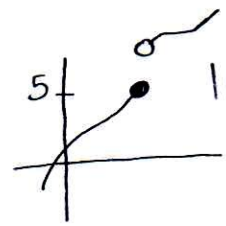
(5) We can't say anything about $\lim_{x \to 1} f(x)$. It can be anything, such as



limit = 5

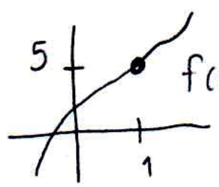


limit ≠ 5

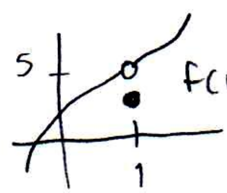


limit does not exist

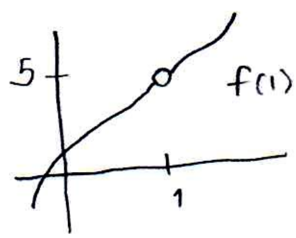
(4) We can't say anything about $f(1)$. It can be anything, such as



$f(1) = 5$



$f(1) \neq 5$



$f(1)$ is undefined

Using limit rules

(1) $\lim_{x \to c} f(x) = 5$, $\lim_{x \to c} g(x) = 2$

1.1 $\lim_{x \to c} f(x)g(x) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right) = 5 \cdot 2 = 10$

1.2 $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \dfrac{5}{2}$

1.3 $\lim_{x \to c} \left(f(x) + 3g(x)\right) = \lim_{x \to c} f(x) + 3 \lim_{x \to c} g(x) = 5 + 3 \cdot 2 = 11$

1.4 $\lim_{x \to c} \dfrac{f(x)}{f(x) - g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} f(x) - \lim_{x \to c} g(x)} = \dfrac{5}{5-2} = \dfrac{5}{3}$

Limit as Average rate of chang

① $f(x) = x^2, \quad x = 1,$

method 1

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{x \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{x \to 0} \frac{2xh + h^2}{h} = \lim_{x \to 0} (2x + h) = 2x$$

Then, plug-in $x = 1$, we get limit $= 2$.

Method 2 Plug-in $x = 1$ at the beginning

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h \to 0} \frac{2h + h^2}{h}$$

$$= \lim_{h \to 0} (2 + h) = 2$$

② $f(x) = x^2, \quad x = -2$

From the previous problem, we have $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = 2x$ for $f(x) = x^2$.

Plug-in $x = 2$, we get limit $= 4$.

③ $f(x) = 3x - 4, \quad x = 2$

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{[3(2+h) - 4] - [3(2) - 4]}{h}$$

$$= \lim_{h \to 0} \frac{6 + 3h - 4 - 6 + 4}{h} = \lim_{h \to 0} \frac{3h}{h} = 3$$

④ $f(x) = \frac{1}{x}$, $x = -2$

$$\lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{\frac{1}{-2+h} - \frac{1}{-2}}{h} = \lim_{h\to 0} \frac{\frac{-2-(-2+h)}{(-2+h)(-2)}}{h}$$

$$= \lim_{h\to 0} \frac{-2+2-h}{h(-2+h)(-2)} = \lim_{h\to 0} \frac{-h}{h(-2+h)(-2)} = \lim_{h\to 0} \frac{-1}{(-2+h)(-2)} = \frac{1}{4}$$

⑤ $f(x) = \sqrt{x}$, $x = 7$

$$\lim_{h\to 0} \frac{f(7+h) - f(7)}{h} = \lim_{h\to 0} \frac{\sqrt{7+h} - \sqrt{7}}{h} = \lim_{h\to 0} \frac{\sqrt{7+h} - \sqrt{7}}{h} \cdot \frac{\sqrt{7+h} + \sqrt{7}}{\sqrt{7+h} + \sqrt{7}}$$

$$= \lim_{h\to 0} \frac{(7+h) - 7}{h(\sqrt{7+h} + \sqrt{7})} = \lim_{h\to 0} \frac{h}{h(\sqrt{7+h} + \sqrt{7})} = \lim_{h\to 0} \frac{1}{\sqrt{7+h} + \sqrt{7}} = \frac{1}{2\sqrt{7}}$$

⑥ $f(x) = \sqrt{3x+1}$, $x = 0$

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{3(0+h)+1} - \sqrt{3\cdot 0+1}}{h} = \lim_{h\to 0} \frac{\sqrt{3h+1} - \sqrt{1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3h+1} - 1}{h} \cdot \frac{\sqrt{3h+1} + 1}{\sqrt{3h+1} + 1} = \lim_{h\to 0} \frac{(3h+1) - 1}{h(\sqrt{3h+1} + 1)} = \lim_{h\to 0} \frac{3h}{h(\sqrt{3h+1} + 1)}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3h+1} + 1} = \frac{3}{\sqrt{0+1} + 1} = \frac{3}{2}$$

Theory & Examples

① Yes, we can tell if $\lim_{x\to a} f(x)$ exists once when we know $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$. This is because $\lim_{x\to a} f(x)$ is equal to both of the one-side limits if they agree. Otherwise, $\lim_{x\to a} f(x)$ does not exist.

② Yes, the reason is given above.

continuity from graphs

① Yes, $f(-1) = (-1)^2 - 1 = 0$

Yes, $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x^2-1) = (-1)^2 - 1 = 0$

Yes, $\lim_{x \to -1^+} f(x) = f(-1) = 0$.

Yes, f is continuous at $x = -1$ from the right.

② Yes, $f(1)$ exists and equals 1.

Yes, $\lim_{x \to 1} f(x) = 2$ (from graph).

NO.

NO.

③ NO

NO

④ f is continuous on $(-1,0)$, $(0,1)$, $(1,2)$, $(2,3)$.

⑤ If we let $f(2) = 0$, then the extended function would be continuous at $x = 2$.

⑥ $f(1)$ should be changed to 2.