8.7 IMPROPER INTEGRALS

1. $\int_0^{\infty} \frac{dx}{x^2+1} = \lim_{b \to \infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b \to \infty} \left[\tan^{-1} x\right]_0^b = \lim_{b \to \infty} \left(\tan^{-1} b - \tan^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

2. $\int_1^{\infty} \frac{dx}{x^{1.001}} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^{1.001}} = \lim_{b \to \infty} \left[-1000x^{-0.001}\right]_1^b = \lim_{b \to \infty} \left(\frac{-1000}{b^{0.001}} + 1000\right) = 1000$

3. $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b \to 0^+} \int_b^1 x^{-1/2}\, dx = \lim_{b \to 0^+} \left[2x^{1/2}\right]_b^1 = \lim_{b \to 0^+} \left(2 - 2\sqrt{b}\right) = 2 - 0 = 2$

4. $\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{b \to 4^-} \int_0^b (4-x)^{-1/2} dx = \lim_{b \to 4^-} \left[-2\sqrt{4-b} - \left(-2\sqrt{4}\right)\right] = 0 + 4 = 4$

5. $\int_{-1}^1 \frac{dx}{x^{2/3}} = \int_{-1}^0 \frac{dx}{x^{2/3}} + \int_0^1 \frac{dx}{x^{2/3}} = \lim_{b \to 0^-} \left[3x^{1/3}\right]_{-1}^b + \lim_{c \to 0^+} \left[3x^{1/3}\right]_c^1$
$= \lim_{b \to 0^-} \left[3b^{1/3} - 3(-1)^{1/3}\right] + \lim_{c \to 0^+} \left[3(1)^{1/3} - 3c^{1/3}\right] = (0+3) + (3-0) = 6$

6. $\int_{-8}^1 \frac{dx}{x^{1/3}} = \int_{-8}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}} = \lim_{b \to 0^-} \left[\frac{3}{2} x^{2/3}\right]_{-8}^b + \lim_{c \to 0^+} \left[\frac{3}{2} x^{2/3}\right]_c^1$
$= \lim_{b \to 0^-} \left[\frac{3}{2} b^{2/3} - \frac{3}{2}(-8)^{2/3}\right] + \lim_{c \to 0^+} \left[\frac{3}{2}(1)^{2/3} - \frac{3}{2} c^{2/3}\right] = \left[0 - \frac{3}{2}(4)\right] + \left(\frac{3}{2} - 0\right) = -\frac{9}{2}$

7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b \to 1^-} \left[\sin^{-1} x\right]_0^b = \lim_{b \to 1^-} \left(\sin^{-1} b - \sin^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

8. $\int_0^1 \frac{dr}{r^{0.999}} = \lim_{b \to 0^+} \left[1000r^{0.001}\right]_b^1 = \lim_{b \to 0} \left(1000 - 1000b^{0.001}\right) = 1000 - 0 = 1000$

9. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1} = \int_{-\infty}^{-2} \frac{dx}{x-1} - \int_{-\infty}^{-2} \frac{dx}{x+1} = \lim_{b \to -\infty} \left[\ln|x-1|\right]_b^{-2} - \lim_{b \to -\infty} \left[\ln|x+1|\right]_b^{-2} = \lim_{b \to -\infty} \left[\ln\left|\frac{x-1}{x+1}\right|\right]_b^{-2}$
$= \lim_{b \to -\infty} \left(\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right) = \ln 3 - \ln\left(\lim_{b \to -\infty} \frac{b-1}{b+1}\right) = \ln 3 - \ln 1 = \ln 3$

10. $\int_{-\infty}^2 \frac{2\,dx}{x^2+4} = \lim_{b \to -\infty} \left[\tan^{-1} \frac{x}{2}\right]_b^2 = \lim_{b \to -\infty} \left(\tan^{-1} 1 - \tan^{-1} \frac{b}{2}\right) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}$

11. $\int_2^{\infty} \frac{2\,dv}{v^2-v} = \lim_{b \to \infty} \left[2\ln\left|\frac{v-1}{v}\right|\right]_2^b = \lim_{b \to \infty} \left(2\ln\left|\frac{b-1}{b}\right| - 2\ln\left|\frac{2-1}{2}\right|\right) = 2\ln(1) - 2\ln\left(\frac{1}{2}\right) = 0 + 2\ln 2 = \ln 4$

12. $\int_2^{\infty} \frac{2\,dt}{t^2-1} = \lim_{b \to \infty} \left[\ln\left|\frac{t-1}{t+1}\right|\right]_2^b = \lim_{b \to \infty} \left(\ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right) = \ln(1) - \ln\left(\frac{1}{3}\right) = 0 + \ln 3 = \ln 3$

13. $\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = \int_{-\infty}^0 \frac{2x\,dx}{(x^2+1)^2} + \int_0^{\infty} \frac{2x\,dx}{(x^2+1)^2}; \begin{bmatrix} u = x^2+1 \\ du = 2x\,dx \end{bmatrix} \to \int_{\infty}^1 \frac{du}{u^2} + \int_1^{\infty} \frac{du}{u^2} = \lim_{b \to \infty} \left[-\frac{1}{u}\right]_b^1 + \lim_{c \to \infty} \left[-\frac{1}{u}\right]_1^c$
$= \lim_{b \to \infty} \left(-1 + \frac{1}{b}\right) + \lim_{c \to \infty} \left[-\frac{1}{c} - (-1)\right] = (-1+0) + (0+1) = 0$

14. $\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}} = \int_{-\infty}^0 \frac{x\,dx}{(x^2+4)^{3/2}} + \int_0^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}; \begin{bmatrix} u = x^2+4 \\ du = 2x\,dx \end{bmatrix} \to \int_{\infty}^4 \frac{du}{2u^{3/2}} + \int_4^{\infty} \frac{du}{2u^{3/2}}$
$= \lim_{b \to \infty} \left[-\frac{1}{\sqrt{u}}\right]_b^4 + \lim_{c \to \infty} \left[-\frac{1}{\sqrt{u}}\right]_4^c = \lim_{b \to \infty} \left(-\frac{1}{2} + \frac{1}{\sqrt{b}}\right) + \lim_{c \to \infty} \left(-\frac{1}{\sqrt{c}} + \frac{1}{2}\right) = \left(-\frac{1}{2} + 0\right) + \left(0 + \frac{1}{2}\right) = 0$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta;\ \begin{bmatrix} u=\theta^2+2\theta \\ du=2(\theta+1)\,d\theta \end{bmatrix} \rightarrow \int_0^3 \frac{du}{2\sqrt{u}} = \lim_{b\to 0^+} \int_b^3 \frac{du}{2\sqrt{u}} = \lim_{b\to 0^+} \left[\sqrt{u}\right]_b^3 = \lim_{b\to 0^+} \left(\sqrt{3}-\sqrt{b}\right) = \sqrt{3}-0$
$= \sqrt{3}$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds = \frac{1}{2}\int_0^2 \frac{2s\,ds}{\sqrt{4-s^2}} + \int_0^2 \frac{ds}{\sqrt{4-s^2}};\ \begin{bmatrix} u=4-s^2 \\ du=-2s\,ds \end{bmatrix} \rightarrow -\frac{1}{2}\int_4^0 \frac{du}{\sqrt{u}} + \lim_{c\to 2^-}\int_0^c \frac{ds}{\sqrt{4-s^2}}$
$= \lim_{b\to 0^+}\int_b^4 \frac{du}{2\sqrt{u}} + \lim_{c\to 2^-}\int_0^c \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to 0^+}\left[\sqrt{u}\right]_b^4 + \lim_{c\to 2^-}\left[\sin^{-1}\frac{s}{2}\right]_0^c$
$= \lim_{b\to 0^+}\left(2-\sqrt{b}\right) + \lim_{c\to 2^-}\left(\sin^{-1}\frac{c}{2}-\sin^{-1}0\right) = (2-0)+\left(\frac{\pi}{2}-0\right) = \frac{4+\pi}{2}$

17. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}};\ \begin{bmatrix} u=\sqrt{x} \\ du=\frac{dx}{2\sqrt{x}} \end{bmatrix} \rightarrow \int_0^\infty \frac{2\,du}{u^2+1} = \lim_{b\to\infty}\int_0^b \frac{2\,du}{u^2+1} = \lim_{b\to\infty}\left[2\tan^{-1}u\right]_0^b$
$= \lim_{b\to\infty}\left(2\tan^{-1}b - 2\tan^{-1}0\right) = 2\left(\frac{\pi}{2}\right) - 2(0) = \pi$

18. $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{b\to 1^+}\int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty}\int_2^c \frac{dx}{x\sqrt{x^2-1}}$
$= \lim_{b\to 1^+}\left[\sec^{-1}|x|\right]_b^2 + \lim_{c\to\infty}\left[\sec^{-1}|x|\right]_2^c = \lim_{b\to 1^+}\left(\sec^{-1}2-\sec^{-1}b\right) + \lim_{c\to\infty}\left(\sec^{-1}c-\sec^{-1}2\right)$
$= \left(\frac{\pi}{3}-0\right)+\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{\pi}{2}$

19. $\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1}v)} = \lim_{b\to\infty}\left[\ln|1+\tan^{-1}v|\right]_0^b = \lim_{b\to\infty}\left[\ln|1+\tan^{-1}b|\right] - \ln|1+\tan^{-1}0|$
$= \ln\left(1+\frac{\pi}{2}\right) - \ln(1+0) = \ln\left(1+\frac{\pi}{2}\right)$

20. $\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx = \lim_{b\to\infty}\left[8\left(\tan^{-1}x\right)^2\right]_0^b = \lim_{b\to\infty}\left[8\left(\tan^{-1}b\right)^2\right] - 8\left(\tan^{-1}0\right)^2 = 8\left(\frac{\pi}{2}\right)^2 - 8(0) = 2\pi^2$

21. $\int_{-\infty}^0 \theta e^\theta\,d\theta = \lim_{b\to-\infty}\left[\theta e^\theta - e^\theta\right]_b^0 = \left(0\cdot e^0 - e^0\right) - \lim_{b\to-\infty}\left[be^b - e^b\right] = -1 - \lim_{b\to-\infty}\left(\frac{b-1}{e^{-b}}\right)$
$= -1 - \lim_{b\to-\infty}\left(\frac{1}{-e^{-b}}\right)$ (l'Hôpital's rule for $\frac{\infty}{\infty}$ form)
$= -1 - 0 = -1$

22. $\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta = \lim_{b\to\infty}\int_0^b 2e^{-\theta}\sin\theta\,d\theta$
$= \lim_{b\to\infty} 2\left[\frac{e^{-\theta}}{1+1}\left(-\sin\theta-\cos\theta\right)\right]_0^b$ (FORMULA 107 with $a=-1$, $b=1$)
$= \lim_{b\to\infty}\frac{-2(\sin b+\cos b)}{2e^b} + \frac{2(\sin 0+\cos 0)}{2e^0} = 0 + \frac{2(0+1)}{2} = 1$

23. $\int_{-\infty}^0 e^{-|x|}\,dx = \int_{-\infty}^0 e^x\,dx = \lim_{b\to-\infty}\left[e^x\right]_b^0 = \lim_{b\to-\infty}\left(1-e^b\right) = (1-0) = 1$

24. $\int_{-\infty}^\infty 2xe^{-x^2}\,dx = \int_{-\infty}^0 2xe^{-x^2}\,dx + \int_0^\infty 2xe^{-x^2}\,dx = \lim_{b\to-\infty}\left[-e^{-x^2}\right]_b^0 + \lim_{c\to\infty}\left[-e^{-x^2}\right]_0^c$
$= \lim_{b\to-\infty}\left[-1-\left(-e^{-b^2}\right)\right] + \lim_{c\to\infty}\left[-e^{-c^2}-(-1)\right] = (-1-0)+(0+1) = 0$

25. $\int_0^1 x\ln x\,dx = \lim_{b\to 0^+}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_b^1 = \left(\frac{1}{2}\ln 1 - \frac{1}{4}\right) - \lim_{b\to 0^+}\left(\frac{b^2}{2}\ln b - \frac{b^2}{4}\right) = -\frac{1}{4} - \lim_{b\to 0^+}\frac{\ln b}{\left(\frac{2}{b^2}\right)} + 0$
$= -\frac{1}{4} - \lim_{b\to 0^+}\frac{\left(\frac{1}{b}\right)}{\left(-\frac{4}{b^3}\right)} = -\frac{1}{4} + \lim_{b\to 0^+}\left(\frac{b^2}{4}\right) = -\frac{1}{4} + 0 = -\frac{1}{4}$

26. $\int_0^1 (-\ln x)\,dx = \lim_{b \to 0} [x - x \ln x]_b^1 = [1 - 1 \ln 1] - \lim_{b \to 0^+} [b - b \ln b] = 1 - 0 + \lim_{b \to 0^+} \frac{\ln b}{\left(\frac{1}{b}\right)} = 1 + \lim_{b \to 0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$
$= 1 - \lim_{b \to 0^+} b = 1 - 0 = 1$

27. $\int_0^2 \frac{ds}{\sqrt{4-s^2}} = \lim_{b \to 2^-} \left[\sin^{-1} \frac{s}{2}\right]_0^b = \lim_{b \to 2^-} \left(\sin^{-1} \frac{b}{2}\right) - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

28. $\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}} = \lim_{b \to 1^-} \left[2 \sin^{-1} (r^2)\right]_0^b = \lim_{b \to 1^-} \left[2 \sin^{-1} (b^2)\right] - 2 \sin^{-1} 0 = 2 \cdot \frac{\pi}{2} - 0 = \pi$

29. $\int_1^2 \frac{ds}{s\sqrt{s^2-1}} = \lim_{b \to 1} \left[\sec^{-1} s\right]_b^2 = \sec^{-1} 2 - \lim_{b \to 1^+} \sec^{-1} b = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

30. $\int_2^4 \frac{dt}{t\sqrt{t^2-4}} = \lim_{b \to 2^+} \left[\frac{1}{2} \sec^{-1} \frac{t}{2}\right]_b^4 = \lim_{b \to 2^+} \left[\left(\frac{1}{2} \sec^{-1} \frac{4}{2}\right) - \frac{1}{2} \sec^{-1} \left(\frac{b}{2}\right)\right] = \frac{1}{2}\left(\frac{\pi}{3}\right) - \frac{1}{2} \cdot 0 = \frac{\pi}{6}$

31. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}} = \lim_{b \to 0^-} \int_{-1}^b \frac{dx}{\sqrt{-x}} + \lim_{c \to 0^+} \int_c^4 \frac{dx}{\sqrt{x}} = \lim_{b \to 0^-} \left[-2\sqrt{-x}\right]_{-1}^b + \lim_{c \to 0^+} \left[2\sqrt{x}\right]_c^4$
$= \lim_{b \to 0^-} \left(-2\sqrt{-b}\right) - \left(-2\sqrt{-(-1)}\right) + 2\sqrt{4} - \lim_{c \to 0^+} 2\sqrt{c} = 0 + 2 + 2 \cdot 2 - 0 = 6$

32. $\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{b \to 1^-} \left[-2\sqrt{1-x}\right]_0^b + \lim_{c \to 1^+} \left[2\sqrt{x-1}\right]_c^2$
$= \lim_{b \to 1^-} \left(-2\sqrt{1-b}\right) - \left(-2\sqrt{1-0}\right) + 2\sqrt{2-1} - \lim_{c \to 1^+} \left(2\sqrt{c-1}\right) = 0 + 2 + 2 - 0 = 4$

33. $\int_{-1}^\infty \frac{d\theta}{\theta^2 + 5\theta + 6} = \lim_{b \to \infty} \left[\ln \left|\frac{\theta+2}{\theta+3}\right|\right]_{-1}^b = \lim_{b \to \infty} \left[\ln \left|\frac{b+2}{b+3}\right|\right] - \ln \left|\frac{-1+2}{-1+3}\right| = 0 - \ln\left(\frac{1}{2}\right) = \ln 2$

34. $\int_0^\infty \frac{dx}{(x+1)(x^2+1)} = \lim_{b \to \infty} \left[\frac{1}{2} \ln |x+1| - \frac{1}{4} \ln (x^2+1) + \frac{1}{2} \tan^{-1} x\right]_0^b = \lim_{b \to \infty} \left[\frac{1}{2} \ln \left(\frac{x+1}{\sqrt{x^2+1}}\right) + \frac{1}{2} \tan^{-1} x\right]_0^b$
$= \lim_{b \to \infty} \left[\frac{1}{2} \ln \left(\frac{b+1}{\sqrt{b^2+1}}\right) + \frac{1}{2} \tan^{-1} b\right] - \left[\frac{1}{2} \ln \frac{1}{\sqrt{1}} + \frac{1}{2} \tan^{-1} 0\right] = \frac{1}{2} \ln 1 + \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \ln 1 - \frac{1}{2} \cdot 0 = \frac{\pi}{4}$

35. $\int_0^{\pi/2} \tan \theta\,d\theta = \lim_{b \to \frac{\pi}{2}^-} \left[-\ln |\cos \theta|\right]_0^b = \lim_{b \to \frac{\pi}{2}^-} \left[-\ln |\cos b|\right] + \ln 1 = \lim_{b \to \frac{\pi}{2}^-} \left[-\ln |\cos b|\right] = +\infty$, the integral diverges

36. $\int_0^{\pi/2} \cot \theta\,d\theta = \lim_{b \to 0^+} \left[\ln |\sin \theta|\right]_b^{\pi/2} = \ln 1 - \lim_{b \to 0^+} \left[\ln |\sin b|\right] = -\lim_{b \to 0^+} \left[\ln |\sin b|\right] = +\infty$, the integral diverges

37. $\int_0^\pi \frac{\sin \theta\,d\theta}{\sqrt{\pi - \theta}}$; $[\pi - \theta = x] \rightarrow -\int_\pi^0 \frac{\sin x\,dx}{\sqrt{x}} = \int_0^\pi \frac{\sin x\,dx}{\sqrt{x}}$. Since $0 \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ for all $0 \le x \le \pi$ and $\int_0^\pi \frac{dx}{\sqrt{x}}$ converges, then $\int_0^\pi \frac{\sin x}{\sqrt{x}}\,dx$ converges by the Direct Comparison Test.

38. $\int_{-\pi/2}^{\pi/2} \frac{\cos \theta\,d\theta}{(\pi - 2\theta)^{1/3}}$; $\begin{bmatrix} x = \pi - 2\theta \\ \theta = \frac{\pi}{2} - \frac{x}{2} \\ d\theta = -\frac{dx}{2} \end{bmatrix} \rightarrow \int_{2\pi}^0 \frac{-\cos\left(\frac{\pi}{2} - \frac{x}{2}\right)\,dx}{2x^{1/3}} = \int_0^{2\pi} \frac{\sin\left(\frac{x}{2}\right)\,dx}{2x^{1/3}}$. Since $0 \le \frac{\sin \frac{x}{2}}{2x^{1/3}} \le \frac{1}{2x^{1/3}}$ for all $0 \le x \le 2\pi$ and $\int_0^{2\pi} \frac{dx}{2x^{1/3}}$ converges, then $\int_0^{2\pi} \frac{\sin \frac{x}{2}\,dx}{2x^{1/3}}$ converges by the Direct Comparison Test.

39. $\int_0^{\ln 2} x^{-2} e^{-1/x}\,dx$; $\left[\frac{1}{x} = y\right] \rightarrow \int_\infty^{1/\ln 2} \frac{y^2 e^{-y}\,dy}{-y^2} = \int_{1/\ln 2}^\infty e^{-y}\,dy = \lim_{b \to \infty} \left[-e^{-y}\right]_{1/\ln 2}^b = \lim_{b \to \infty} \left[-e^{-b}\right] - \left[-e^{-1/\ln 2}\right]$
$= 0 + e^{-1/\ln 2} = e^{-1/\ln 2}$, so the integral converges.

40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$; $\left[y = \sqrt{x}\right] \rightarrow 2\int_0^1 e^{-y}\,dy = 2 - \frac{2}{e}$, so the integral converges.

41. $\int_0^{\pi} \frac{dt}{\sqrt{t+\sin t}}$. Since for $0 \le t \le \pi$, $0 \le \frac{1}{\sqrt{t+\sin t}} \le \frac{1}{\sqrt{t}}$ and $\int_0^{\pi} \frac{dt}{\sqrt{t}}$ converges, then the original integral converges as well by the Direct Comparison Test.

42. $\int_0^1 \frac{dt}{t-\sin t}$; let $f(t) = \frac{1}{t-\sin t}$ and $g(t) = \frac{1}{t^3}$, then $\lim\limits_{t\to 0} \frac{f(t)}{g(t)} = \lim\limits_{t\to 0} \frac{t^3}{t-\sin t} = \lim\limits_{t\to 0} \frac{3t^2}{1-\cos t} = \lim\limits_{t\to 0} \frac{6t}{\sin t}$ $= \lim\limits_{t\to 0} \frac{6}{\cos t} = 6$. Now, $\int_0^1 \frac{dt}{t^3} = \lim\limits_{b\to 0^+} \left[-\frac{1}{2t^2}\right]_b^1 = -\frac{1}{2} - \lim\limits_{b\to 0^+} \left[-\frac{1}{2b^2}\right] = +\infty$, which diverges $\Rightarrow \int_0^1 \frac{dt}{t-\sin t}$ diverges by the Limit Comparison Test.

43. $\int_0^2 \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$ and $\int_0^1 \frac{dx}{1-x^2} = \lim\limits_{b\to 1^-} \left[\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right]_0^b = \lim\limits_{b\to 1^-} \left[\frac{1}{2}\ln\left|\frac{1+b}{1-b}\right|\right] - 0 = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1-x^2}$ diverges as well.

44. $\int_0^2 \frac{dx}{1-x} = \int_0^1 \frac{dx}{1-x} + \int_1^2 \frac{dx}{1-x}$ and $\int_0^1 \frac{dx}{1-x} = \lim\limits_{b\to 1^-} \left[-\ln(1-x)\right]_0^b = \lim\limits_{b\to 1^-} \left[-\ln(1-b)\right] - 0 = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1-x}$ diverges as well.

45. $\int_{-1}^1 \ln|x|\,dx = \int_{-1}^0 \ln(-x)\,dx + \int_0^1 \ln x\,dx$; $\int_0^1 \ln x\,dx = \lim\limits_{b\to 0^+} \left[x\ln x - x\right]_b^1 = [1\cdot 0 - 1] - \lim\limits_{b\to 0} [b\ln b - b]$ $= -1 - 0 = -1$; $\int_{-1}^0 \ln(-x)\,dx = -1 \Rightarrow \int_{-1}^1 \ln|x|\,dx = -2$ converges.

46. $\int_{-1}^1 (-x\ln|x|)\,dx = \int_{-1}^0 [-x\ln(-x)]\,dx + \int_0^1 (-x\ln x)\,dx = \lim\limits_{b\to 0^+} \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_b^1 - \lim\limits_{c\to 0^+} \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_c^1$ $= \left[\frac{1}{2}\ln 1 - \frac{1}{4}\right] - \lim\limits_{b\to 0^+} \left[\frac{b^2}{2}\ln b - \frac{b^2}{4}\right] - \left[\frac{1}{2}\ln 1 - \frac{1}{4}\text{-}\right] + \lim\limits_{c\to 0^+} \left[\frac{c^2}{2}\ln c - \frac{c^2}{4}\right] = -\frac{1}{4} - 0 + \frac{1}{4} + 0 = 0 \Rightarrow$ the integral converges (see Exercise 25 for the limit calculations).

47. $\int_1^{\infty} \frac{dx}{1+x^3}$; $0 \le \frac{1}{x^3+1} \le \frac{1}{x^3}$ for $1 \le x < \infty$ and $\int_1^{\infty} \frac{dx}{x^3}$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{1+x^3}$ converges by the Direct Comparison Test.

48. $\int_4^{\infty} \frac{dx}{\sqrt{x}-1}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x}-1}\right)}{\left(\frac{1}{\sqrt{x}}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}-1} = \lim\limits_{x\to\infty} \frac{1}{1-\frac{1}{\sqrt{x}}} = \frac{1}{1-0} = 1$ and $\int_4^{\infty} \frac{dx}{\sqrt{x}} = \lim\limits_{b\to\infty} \left[2\sqrt{x}\right]_4^b = \infty$, which diverges $\Rightarrow \int_4^{\infty} \frac{dx}{\sqrt{x}-1}$ diverges by the Limit Comparison Test.

49. $\int_2^{\infty} \frac{dv}{\sqrt{v-1}}$; $\lim\limits_{v\to\infty} \frac{\left(\frac{1}{\sqrt{v-1}}\right)}{\left(\frac{1}{\sqrt{v}}\right)} = \lim\limits_{v\to\infty} \frac{\sqrt{v}}{\sqrt{v-1}} = \lim\limits_{v\to\infty} \frac{1}{\sqrt{1-\frac{1}{v}}} = \frac{1}{\sqrt{1-0}} = 1$ and $\int_2^{\infty} \frac{dv}{\sqrt{v}} = \lim\limits_{b\to\infty} \left[2\sqrt{v}\right]_2^b = \infty$, which diverges $\Rightarrow \int_2^{\infty} \frac{dv}{\sqrt{v-1}}$ diverges by the Limit Comparison Test.

50. $\int_0^{\infty} \frac{d\theta}{1+e^{\theta}}$; $0 \le \frac{1}{1+e^{\theta}} \le \frac{1}{e^{\theta}}$ for $0 \le \theta < \infty$ and $\int_0^{\infty} \frac{d\theta}{e^{\theta}} = \lim\limits_{b\to\infty} \left[-e^{-\theta}\right]_0^b = \lim\limits_{b\to\infty} \left(-e^{-b}+1\right) = 1 \Rightarrow \int_0^{\infty} \frac{d\theta}{e^{\theta}}$ converges $\Rightarrow \int_0^{\infty} \frac{d\theta}{1+e^{\theta}}$ converges by the Direct Comparison Test.

51. $\int_0^{\infty} \frac{dx}{\sqrt{x^6+1}} = \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^{\infty} \frac{dx}{\sqrt{x^6+1}} < \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^{\infty} \frac{dx}{x^3}$ and $\int_1^{\infty} \frac{dx}{x^3} = \lim\limits_{b\to\infty} \left[-\frac{1}{2x^2}\right]_1^b$ $= \lim\limits_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{2}\right) = \frac{1}{2} \Rightarrow \int_0^{\infty} \frac{dx}{\sqrt{x^6+1}}$ converges by the Direct Comparison Test.

52. $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$; $\int_2^\infty \frac{1}{x}\,dx = \lim\limits_{b\to\infty} [\ln b]_2^b = \infty$, which diverges $\Rightarrow \int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges by the Limit Comparison Test.

53. $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$; $\lim\limits_{x\to\infty} \frac{\left(\frac{\sqrt{x}}{x^2}\right)}{\left(\frac{\sqrt{x+1}}{x^2}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+1}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}} = 1$; $\int_1^\infty \frac{\sqrt{x}}{x^2}\,dx = \int_1^\infty \frac{dx}{x^{3/2}}$ $= \lim\limits_{b\to\infty} \left[-2x^{-1/2}\right]_1^b = \lim\limits_{b\to\infty} \left(\frac{-2}{\sqrt{b}} + 2\right) = 2 \Rightarrow \int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ converges by the Limit Comparison Test.

54. $\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{x}{\sqrt{x^4-1}}\right)}{\left(\frac{x}{\sqrt{x^4}}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{x^4}}{\sqrt{x^4-1}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^4}}} = 1$; $\int_2^\infty \frac{x\,dx}{\sqrt{x^4}} = \int_2^\infty \frac{dx}{x} = \lim\limits_{b\to\infty} [\ln x]_2^b = \infty$, which diverges $\Rightarrow \int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$ diverges by the Limit Comparison Test.

55. $\int_\pi^\infty \frac{2+\cos x}{x}\,dx$; $0 < \frac{1}{x} \le \frac{2+\cos x}{x}$ for $x \ge \pi$ and $\int_\pi^\infty \frac{dx}{x} = \lim\limits_{b\to\infty} [\ln x]_\pi^b = \infty$, which diverges $\Rightarrow \int_\pi^\infty \frac{2+\cos x}{x}\,dx$ diverges by the Direct Comparison Test.

56. $\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$; $0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$ for x $\quad \pi$ and $\int_\pi^\infty \frac{2}{x^2}\,dx = \lim\limits_{b\to\infty} \left[-\frac{2}{x}\right]_\pi^b = \lim\limits_{b\to\infty} \left(-\frac{2}{b} + \frac{2}{\pi}\right) = \frac{2}{\pi}$ $\Rightarrow \int_\pi^\infty \frac{2\,dx}{x^2}$ converges $\Rightarrow \int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$ converges by the Direct Comparison Test.

57. $\int_4^\infty \frac{2\,dt}{t^{3/2}-1}$; $\lim\limits_{t\to\infty} \frac{t^{3/2}}{t^{3/2}-1} = 1$ and $\int_4^\infty \frac{2\,dt}{t^{3/2}} = \lim\limits_{b\to\infty} \left[-4t^{-1/2}\right]_4^b = \lim\limits_{b\to\infty} \left(\frac{-4}{\sqrt{b}} + 2\right) = 2 \Rightarrow \int_4^\infty \frac{2\,dt}{t^{3/2}}$ converges $\Rightarrow \int_4^\infty \frac{2\,dt}{t^{3/2}+1}$ converges by the Limit Comparison Test.

58. $\int_2^\infty \frac{dx}{\ln x}$; $0 < \frac{1}{x} < \frac{1}{\ln x}$ for $x > 2$ and $\int_2^\infty \frac{dx}{x}$ diverges $\Rightarrow \int_2^\infty \frac{dx}{\ln x}$ diverges by the Direct Comparison Test.

59. $\int_1^\infty \frac{e^x}{x}\,dx$; $0 < \frac{1}{x} < \frac{e^x}{x}$ for $x > 1$ and $\int_1^\infty \frac{dx}{x}$ diverges $\Rightarrow \int_1^\infty \frac{e^x\,dx}{x}$ diverges by the Direct Comparison Test.

60. $\int_{e^e}^\infty \ln(\ln x)\,dx$; $[x = e^y] \rightarrow \int_e^\infty (\ln y)\,e^y\,dy$; $0 < \ln y < (\ln y)\,e^y$ for $y \ge e$ and $\int_e^\infty \ln y\,dy = \lim\limits_{b\to\infty} [y\ln y - y]_e^b = \infty$, which diverges $\Rightarrow \int_e^\infty \ln e^y\,dy$ diverges $\Rightarrow \int_{e^e}^\infty \ln(\ln x)\,dx$ diverges by the Direct Comparison Test.

61. $\int_1^\infty \frac{dx}{\sqrt{e^x - x}}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{\sqrt{e^x-x}}\right)}{\left(\frac{1}{\sqrt{e^x}}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{e^x}}{\sqrt{e^x-x}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1-\frac{x}{e^x}}} = \frac{1}{\sqrt{1-0}} = 1$; $\int_1^\infty \frac{dx}{\sqrt{e^x}} = \int_1^\infty e^{-x/2}\,dx$ $= \lim\limits_{b\to\infty} \left[-2e^{-x/2}\right]_1^b = \lim\limits_{b\to\infty} \left(-2e^{-b/2} + 2e^{-1/2}\right) = \frac{2}{\sqrt{e}} \Rightarrow \int_1^\infty e^{-x/2}\,dx$ converges $\Rightarrow \int_1^\infty \frac{dx}{\sqrt{e^x-x}}$ converges by the Limit Comparison Test.

62. $\int_1^\infty \frac{dx}{e^x - 2^x}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{e^x-2^x}\right)}{\left(\frac{1}{e^x}\right)} = \lim\limits_{x\to\infty} \frac{e^x}{e^x-2^x} = \lim\limits_{x\to\infty} \frac{1}{1-\left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = 1$ and $\int_1^\infty \frac{dx}{e^x} = \lim\limits_{b\to\infty} \left[-e^{-x}\right]_1^b$ $= \lim\limits_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = \frac{1}{e} \Rightarrow \int_1^\infty \frac{dx}{e^x}$ converges $\Rightarrow \int_1^\infty \frac{dx}{e^x-2^x}$ converges by the Limit Comparison Test.

63. $\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}} = 2\int_0^\infty \frac{dx}{\sqrt{x^4+1}}$; $\int_0^\infty \frac{dx}{\sqrt{x^4+1}} = \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^\infty \frac{dx}{\sqrt{x^4+1}} < \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^\infty \frac{dx}{x^2}$ and $\int_1^\infty \frac{dx}{x^2} = \lim\limits_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim\limits_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1 \Rightarrow \int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$ converges by the Direct Comparison Test.

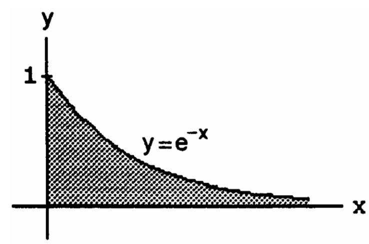
64. $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = 2\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$; $0 < \frac{1}{e^x + e^{-x}} < \frac{1}{e^x}$ for $x > 0$; $\int_0^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow$ $2\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$ converges by the Direct Comparison Test.

65. (a) $\int_1^2 \frac{dx}{x(\ln x)^p}$; $[t = \ln x]$ $\rightarrow \int_0^{\ln 2} \frac{dt}{t^p} = \lim_{b \to 0^+} \left[\frac{1}{-p+1} t^{1-p}\right]_b^{\ln 2} = \lim_{b \to 0^+} \frac{b^{1-p}}{p-1} + \frac{1}{1-p} (\ln 2)^{1-p}$
$\Rightarrow$ the integral converges for $p < 1$ and diverges for $p \geq 1$

(b) $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$; $[t = \ln x]$ $\rightarrow \int_{\ln 2}^{\infty} \frac{dt}{t^p}$ and this integral is essentially the same as in Exercise 65(a): it converges for $p > 1$ and diverges for $p \leq 1$

66. $\int_0^{\infty} \frac{2x\,dx}{x^2+1} = \lim_{b \to \infty} \left[\ln(x^2+1)\right]_0^b = \lim_{b \to \infty} \left[\ln(b^2+1)\right] - 0 = \lim_{b \to \infty} \ln(b^2+1) = \infty \Rightarrow$ the integral $\int_{-\infty}^{\infty} \frac{2x}{x^2+1}\,dx$ diverges. But $\lim_{b \to \infty} \int_{-\infty}^{b} \frac{2x\,dx}{x^2+1} = \lim_{b \to \infty} \left[\ln(x^2+1)\right]_{-b}^b = \lim_{b \to \infty} \left[\ln(b^2+1) - \ln(b^2+1)\right] = \lim_{b \to \infty} \ln\left(\frac{b^2+1}{b^2+1}\right)$
$= \lim_{b \to \infty} (\ln 1) = 0$

67. $A = \int_0^{\infty} e^{-x}\,dx = \lim_{b \to \infty} \left[-e^{-x}\right]_0^b = \lim_{b \to \infty} (-e^{-b}) - (-e^{-0})$
$= 0 + 1 = 1$



68. $\bar{x} = \frac{1}{A}\int_0^{\infty} xe^{-x}\,dx = \lim_{b \to \infty} \left[-xe^{-x} - e^{-x}\right]_0^b = \lim_{b \to \infty} (-be^{-b} - e^{-b}) - (-0 \cdot e^{-0} - e^{-0}) = 0 + 1 = 1;$
$\bar{y} = \frac{1}{2A}\int_0^{\infty} (e^{-x})^2\,dx = \frac{1}{2}\int_0^{\infty} e^{-2x}\,dx = \lim_{b \to \infty} \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_0^b = \lim_{b \to \infty} \frac{1}{2}\left(-\frac{1}{2}e^{-2b}\right) - \frac{1}{2}\left(-\frac{1}{2}e^{-2\cdot 0}\right) = 0 + \frac{1}{4} = \frac{1}{4}$

69. $V = \int_0^{\infty} 2\pi xe^{-x}\,dx = 2\pi\int_0^{\infty} xe^{-x}\,dx = 2\pi \lim_{b \to \infty} \left[-xe^{-x} - e^{-x}\right]_0^b = 2\pi\left[\lim_{b \to \infty} (-be^{-b} - e^{-b}) - 1\right] = 2\pi$

70. $V = \int_0^{\infty} \pi(e^{-x})^2\,dx = \pi\int_0^{\infty} e^{-2x}\,dx = \pi \lim_{b \to \infty} \left[-\frac{1}{2}e^{-2x}\right]_0^b = \pi \lim_{b \to \infty} \left(-\frac{1}{2}e^{-2b} + \frac{1}{2}\right) = \frac{\pi}{2}$

71. $A = \int_0^{\pi/2} (\sec x - \tan x)\,dx = \lim_{b \to \frac{\pi}{2}^-} \left[\ln|\sec x + \tan x| - \ln|\sec x|\right]_0^b = \lim_{b \to \frac{\pi}{2}^-} \left(\ln\left|1 + \frac{\tan b}{\sec b}\right| - \ln|1+0|\right)$
$= \lim_{b \to \frac{\pi}{2}^-} \ln|1 + \sin b| = \ln 2$

72. (a) $V = \int_0^{\pi/2} \pi\sec^2 x\,dx - \int_0^{\pi/2} \pi\tan^2 x\,dx = \pi\int_0^{\pi/2} (\sec^2 x - \tan^2 x)\,dx = \int_0^{\pi/2} \pi\left[\sec^2 x - (\sec^2 x - 1)\right]dx$
$= \pi\int_0^{\pi/2} dx = \frac{\pi^2}{2}$

(b) $S_{outer} = \int_0^{\pi/2} 2\pi\sec x\sqrt{1 + \sec^2 x\tan^2 x}\,dx$ $\quad \int_0^{\pi/2} 2\pi\sec x(\sec x\tan x)\,dx = \pi \lim_{b \to \frac{\pi}{2}^-} \left[\tan^2 x\right]_0^b$
$= \pi\left[\lim_{b \to \frac{\pi}{2}^-} [\tan^2 b] - 0\right] = \pi \lim_{b \to \frac{\pi}{2}^-} (\tan^2 b) = \infty \Rightarrow S_{outer}$ diverges; $S_{inner} = \int_0^{\pi/2} 2\pi\tan x\sqrt{1 + \sec^4 x}\,dx$
$\int_0^{\pi/2} 2\pi\tan x\sec^2 x\,dx = \pi \lim_{b \to \frac{\pi}{2}^-} \left[\tan^2 x\right]_0^b = \pi\left[\lim_{b \to \frac{\pi}{2}^-} [\tan^2 b] - 0\right] = \pi \lim_{b \to \frac{\pi}{2}^-} (\tan^2 b) = \infty$
$\Rightarrow S_{inner}$ diverges