Exercises on integration

1. Evaluate the indefinite integrals below by using the given substitutions to reduce the integrals to standard form.

   (a) $\displaystyle\int 7\sqrt{7x-1}\,dx, \quad u = 7x-1$

   (b) $\displaystyle\int \frac{4x^3}{(x^4+1)^2}\,dx, \quad u = x^4+1$

   (c) $\displaystyle\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx, \quad u = 1+\sqrt{x}$

   (d) $\displaystyle\int x\sin(2x^2)\,dx, \quad u = 2x^2$

   (e) $\displaystyle\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1-\cos\frac{t}{2}$

   (f) $\displaystyle\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy, \quad u = y^4+4y^2+1$

   (g) $\displaystyle\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)\,dx, \quad u = -\frac{1}{x}$

   (h) $\displaystyle\int \frac{1}{\sqrt{5x+8}}\,dx$

   - Using $u = 5x+8$
   - Using $u = \sqrt{5x+8}$

2. If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what this mean if you try the sequences of substitutions in the exercises below.

   (a) $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$

   1. $u = \tan x$, followed by $v = u^3$, then by $w = 2+v$.
   2. $u = \tan^3 x$, followed by $v = 2+u$.
   3. $u = 2+\tan^3 x$.

   (b) $\displaystyle\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$

   1. $u = x-1$, followed by $v = \sin u$, then by $w = 1+v^2$.
   2. $u = \sin(x-1)$, followed by $v = 1+u^2$.
   3. $u = 1+\sin^2(x-1)$.

3. The velocity of a particle moving back and forth on a line is $v = ds/dt = 6\sin 2t$ m/sec for all t. If $s = 0$ when $t = 0$, find the values of s when $t = \pi/2$ sec.

4. The acceleration of a particle moving back and forth on a line is $a = d^2s/dt^2 = \pi^2 \cos \pi t$ m/sec^2 for all t. If $s = 0$ and $v = 8$ m/sec when $t = 0$, find s when $t = 1$ sec.

5. It looks as if we can integrate $2 \sin x \cos x$ with respect to x in three difference ways:

   (a)
$$\begin{aligned} \int 2 \sin x \cos x \, dx &= \int 2u \, du \qquad (u = \sin x) \\ &= u^2 + C_1 = \sin^2 x + C_1 \end{aligned}$$

   (b)
$$\begin{aligned} \int 2 \sin x \cos x \, dx &= \int -2u \, du \qquad (u = \cos x) \\ &= -u^2 + C_2 = -\cos^2 x + C_2 \end{aligned}$$

   (c)
$$\begin{aligned} \int 2 \sin x \cos x \, dx &= \int \sin 2x \, dx \qquad (2 \sin x \cos x = \sin 2x) \\ &= -\frac{\cos 2x}{2} + C_3 \end{aligned}$$

   Can all three integration be correct? Give reasons for your answer.

6. Evaluate the following integrals. Some integrals may not require integration by parts.

   (a) $\displaystyle\int \frac{\ln x}{x^2} \, dx$

   (b) $\displaystyle\int x^3 \sqrt{x^2 + 1} \, dx$

   (c) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$

   (d) $\displaystyle\int \sqrt{x} e^{\sqrt{x}} \, dx$

   (e) $\displaystyle\int \cos \sqrt{x} \, dx$

   (f) $\displaystyle\int_{2/\sqrt{3}}^{2} t \sec^{-1} t \, dt$

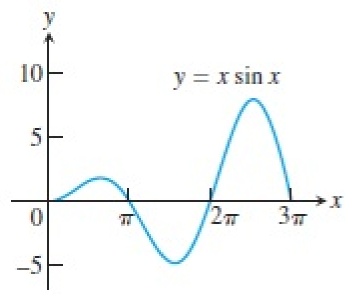
7. **Finding area** Find the area of the region enclosed by the curve $y = x \sin x$ and the $x-$axis (see the accompanying figure) for

   (a) $0 \le x \le \pi$.

   (b) $\pi \le x \le 2\pi$.

(c) $2\pi \leq x \leq 3\pi$.

(d) What pattern do you see here? What is the area between the curve and the $x-$axis for $n\pi \leq x \leq (n+1)\pi$, n an arbitrary nonegative integer? Give reasons for your answer.



8. **Finding area** Find the area of the region enclosed by the curve $y = x\cos x$ and the $x-$axis (see the accompanying figure) for

(a) $\pi/2 \leq x \leq 3\pi/2$.

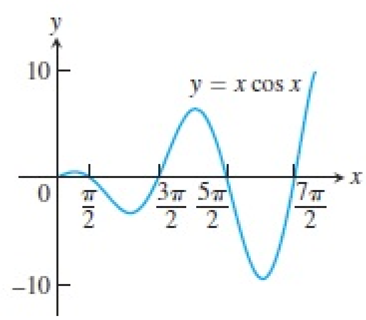
(b) $3\pi/2 \leq x \leq 5\pi/2$.

(c) $5\pi/2 \leq x \leq 7\pi/2$.

(d) What pattern do you see? What is the area between the curve and the $x-$axis for

$$\left(\frac{2n-1}{2}\right)\pi \leq x \leq \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



9. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

10. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = 1$

(a) about the $y-$axis.

(b) about the line $x = 1$.

11. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about
    (a) the $y-$axis.
    (b) the line $x = \pi/2$.

12. **Finding volume** Find the volume of the solid generated by revolving the region bounded by the $x-$axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about
    (a) the $y-$axis.
    (b) the line $x = \pi$.

13. Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.
    (a) Find the area of the region.
    (b) Find the volume of the solid formed by revolving this region about the $x-$axis.
    (c) Find the volume of the solid formed by revolving this region about the line $x = -2$.
    (d) Find the centroid of the region.

14. Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$, and $x = 1$.
    (a) Find the area of the region.
    (b) Find the volume of the solid formed by revolving this region about the $y-$axis.

15. Use integration by parts to establish the reduction formulas.
    (a) $\displaystyle\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$
    (b) $\displaystyle\int x^n e^{ax} \, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, \quad a \ne 0$

16. Use integration by parts to obtain the formula
$$\int \sqrt{1 - x^2} \, dx = \frac{1}{2} x \sqrt{1 - x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1 - x^2}} \, dx.$$

17. **Arc length** Find the length of the curve
$$y = \ln(\sec x), \quad 0 \le x \le \pi/4.$$

18. **Volume** Find the volume generated by revolving one arch of the curve $y = \sin x$ about the $x-$axis.

19. **Volume** Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sin x + \sec x$, $y = 0, x = 0$, and $x = \pi/3$ about the $x-$axis.

20. **Area** Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9 - x^2}/3$.

21. **Area** Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

22. Evaluate $\displaystyle\int x^3\sqrt{1-x^2}\,dx$ using

    (a) integration by parts.

    (b) a $u-$substituion.

    (c) a trigonometric substitution.

23. Evaluate the integrals

    (a) $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx \qquad (\textit{Hint:}\ \text{Let } x+1 = u^2.)$

    (b) $\displaystyle\int \frac{1}{x(x^4+1)}\,dx \qquad \left(\textit{Hint:}\ \text{Multiply by} \tfrac{x^3}{x^3}.\right)$

24. Solve the following initial value problems for x as a function of t.

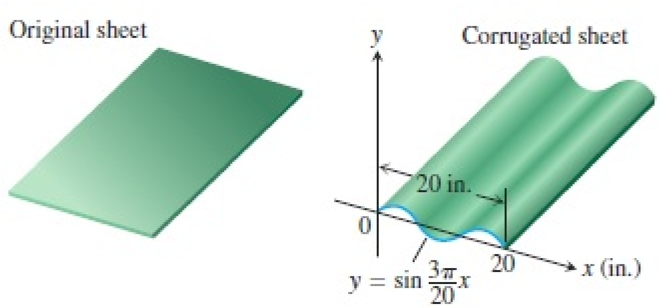
    (a) $(t^2 - 3t + 2)\dfrac{dx}{dt} = 1, \quad (t > 2), \quad x(3) = 0.$

    (b) $(t^2 + 2t)\dfrac{dx}{dt} = 2x + 2, \quad (t, x > 0), \quad x(1) = 1.$

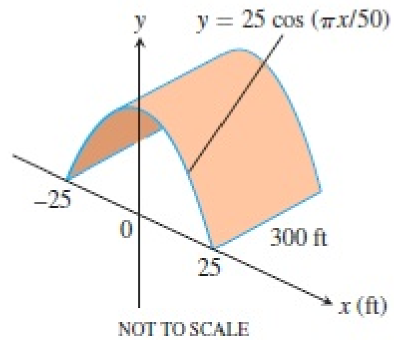
25. Your metal fabrication company is bidding for a contract to make sheets of corrugated iron roofing like the one shown here. The cross-sections of the corrugated sheets are to conform to the curve

$$y = \sin\frac{3\pi}{20}x, \quad 0 \le x \le 20 \text{ in.}$$

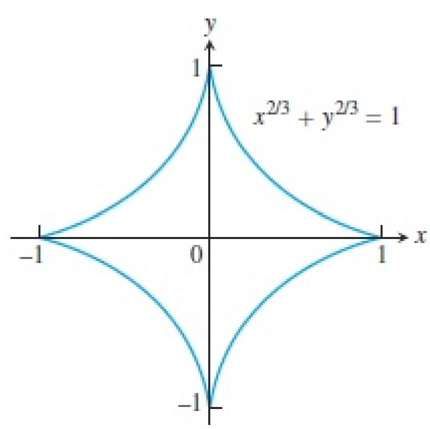
If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? To find out, use numerical integration to approximate the length of the sine curve to two decimal places.



26. Your engineering firm is bidding for the contract to construct the tunnel shown here. The tunnel is 300 ft long and 50 ft wide at the base. The cross-section is shaped like arch of the curve $y = 25\cos(\pi x/50)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$1.75 per square foot to apply. How much will it cost to apply the sealer? (*Hint:* Use numerical integration to find the length of the cosine curve.)



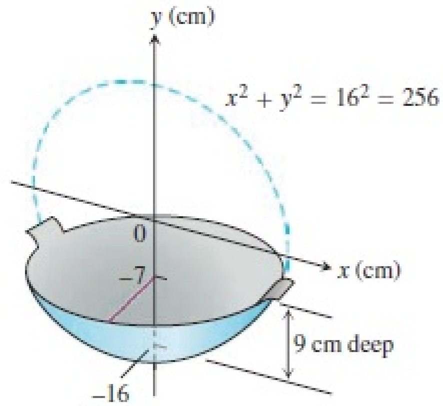
27. **The length of an astroid** The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (see the accompanying figure). Find the length of this particular astroid by finding the length of half the first-quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \le x \le 1$, and multiplying by 8.



28. **Distance between two points** Assume that the two points (x_1, y_1) and (x_2, y_2) lie on the graph of the straight line $y = mx + b$. Use the arc length formula to find the distance between the two points.

29. **Designing a wok** You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentations at home persuades you that you can get one that holds about 3L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, as shown here, and calculate

its volume with an integral. To the nearest cubic centimeter, what volume do you really get? (1L = 1000 cm^3.)



30. In statistics, we define the *mean* $\bar{x}$ and the *variance* s^2 of a sequence of numbers $x_1, x_2, \ldots, x_n$ by

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}, \quad s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}.$$

Use the definitions above to show that

(a) $\sum_{i=1}^{n} (x_i - \bar{x}) = 0$

(b) $s^2 = \dfrac{\sum_{i=1}^{n} x_i^2}{n} - (\bar{x})^2$

31. A grocer stacks oranges in a pyramid-like pile. If the bottom layer is rectangular with 10 rows of 16 oranges and the top layer has a single row of oranges, how many oranges are in the stack?

32. Water leaks out of a 200-gallon storage tank (initially full) at the rate $V'(t) = 20 - t$, where t is measured in hours and V in gallons.

(a) How much water leaked out between 10 and 20 hours?

(b) How long will it take the tank to drain completely?

33. Let $f(x) = -x + 2$. Find c such that $f(c)$ is the average value of f on $[-2, 2]$.

34. Find area under the graph of $f(x) = |x - 1|, \quad 0 \le x \le 2$.

35. Use area to evaluate $\displaystyle\int_0^5 1 + \sqrt{25 - x^2}\, dx$.

36. Find the derivatives of the following functions.

(a) $F(x) = \int_{x}^{e^x} \sin(t^2)\,dt$

(b) $G(x) = \int_{1}^{x^2} x\sqrt{t^2-1}\,dt$

37. Let $f(x) = 3x^2$. Determine c such that $f(c)$ is the average value of f on $[0, 2]$.

38. Evaluate the following integrals.

(a) $\int_1^8 \frac{(1+2x)(1-x^2)}{x^{2/3}}\,dx$

(b) $\int_{1/2}^{e/2} x^2 \ln(2x)\,dx$

(c) $\int_0^{\sqrt{3}} \frac{1}{(x^2+3)^{3/2}}\,dx$

39. Sketch the region $\mathcal{R}$ bounded by the graph of $y = 4x^2$, the y-axis, and the horizontal line $y = 1$, and then find the area of $\mathcal{R}$.

40. Evaluate the following improper integrals.

(a) $\int_0^{\infty} e^{-|x|}\,dx$

(b) $\int_0^1 \frac{1-x}{\sqrt{2x-x^2}}\,dx$

(c) $\int_0^{\infty} \frac{1}{x^2-1}\,dx$

41. Find Laplace transform of $f(t) = t^2$.