$\boxed{\text{Exercises } 10.8}$ Solutions to selected problems

② $f(x) = \sin x, \quad a = 0$ $\qquad f(0) = \sin 0 = 0$

$f'(x) = \cos x \qquad f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \qquad f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -\cos 0 = -1$

$T_0(x) = 0$

$T_1(x) = T_0 + 1\cdot(x-0) = x$

$T_2(x) = T_1(x) + \frac{0}{2!}\cdot(x-0)^2 = x + 0 = x$

$T_3(x) = T_2(x) + \frac{(-1)}{3!}(x-0)^3 = x - \frac{1}{6}x^3$

④ $f(x) = \ln(x+1), \quad a = 0$ $\qquad f(0) = \ln(1) = 0$

$f'(x) = \frac{1}{(x+1)} = (x+1)^{-1} \qquad f'(0) = \frac{1}{0+1} = 1$

$f''(x) = (-1)(x+1)^{-2} \qquad f''(0) = (-1)(0+1)^{-2} = -1$

$f'''(x) = 2(x+1)^{-3} \qquad f'''(0) = 2(0+1)^{-3} = 2$

$T_0(x) = 0$

$T_1(x) = T_0(x) + 1\cdot(x-0) = x$

$T_2(x) = T_1(x) + \frac{(-1)}{2!}(x-0)^2 = x - \frac{1}{2}x^2$

$T_3(x) = T_2(x) + \frac{2}{3!}(x-0)^3 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

⑥ $f(x) = \frac{1}{x+2} = (x+2)^{-1}, \quad a = 0$ $\qquad f(0) = \frac{1}{2}$

$f'(x) = (-1)(x+2)^{-2} \qquad f'(0) = -\frac{1}{4}$

$f''(x) = 2(x+2)^{-3} \qquad f''(0) = \frac{2}{8} = \frac{1}{4}$

$f'''(x) = -6(x+2)^{-4} \qquad f'''(0) = \frac{-6}{2^4} = \frac{-6}{16} = -\frac{3}{8}$

$T_0(x) = \frac{1}{2}$

$T_1(x) = T_0(x) + \left(-\frac{1}{4}\right)(x-0) = \frac{1}{2} - \frac{1}{4}x$

$T_2(x) = T_1(x) + \frac{\frac{1}{4}}{2!}(x-0)^2 = \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2$

$T_3(x) = T_2(x) + \frac{-\frac{3}{8}}{3!}(x-0)^3 = \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3$

⑧ $f(x) = \tan x, \quad a = \frac{\pi}{4}$

$f'(x) = \sec^2 x$

$f''(x) = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x$

$f'''(x) = 2\sec^2 x \cdot \sec^2 x + \tan x \cdot 4\sec x \cdot \sec x \tan x = 2\sec^4 x + 4\sec^2 x \tan^2 x$

$f(\frac{\pi}{4}) = \tan\frac{\pi}{4} = 1$

$f'(\frac{\pi}{4}) = \sec^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$

$f''(\frac{\pi}{4}) = 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 2\cdot 2\cdot 1 = 4$

$f'''(\frac{\pi}{4}) = 2\sec^4 x + 4\sec^2 x\tan^2 x = 2\cdot 4 + 4\cdot 2\cdot 1 = 16$

$T_0(x) = 1$

$T_1(x) = T_0(x) + 2(x-\frac{\pi}{4}) = 1 + 2(x-\frac{\pi}{4})$

$T_2(x) = T_1(x) + \frac{4}{2!}(x-\frac{\pi}{4})^2 = 1 + 2(x-\frac{\pi}{4}) + 2(x-\frac{\pi}{4})^2$

$T_3(x) = T_2(x) + \frac{16}{3!}(x-\frac{\pi}{4})^3 = 1 + 2(x-\frac{\pi}{4}) + 2(x-\frac{\pi}{4})^2 + \frac{8}{3}(x-\frac{\pi}{4})^3$

(10) $f(x) = \sqrt{1-x} = (1-x)^{\frac{1}{2}}$, $a = 0$

$f'(x) = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) = -\frac{1}{2}(1-x)^{-\frac{1}{2}}$

$f''(x) = (\frac{1}{2})(-\frac{1}{2})(1-x)^{-\frac{3}{2}}(-1) = -\frac{1}{4}(1-x)^{-3/2}$

$f'''(x) = (-\frac{1}{4})(-\frac{3}{2})(1-x)^{-\frac{5}{2}}(-1) = -\frac{3}{8}(1-x)^{-\frac{5}{2}}$

$f(0) = \sqrt{1-0} = 1$

$f'(0) = -\frac{1}{2}(1-0)^{-\frac{1}{2}} = -\frac{1}{2}$

$f''(0) = -\frac{1}{4}(1-0)^{-3/2} = -\frac{1}{4}$

$f'''(0) = -\frac{3}{8}(1-0)^{-\frac{5}{2}} = -\frac{3}{8}$

$T_0(x) = 1$

$T_1(x) = T_0(x) + (-\frac{1}{2})(x-0) = 1 - \frac{1}{2}x$

$T_2(x) = T_1(x) + \frac{-\frac{1}{4}}{2!}(x-0)^2 = 1 - \frac{1}{2}x - \frac{1}{8}x^2$

$T_3(x) = T_2(x) + \frac{-\frac{3}{8}}{3!}(x-0)^3 = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$

(12) $f(x) = xe^x$

$f'(x) = xe^x + e^x$

$f''(x) = xe^x + e^x + e^x = xe^x + 2e^x$

$f'''(x) = xe^x + e^x + 2e^x = xe^x + 3e^x$

$\vdots$

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 2$

$f'''(0) = 3$

$\vdots$

Note that $f^{(n)}(x) = xe^x + ne^x$, and that $f^{(n)}(0) = n$

so, $P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)(x-0)^n}{n!} = \sum_{n=0}^{\infty} \frac{n x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n-1)!}$

(24) $f(x) = 2x^3 + x^2 + 3x - 8$, $a = 1$,

$f'(x) = 6x^2 + 2x + 3$

$f''(x) = 12x + 2$

$f'''(x) = 12$

$f^{(4)}(x) = 0$

$\vdots$

$f(1) = 2 + 1 + 3 - 8 = -2$

$f'(1) = 6 + 2 + 3 = 11$

$f''(1) = 12 + 2 = 14$

$f'''(1) = 12$

$f^{(4)}(1) = 0$

$\vdots$

Note that $f^{(n)}(1) = 0$ for $n \geq 4$

So, $$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{3} \frac{f^{(n)}(1)(x-1)^n}{n!} = -2 + 11(x-1) + \frac{14}{2!}(x-1)^2 + \frac{12}{3!}(x-1)^3$$

$$= -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$$