① $\lim_{\Delta x \to 0^-} \frac{f(2+\Delta x)-f(2)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{4|(2+\Delta x)-2|+1-(2^2-3)}{\Delta x}$

$= \lim_{\Delta x \to 0^-} \frac{4|\Delta x|+1-1}{\Delta x}$

$= \lim_{\Delta x \to 0^-} \frac{4|\Delta x|}{\Delta x}$

$= \lim_{\Delta x \to 0^-} \frac{4(-\cancel{\Delta x})}{\cancel{\Delta x}}$

$= -4$

$\lim_{\Delta x \to 0^+} \frac{f(2+\Delta x)-f(2)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{(2+\Delta x)^2-3-(2^2-3)}{\Delta x}$

$= \lim_{\Delta x \to 0^+} \frac{4+4\Delta x+(\Delta x)^2-3-1}{\Delta x}$

$= \lim_{\Delta x \to 0^+} (4+\Delta x)$

$= 4$

Since $\lim_{\Delta x \to 0^-} \frac{f(2+\Delta x)-f(2)}{\Delta x} = -4 \neq 4 = \lim_{\Delta x \to 0^+} \frac{f(2+\Delta x)-f(2)}{\Delta x}$, $f'(2)$ does not exist.

② $y = y_0(5t^2-3t)$

$y' = y_0(10t-3)$

Since $y_0 = 100$, we obtain $y' = 100(10t-3)$.

Rate of Change at $t=40$ is $y'\big|_{t=40} = 100(10\cdot 40-3) = 39700$

③ The slope of the line through $(0,3)$ and $(5,-2)$ is $\frac{3-(-2)}{0-5} = -1$

The line through $(0,3)$ and $(5,-2)$ is $y-3=-x \Rightarrow y = 3-x$

From $y = \frac{4}{x+1}$, we obtain $y' = \frac{-4}{(x+1)^2}$.

We want to find x_0 such that $y'\big|_{x=x_0} = -1$.

$\frac{-4}{(x+1)^2} = -1 \Rightarrow (x+1)^2 = 4 \Rightarrow x^2+2x-3=0 \Rightarrow (x+3)(x-1)=0$

$\Rightarrow x = 1, -3$

Plug-in $x = 1$ into $y = \frac{4}{x+1}$ to get $y = \frac{4}{2} = 2$

The point $(1, 2)$ is on the curve and the line.

Plug-in $x = -3$ into $y = \frac{4}{x+1}$ to get $y = \frac{4}{-2} = -2$

The point $(-3, -2)$ is Not on the line $y = 3 - x$

so, the point we're looking for is $\boxed{(1, 2)}$

4.1 $y = x^2 + \sqrt{2x-1} - \cot x + 2$

$y' = 2x + \frac{1}{2\sqrt{2x-1}} \cdot 2 + \operatorname{cosec}^2 x$

4.2 $y = \frac{e}{x} + \log_2 x - \sec(x^2+1)$

$= \frac{-e}{x^2} + \frac{1}{x \ln 2} - \left[\sec(x^2+1)\tan(x^2+1)\right] \cdot 2x$

4.3 $y = \frac{\pi^x}{\arccos x} + (\ln x)^2$

$= \frac{(\arccos x)\pi^x \ln \pi - \pi^x \cdot \left(\frac{-1}{\sqrt{1-x^2}}\right)}{(\arccos x)^2} + \frac{2\ln x}{x}$

4.4 $y = \tan(3x^2 + x - 1) + 2^{\cos x}$

$= \left[\sec^2(3x^2+x-1)\right](6x+1) - 2^{\cos x} \ln 2 \sin x$

4.5 $y = (\ln x) \cdot (\operatorname{cosec} 2x)$

$y' = (\ln x)(-2\operatorname{cosec} 2x \cot 2x) + \frac{\operatorname{cosec} 2x}{x}$

4.6 $y = \sin\left(\frac{1}{x}\right) - e^{\frac{1}{x}}$

$y' = \left[\cos\left(\frac{1}{x}\right)\right]\left(\frac{-1}{x^2}\right) + \frac{e^{\frac{1}{x}}}{x^2}$

(5) $f(x) = (2x)^{100} - x^{99} + 1 = 2^{100} x^{100} - x^{99} + 1$

$f^{(1)}(x) = 2^{100} \cdot 100x^{99} - 99x^{98}$

$f^{(2)}(x) = 2^{100} \cdot 100 \cdot 99 x^{98} - 99 \cdot 98 x^{97}$

$f^{(3)}(x) = 2^{100} \cdot 100 \cdot 99 \cdot 98 x^{97} - 99 \cdot 98 \cdot 97 x^{96}$

$\vdots$

$f^{(n)}(x) = 2^{100} \cdot 100 \cdot 99 \cdot 98 \cdots (100-n+1) x^{100-n} - 99 \cdot 98 \cdot 97 \cdots (99-n+1) x^{99-n}$

$f^{(100)}(x) = 2^{100} \cdot 100 \cdot 99 \cdot 98 \cdots 1 = 2^{100} \cdot 100!$

5.1 $k = 100$, 5.2 $f^{(100)}(x) = 2^{100} \cdot 100!$

(6) $\ln(x+y) = x\sin y + 7$

$\frac{1}{x+y} \cdot \left(1 + \frac{dy}{dx}\right) = x\cos y \frac{dy}{dx} + \sin y$

$\frac{1}{x+y} + \frac{1}{x+y} \cdot \frac{dy}{dx} = x\cos y \frac{dy}{dx} + \sin y$

$\left(\frac{1}{x+y} - x\cos y\right) \frac{dy}{dx} = \sin y - \frac{1}{x+y}$

$\frac{dy}{dx} = \left(\sin y - \frac{1}{x+y}\right) \Big/ \left(\frac{1}{x+y} - x\cos y\right)$

(7) $y = \frac{x^{\sin x} \cdot e^{ex+e}}{\sqrt[3]{\ln(3x)}}$

$\ln y = \sin x \ln x + (ex+e)\ln e - \frac{1}{3}\ln(\ln 3x)$

$\ln y = \sin x \ln x + ex + e - \frac{1}{3}\ln(\ln 3x)$

$\frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + \ln x \cos x + e - \frac{1}{3}\frac{1}{\ln 3x} \cdot \frac{1}{3x}$

$\frac{dy}{dx} = \left(\frac{\sin x}{x} + \ln x \cos x + e - \frac{1}{3} \cdot \frac{1}{\ln 3x} \cdot \frac{1}{3x}\right) \frac{x^{\sin x} e^{ex+e}}{\sqrt[3]{\ln(3x)}}$

(8) $f(a+dx) - f(a) \approx df = f'(a)\,dx$

Let $a = 2$, $dx = -0.1$, $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

$f(2+(-0.1)) - f(2) \approx f'(2)(-0.1)$

$f(1.9) - \ln 2 \approx \frac{1}{2}\cdot(-0.1)$

$\ln(1.9) \approx \frac{1}{2}(-0.1) + \ln 2 = -0.05 + 0.69 = 0.64$

(9) (9.1) $V = \frac{1}{2}\cdot(\text{Volume of sphere}) = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3$

(9.2) $dV = 2\pi r^2\,dr$

$r = 10$, $dr = 0.05$

$\therefore dV = 2\pi(10)^2(0.05) = 10\pi$

Error $= \pm 10\pi$

(10) Let $f(x) = x^3 - \frac{78}{3}$, then the solution to $f(x) = 0$ is $\sqrt[3]{\frac{78}{3}}$.

$f'(x) = 3x^2$

Using $x_0 = 3$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{f(3)}{f'(3)}$$

$$= 3 - \frac{3^3 - \frac{78}{3}}{3\cdot 3^2}$$

$$= 3 - \frac{1}{27}$$

$$= \frac{80}{27} \approx 2.96$$

$$\begin{array}{r} 2.96 \\ 27\,\overline{)\,80} \\ \underline{54} \\ 260 \\ \underline{243} \\ 170 \\ \underline{162} \\ 80 \end{array}$$

(11)

$f(x) = \cos x \qquad f(\pi) = -1$

$f'(x) = -\sin x \qquad f'(\pi) = 0$

$f''(x) = -\cos x \qquad f''(\pi) = 1$

$f'''(x) = \sin x \qquad f'''(\pi) = 0$

11.1

$$T_3(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2}(x-\pi)^2 + \frac{f'''(\pi)}{3!}(x-\pi)^3$$

$$T_3(x) = -1 + 0 + \frac{1}{2}(x-\pi)^2 + 0$$

$$T_3(x) = -1 + \frac{1}{2}(x-\pi)^2$$

11.2 From 11.2, we have

$$T_3(x) = -1 + \frac{1}{2}(x-3.14)^2$$

$$T_3(3.04) = -1 + \frac{1}{2}(3.04-3.14)^2$$

$$= -1 + \frac{1}{2}(-0.10)^2$$

$$= -1 + \frac{1}{2}(0.01)$$

$$= -1 + 0.005$$

$$= -0.995$$

12.1 $\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \frac{0}{1-(-1)} = 0$

12.2 $\lim_{x\to+\infty} \left[\ln\left(\frac{1}{x}\right) - \frac{x^3}{2x+1} \right] = -\infty - (+\infty) = -\infty$

13.1 $\lim_{x\to 0} \frac{7^x - 5^x}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

$$\lim_{x\to 0} \frac{7^x - 5^x}{x} = \lim_{x\to 0} \frac{7^x \ln 7 - 5^x \ln 5}{1} = \ln 7 - \ln 5$$

13.2 $\lim_{x\to 0^+} \left(\frac{1}{1-e^{-x}} - \frac{1}{x} \right) = -\infty - (+\infty) = -\infty$

13.3 $\lim_{x\to\frac{\pi}{2}} \sin(4x-\pi)\cot(2x-\pi) = \lim_{x\to\frac{\pi}{2}} \frac{\sin(4x-\pi)\cos(2x-\pi)}{\sin(2x-\pi)}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-2\sin(4x-\pi)\sin(2x-\pi) + 4\cos(2x-\pi)\cos(4x-\pi)}{\cos(2x-\pi)}$$

$$= \frac{-4}{1} = -4$$

13.4 $\lim_{x\to+\infty} \left(\frac{x+e}{x+2e}\right)^x$ (1^∞ form)

let $y = \left(\frac{x+e}{x+2e}\right)^x$

$$\ln y = x \ln\left(\frac{x+e}{x+2e}\right)$$

$$\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty} x\ln\left(\frac{x+e}{x+2e}\right) \quad (\infty\cdot 0 \text{ form})$$

$$= \lim_{x\to+\infty} \frac{\ln\left(\frac{x+e}{x+2e}\right)}{1/x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to+\infty} \frac{\frac{e}{(x+e)(x+2e)}}{-1/x^2} \quad \left(\frac{d}{dx}\ln\left(\frac{x+e}{x+2e}\right) = \frac{e}{(x+e)(x+2e)}\right)$$

$$= \lim_{x\to+\infty} \frac{-ex^2}{(x+e)(x+2e)}$$

$$= -e \implies \text{so } \lim_{x\to+\infty} y = e^{-e}$$