

8.6 NUMERICAL INTEGRATION

1. $\int_1^2 x \, dx$

- I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;
 $\sum mf(x_i) = 12 \Rightarrow T = \frac{1}{8}(12) = \frac{3}{2}$;
 $f(x) = x \Rightarrow f'(x) = 1 \Rightarrow f'' = 0 \Rightarrow M = 0$
 $\Rightarrow |E_T| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	2	5/2
x_2	3/2	3/2	2	3
x_3	7/4	7/4	2	7/2
x_4	2	2	1	2

(b) $\int_1^2 x \, dx = \left[\frac{x^2}{2} \right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2} \Rightarrow |E_T| = \int_1^2 x \, dx - T = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

- II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;
 $\sum mf(x_i) = 18 \Rightarrow S = \frac{1}{12}(18) = \frac{3}{2}$;
 $f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	4	5
x_2	3/2	3/2	2	3
x_3	7/4	7/4	4	7
x_4	2	2	1	2

(b) $\int_1^2 x \, dx = \frac{3}{2} \Rightarrow |E_S| = \int_1^2 x \, dx - S = \frac{3}{2} - \frac{3}{2} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

2. $\int_1^3 (2x - 1) \, dx$

- I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;
 $\sum mf(x_i) = 24 \Rightarrow T = \frac{1}{4}(24) = 6$;
 $f(x) = 2x - 1 \Rightarrow f'(x) = 2 \Rightarrow f'' = 0 \Rightarrow M = 0$
 $\Rightarrow |E_T| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	2	4
x_2	2	3	2	6
x_3	5/2	4	2	8
x_4	3	5	1	5

(b) $\int_1^3 (2x - 1) \, dx = [x^2 - x]_1^3 = (9 - 3) - (1 - 1) = 6 \Rightarrow |E_T| = \int_1^3 (2x - 1) \, dx - T = 6 - 6 = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

- II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
 $\sum mf(x_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6$;
 $f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	4	8
x_2	2	3	2	6
x_3	5/2	4	4	16
x_4	3	5	1	5

(b) $\int_1^3 (2x - 1) \, dx = 6 \Rightarrow |E_S| = \int_1^3 (2x - 1) \, dx - S = 6 - 6 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

3. $\int_{-1}^1 (x^2 + 1) \, dx$

- I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;
 $\sum mf(x_i) = 11 \Rightarrow T = \frac{1}{4}(11) = 2.75$;
 $f(x) = x^2 + 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2 \Rightarrow M = 2$
 $\Rightarrow |E_T| \leq \frac{1-(-1)}{12} \left(\frac{1}{2}\right)^2 (2) = \frac{1}{12}$ or 0.08333

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	-1	2	1	2
x_1	-1/2	5/4	2	5/2
x_2	0	1	2	2
x_3	1/2	5/4	2	5/2
x_4	1	2	1	2

(b) $\int_{-1}^1 (x^2 + 1) \, dx = \left[\frac{x^3}{3} + x \right]_{-1}^1 = \left(\frac{1}{3} + 1\right) - \left(-\frac{1}{3} - 1\right) = \frac{8}{3} \Rightarrow E_T = \int_{-1}^1 (x^2 + 1) \, dx - T = \frac{8}{3} - \frac{11}{4} = -\frac{1}{12}$
 $\Rightarrow |E_T| = \left| -\frac{1}{12} \right| \approx 0.08333$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{1}{8}\right) \times 100 \approx 3\%$

- II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
 $\sum mf(x_i) = 16 \Rightarrow S = \frac{1}{6}(16) = \frac{8}{3} = 2.66667$;
 $f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$
- (b) $\int_{-1}^1 (x^2 + 1) dx = \left[\frac{x^3}{3} + x \right]_{-1}^1 = \frac{8}{3}$
 $\Rightarrow |E_S| = \int_{-1}^1 (x^2 + 1) dx - S = \frac{8}{3} - \frac{8}{3} = 0$
- (c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	-1	2	1	2
x_1	-1/2	5/4	4	5
x_2	0	1	2	2
x_3	1/2	5/4	4	5
x_4	1	2	1	2

4. $\int_{-2}^0 (x^2 - 1) dx$

- I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{0-(-2)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$
 $\sum mf(x_i) = 3 \Rightarrow T = \frac{1}{4}(3) = \frac{3}{4}$;
 $f(x) = x^2 - 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2$
 $\Rightarrow M = 2 \Rightarrow |E_T| \leq \frac{0-(-2)}{12} \left(\frac{1}{2}\right)^2(2) = \frac{1}{12} = 0.08333$
- (b) $\int_{-2}^0 (x^2 - 1) dx = \left[\frac{x^3}{3} - x \right]_{-2}^0 = 0 - \left(-\frac{8}{3} + 2\right) = \frac{2}{3} \Rightarrow E_T = \int_{-2}^0 (x^2 - 1) dx - T = \frac{2}{3} - \frac{3}{4} = -\frac{1}{12}$
 $\Rightarrow |E_T| = \frac{1}{12}$
- (c) $\frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{1/12}{2/3}\right) \times 100 \approx 13\%$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	-2	3	1	3
x_1	-3/2	5/4	2	5/2
x_2	-1	0	2	0
x_3	-1/2	-3/4	2	-3/2
x_4	0	-1	1	-1

- II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{0-(-2)}{4} = \frac{2}{4} = \frac{1}{2}$
 $\Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$; $\sum mf(x_i) = 4 \Rightarrow S = \frac{1}{6}(4) = \frac{2}{3}$;
 $f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$
- (b) $\int_{-2}^0 (x^2 - 1) dx = \frac{2}{3} \Rightarrow |E_S| = \int_{-2}^0 (x^2 - 1) dx - S = \frac{2}{3} - \frac{2}{3} = 0$
- (c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	-2	3	1	3
x_1	-3/2	5/4	4	5
x_2	-1	0	2	0
x_3	-1/2	-3/4	4	-3
x_4	0	-1	1	-1

5. $\int_0^2 (t^3 + t) dt$

- I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2}$
 $\Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$; $\sum mf(t_i) = 25 \Rightarrow T = \frac{1}{4}(25) = \frac{25}{4}$;
 $f(t) = t^3 + t \Rightarrow f'(t) = 3t^2 + 1 \Rightarrow f''(t) = 6t$
 $\Rightarrow M = 12 = f''(2) \Rightarrow |E_T| \leq \frac{2-0}{12} \left(\frac{1}{2}\right)^2(12) = \frac{1}{2}$
- (b) $\int_0^2 (t^3 + t) dt = \left[\frac{t^4}{4} + \frac{t^2}{2} \right]_0^2 = \left(\frac{2^4}{4} + \frac{2^2}{2}\right) - 0 = 6 \Rightarrow |E_T| = \int_0^2 (t^3 + t) dt - T = 6 - \frac{25}{4} = -\frac{1}{4} \Rightarrow |E_T| = \frac{1}{4}$
- (c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{|-1/4|}{6} \times 100 \approx 4\%$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/2	5/8	2	5/4
t_2	1	2	2	4
t_3	3/2	39/8	2	39/4
t_4	2	10	1	10

- II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
 $\sum mf(t_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6$;
 $f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$
- (b) $\int_0^2 (t^3 + t) dt = 6 \Rightarrow |E_S| = \int_0^2 (t^3 + t) dt - S = 6 - 6 = 0$
- (c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/2	5/8	4	5/2
t_2	1	2	2	4
t_3	3/2	39/8	4	39/2
t_4	2	10	1	10

6. $\int_{-1}^1 (t^3 + 1) dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2}$
 $\Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$; $\sum mf(t_i) = 8 \Rightarrow T = \frac{1}{4}(8) = 2$;
 $f(t) = t^3 + 1 \Rightarrow f'(t) = 3t^2 \Rightarrow f''(t) = 6t$
 $\Rightarrow M = 6 = f''(1) \Rightarrow |E_T| \leq \frac{1-(-1)}{12} \left(\frac{1}{2}\right)^2(6) = \frac{1}{4}$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	-1	0	1	0
t_1	-1/2	7/8	2	7/4
t_2	0	1	2	2
t_3	1/2	9/8	2	9/4
t_4	1	2	1	2

(b) $\int_{-1}^1 (t^3 + 1) dt = \left[\frac{t^4}{4} + t\right]_{-1}^1 = \left(\frac{1^4}{4} + 1\right) - \left(\frac{(-1)^4}{4} + (-1)\right) = 2 \Rightarrow |E_T| = \int_{-1}^1 (t^3 + 1) dt - T = 2 - 2 = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2}$
 $\Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$; $\sum mf(t_i) = 12 \Rightarrow S = \frac{1}{6}(12) = 2$;
 $f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	-1	0	1	0
t_1	-1/2	7/8	4	7/2
t_2	0	1	2	2
t_3	1/2	9/8	4	9/2
t_4	1	2	1	2

(b) $\int_{-1}^1 (t^3 + 1) dt = 2 \Rightarrow |E_S| = \int_{-1}^1 (t^3 + 1) dt - S = 2 - 2 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

7. $\int_1^2 \frac{1}{s^2} ds$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;
 $\sum mf(s_i) = \frac{179,573}{44,100} \Rightarrow T = \frac{1}{8} \left(\frac{179,573}{44,100}\right) = \frac{179,573}{352,800}$
 ≈ 0.50899 ; $f(s) = \frac{1}{s^2} \Rightarrow f'(s) = -\frac{2}{s^3}$
 $\Rightarrow f''(s) = \frac{6}{s^4} \Rightarrow M = 6 = f''(1)$
 $\Rightarrow |E_T| \leq \frac{2-1}{12} \left(\frac{1}{4}\right)^2(6) = \frac{1}{32} = 0.03125$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	1	1	1	1
s_1	5/4	16/25	2	32/25
s_2	3/2	4/9	2	8/9
s_3	7/4	16/49	2	32/49
s_4	2	1/4	1	1/4

(b) $\int_1^2 \frac{1}{s^2} ds = \int_1^2 s^{-2} ds = \left[-\frac{1}{s}\right]_1^2 = -\frac{1}{2} - \left(-\frac{1}{1}\right) = \frac{1}{2} \Rightarrow E_T = \int_1^2 \frac{1}{s^2} ds - T = \frac{1}{2} - 0.50899 = -0.00899$
 $\Rightarrow |E_T| = 0.00899$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.00899}{0.5} \times 100 \approx 2\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;
 $\sum mf(s_i) = \frac{264,821}{44,100} \Rightarrow S = \frac{1}{12} \left(\frac{264,821}{44,100}\right) = \frac{264,821}{529,200}$
 ≈ 0.50042 ; $f^{(3)}(s) = -\frac{24}{s^5} \Rightarrow f^{(4)}(s) = \frac{120}{s^6}$
 $\Rightarrow M = 120 \Rightarrow |E_S| \leq \left|\frac{2-1}{180}\right| \left(\frac{1}{4}\right)^4(120)$
 $= \frac{1}{384} \approx 0.00260$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	1	1	1	1
s_1	5/4	16/25	4	64/25
s_2	3/2	4/9	2	8/9
s_3	7/4	16/49	4	64/49
s_4	2	1/4	1	1/4

(b) $\int_1^2 \frac{1}{s^2} ds = \frac{1}{2} \Rightarrow E_S = \int_1^2 \frac{1}{s^2} ds - S = \frac{1}{2} - 0.50042 = -0.00042 \Rightarrow |E_S| = 0.00042$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.0004}{0.5} \times 100 \approx 0.08\%$

8. $\int_2^4 \frac{1}{(s-1)^2} ds$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;
 $\sum mf(s_i) = \frac{1269}{450}$
 $\Rightarrow T = \frac{1}{4} \left(\frac{1269}{450}\right) = \frac{1269}{1800} = 0.70500$;
 $f(s) = (s-1)^{-2} \Rightarrow f'(s) = -\frac{2}{(s-1)^3}$
 $\Rightarrow f''(s) = \frac{6}{(s-1)^4} \Rightarrow M = 6$
 $\Rightarrow |E_T| \leq \frac{4-2}{12} \left(\frac{1}{2}\right)^2(6) = \frac{1}{4} = 0.25$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	2	1	1	1
s_1	5/2	4/9	2	8/9
s_2	3	1/4	2	1/2
s_3	7/2	4/25	2	8/25
s_4	4	1/9	1	1/9

(b) $\int_2^4 \frac{1}{(s-1)^2} ds = \left[\frac{-1}{s-1} \right]_2^4 = \left(\frac{-1}{4-1} \right) - \left(\frac{-1}{2-1} \right) = \frac{2}{3} \Rightarrow E_T = \int_2^4 \frac{1}{(s-1)^2} ds - T = \frac{2}{3} - 0.705 \approx -0.03833$
 $\Rightarrow |E_T| \approx 0.03833$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03833}{\left(\frac{2}{3}\right)} \times 100 \approx 6\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;

$\sum mf(s_i) = \frac{1813}{450}$
 $\Rightarrow S = \frac{1}{6} \left(\frac{1813}{450} \right) = \frac{1813}{2700} \approx 0.67148$;
 $f^{(3)}(s) = \frac{-24}{(s-1)^3} \Rightarrow f^{(4)}(s) = \frac{120}{(s-1)^4} \Rightarrow M = 120$
 $\Rightarrow |E_S| \leq \frac{4-2}{180} \left(\frac{1}{2} \right)^4 (120) = \frac{1}{12} \approx 0.08333$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	2	1	1	1
s_1	5/2	4/9	4	16/9
s_2	3	1/4	2	1/2
s_3	7/2	4/25	4	16/25
s_4	4	1/9	1	1/9

(b) $\int_2^4 \frac{1}{(s-1)^2} ds = \frac{2}{3} \Rightarrow E_S = \int_2^4 \frac{1}{(s-1)^2} ds - S \approx \frac{2}{3} - 0.67148 = -0.00481 \Rightarrow |E_S| \approx 0.00481$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00481}{\left(\frac{2}{3}\right)} \times 100 \approx 1\%$

9. $\int_0^\pi \sin t dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{\Delta x}{2} = \frac{\pi}{8}$;

$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.8284$
 $\Rightarrow T = \frac{\pi}{8} (2 + 2\sqrt{2}) \approx 1.89612$;

$f(t) = \sin t \Rightarrow f'(t) = \cos t \Rightarrow f''(t) = -\sin t$
 $\Rightarrow M = 1 \Rightarrow |E_T| \leq \frac{\pi-0}{12} \left(\frac{\pi}{4} \right)^2 (1) = \frac{\pi^3}{192}$
 ≈ 0.16149

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	$\pi/4$	$\sqrt{2}/2$	2	$\sqrt{2}$
t_2	$\pi/2$	1	2	2
t_3	$3\pi/4$	$\sqrt{2}/2$	2	$\sqrt{2}$
t_4	π	0	1	0

(b) $\int_0^\pi \sin t dt = [-\cos t]_0^\pi = (-\cos \pi) - (-\cos 0) = 2 \Rightarrow |E_T| = \int_0^\pi \sin t dt - T \approx 2 - 1.89612 = 0.10388$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.10388}{2} \times 100 \approx 5\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{12}$;

$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.6569$
 $\Rightarrow S = \frac{\pi}{12} (2 + 4\sqrt{2}) \approx 2.00456$;

$f^{(3)}(t) = -\cos t \Rightarrow f^{(4)}(t) = \sin t$
 $\Rightarrow M = 1 \Rightarrow |E_S| \leq \frac{\pi-0}{180} \left(\frac{\pi}{4} \right)^4 (1) \approx 0.00664$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	$\pi/4$	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_2	$\pi/2$	1	2	2
t_3	$3\pi/4$	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_4	π	0	1	0

(b) $\int_0^\pi \sin t dt = 2 \Rightarrow E_S = \int_0^\pi \sin t dt - S \approx 2 - 2.00456 = -0.00456 \Rightarrow |E_S| \approx 0.00456$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00456}{2} \times 100 \approx 0\%$

10. $\int_0^1 \sin \pi t dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;

$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.828$
 $\Rightarrow T = \frac{1}{8} (2 + 2\sqrt{2}) \approx 0.60355$; $f(t) = \sin \pi t$
 $\Rightarrow f'(t) = \pi \cos \pi t$
 $\Rightarrow f''(t) = -\pi^2 \sin \pi t \Rightarrow M = \pi^2$
 $\Rightarrow |E_T| \leq \frac{1-0}{12} \left(\frac{1}{4} \right)^2 (\pi^2) \approx 0.05140$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/4	$\sqrt{2}/2$	2	$\sqrt{2}$
t_2	1/2	1	2	2
t_3	3/4	$\sqrt{2}/2$	2	$\sqrt{2}$
t_4	1	0	1	0

(b) $\int_0^1 \sin \pi t dt = \left[-\frac{1}{\pi} \cos \pi t \right]_0^1 = \left(-\frac{1}{\pi} \cos \pi \right) - \left(-\frac{1}{\pi} \cos 0 \right) = \frac{2}{\pi} \approx 0.63662 \Rightarrow |E_T| = \int_0^1 \sin \pi t dt - T$
 $\approx \frac{2}{\pi} - 0.60355 = 0.03307$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03307}{\left(\frac{2}{\pi}\right)} \times 100 \approx 5\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;

$$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.65685$$

$$\Rightarrow S = \frac{1}{12} (2 + 4\sqrt{2}) \approx 0.63807;$$

$$f^{(3)}(t) = -\pi^3 \cos \pi t \Rightarrow f^{(4)}(t) = \pi^4 \sin \pi t$$

$$\Rightarrow M = \pi^4 \Rightarrow |E_S| \leq \frac{1-0}{180} \left(\frac{1}{4}\right)^4 (\pi^4) \approx 0.00211$$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/4	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_2	1/2	1	2	2
t_3	3/4	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_4	1	0	1	0

(b) $\int_0^1 \sin \pi t \, dt = \frac{2}{\pi} \approx 0.63662 \Rightarrow E_S = \int_0^1 \sin \pi t \, dt - S \approx \frac{2}{\pi} - 0.63807 = -0.00145 \Rightarrow |E_S| \approx 0.00145$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00145}{(\frac{2}{\pi})} \times 100 \approx 0\%$

11. (a) $M = 0$ (see Exercise 1): Then $n = 1 \Rightarrow \Delta x = 1 \Rightarrow |E_T| = \frac{1}{12} (1)^2(0) = 0 < 10^{-4}$

(b) $M = 0$ (see Exercise 1): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = \frac{1}{2} \Rightarrow |E_S| = \frac{1}{180} \left(\frac{1}{2}\right)^4 (0) = 0 < 10^{-4}$

12. (a) $M = 0$ (see Exercise 2): Then $n = 1 \Rightarrow \Delta x = 2 \Rightarrow |E_T| = \frac{2}{12} (2)^2(0) = 0 < 10^{-4}$

(b) $M = 0$ (see Exercise 2): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180} (1)^4(0) = 0 < 10^{-4}$

13. (a) $M = 2$ (see Exercise 3): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12} \left(\frac{2}{n}\right)^2 (2) = \frac{4}{3n^2} < 10^{-4} \Rightarrow n^2 > \frac{4}{3} (10^4) \Rightarrow n > \sqrt{\frac{4}{3} (10^4)}$
 $\Rightarrow n > 115.4$, so let $n = 116$

(b) $M = 0$ (see Exercise 3): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180} (1)^4(0) = 0 < 10^{-4}$

14. (a) $M = 2$ (see Exercise 4): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12} \left(\frac{2}{n}\right)^2 (2) = \frac{4}{3n^2} < 10^{-4} \Rightarrow n^2 > \frac{4}{3} (10^4) \Rightarrow n > \sqrt{\frac{4}{3} (10^4)}$
 $\Rightarrow n > 115.4$, so let $n = 116$

(b) $M = 0$ (see Exercise 4): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180} (1)^4(0) = 0 < 10^{-4}$

15. (a) $M = 12$ (see Exercise 5): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12} \left(\frac{2}{n}\right)^2 (12) = \frac{8}{n^2} < 10^{-4} \Rightarrow n^2 > 8 (10^4) \Rightarrow n > \sqrt{8 (10^4)}$
 $\Rightarrow n > 282.8$, so let $n = 283$

(b) $M = 0$ (see Exercise 5): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180} (1)^4(0) = 0 < 10^{-4}$

16. (a) $M = 6$ (see Exercise 6): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12} \left(\frac{2}{n}\right)^2 (6) = \frac{4}{n^2} < 10^{-4} \Rightarrow n^2 > 4 (10^4) \Rightarrow n > \sqrt{4 (10^4)}$
 $= 200$, so let $n = 201$

(b) $M = 0$ (see Exercise 6): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180} (1)^4(0) = 0 < 10^{-4}$

17. (a) $M = 6$ (see Exercise 7): Then $\Delta x = \frac{1}{n} \Rightarrow |E_T| \leq \frac{1}{12} \left(\frac{1}{n}\right)^2 (6) = \frac{1}{2n^2} < 10^{-4} \Rightarrow n^2 > \frac{1}{2} (10^4) \Rightarrow n > \sqrt{\frac{1}{2} (10^4)}$
 $\Rightarrow n > 70.7$, so let $n = 71$

(b) $M = 120$ (see Exercise 7): Then $\Delta x = \frac{1}{n} \Rightarrow |E_S| = \frac{1}{180} \left(\frac{1}{n}\right)^4 (120) = \frac{2}{3n^4} < 10^{-4} \Rightarrow n^4 > \frac{2}{3} (10^4)$
 $\Rightarrow n > \sqrt[4]{\frac{2}{3} (10^4)} \Rightarrow n > 9.04$, so let $n = 10$ (n must be even)

18. (a) $M = 6$ (see Exercise 8): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12} \left(\frac{2}{n}\right)^2 (6) = \frac{4}{n^2} < 10^{-4} \Rightarrow n^2 > 4 (10^4) \Rightarrow n > \sqrt{4 (10^4)}$
 $\Rightarrow n > 200$, so let $n = 201$

(b) $M = 120$ (see Exercise 8): Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \leq \frac{2}{180} \left(\frac{2}{n}\right)^4 (120) = \frac{64}{3n^4} < 10^{-4} \Rightarrow n^4 > \frac{64}{3} (10^4)$
 $\Rightarrow n > \sqrt[4]{\frac{64}{3} (10^4)} \Rightarrow n > 21.5$, so let $n = 22$ (n must be even)

19. (a) $f(x) = \sqrt{x+1} \Rightarrow f'(x) = \frac{1}{2}(x+1)^{-1/2} \Rightarrow f''(x) = -\frac{1}{4}(x+1)^{-3/2} = -\frac{1}{4(\sqrt{x+1})^3} \Rightarrow M = \frac{1}{4(\sqrt{1})^3} = \frac{1}{4}$.

Then $\Delta x = \frac{3}{n} \Rightarrow |E_T| \leq \frac{3}{12} \left(\frac{3}{n}\right)^2 \left(\frac{1}{4}\right) = \frac{9}{16n^2} < 10^{-4} \Rightarrow n^2 > \frac{9}{16}(10^4) \Rightarrow n > \sqrt{\frac{9}{16}(10^4)} \Rightarrow n > 75$,
so let $n = 76$

(b) $f^{(3)}(x) = \frac{3}{8}(x+1)^{-5/2} \Rightarrow f^{(4)}(x) = -\frac{15}{16}(x+1)^{-7/2} = -\frac{15}{16(\sqrt{x+1})^7} \Rightarrow M = \frac{15}{16(\sqrt{1})^7} = \frac{15}{16}$. Then $\Delta x = \frac{3}{n}$
 $\Rightarrow |E_S| \leq \frac{3}{180} \left(\frac{3}{n}\right)^4 \left(\frac{15}{16}\right) = \frac{3^5(15)}{16(180)n^4} < 10^{-4} \Rightarrow n^4 > \frac{3^5(15)(10^4)}{16(180)} \Rightarrow n > \sqrt[4]{\frac{3^5(15)(10^4)}{16(180)}} \Rightarrow n > 10.6$, so let
 $n = 12$ (n must be even)

20. (a) $f(x) = \frac{1}{\sqrt{x+1}} \Rightarrow f'(x) = -\frac{1}{2}(x+1)^{-3/2} \Rightarrow f''(x) = \frac{3}{4}(x+1)^{-5/2} = \frac{3}{4(\sqrt{x+1})^5} \Rightarrow M = \frac{3}{4(\sqrt{1})^5} = \frac{3}{4}$.

Then $\Delta x = \frac{3}{n} \Rightarrow |E_T| \leq \frac{3}{12} \left(\frac{3}{n}\right)^2 \left(\frac{3}{4}\right) = \frac{3^4}{48n^2} < 10^{-4} \Rightarrow n^2 > \frac{3^4(10^4)}{48} \Rightarrow n > \sqrt{\frac{3^4(10^4)}{48}} \Rightarrow n > 129.9$, so let $n = 130$

(b) $f^{(3)}(x) = -\frac{15}{8}(x+1)^{-7/2} \Rightarrow f^{(4)}(x) = \frac{105}{16}(x+1)^{-9/2} = \frac{105}{16(\sqrt{x+1})^9} \Rightarrow M = \frac{105}{16(\sqrt{1})^9} = \frac{105}{16}$. Then $\Delta x = \frac{3}{n}$
 $\Rightarrow |E_S| \leq \frac{3}{180} \left(\frac{3}{n}\right)^4 \left(\frac{105}{16}\right) = \frac{3^5(105)}{16(180)n^4} < 10^{-4} \Rightarrow n^4 > \frac{3^5(105)(10^4)}{16(180)} \Rightarrow n > \sqrt[4]{\frac{3^5(105)(10^4)}{16(180)}} \Rightarrow n > 17.25$, so
let $n = 18$ (n must be even)

21. (a) $f(x) = \sin(x+1) \Rightarrow f'(x) = \cos(x+1) \Rightarrow f''(x) = -\sin(x+1) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \leq \frac{2}{12} \left(\frac{2}{n}\right)^2 (1) = \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8(10^4)}{12} \Rightarrow n > \sqrt{\frac{8(10^4)}{12}} \Rightarrow n > 81.6$, so let $n = 82$

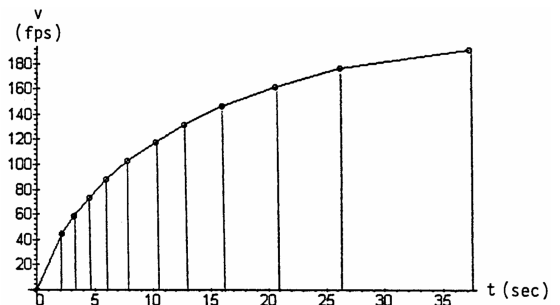
(b) $f^{(3)}(x) = -\cos(x+1) \Rightarrow f^{(4)}(x) = \sin(x+1) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \leq \frac{2}{180} \left(\frac{2}{n}\right)^4 (1) = \frac{32}{180n^4} < 10^{-4} \Rightarrow n^4 > \frac{32(10^4)}{180} \Rightarrow n > \sqrt[4]{\frac{32(10^4)}{180}} \Rightarrow n > 6.49$, so let $n = 8$ (n must be even)

22. (a) $f(x) = \cos(x+\pi) \Rightarrow f'(x) = -\sin(x+\pi) \Rightarrow f''(x) = -\cos(x+\pi) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n}$
 $\Rightarrow |E_T| \leq \frac{2}{12} \left(\frac{2}{n}\right)^2 (1) = \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8(10^4)}{12} \Rightarrow n > \sqrt{\frac{8(10^4)}{12}} \Rightarrow n > 81.6$, so let $n = 82$

(b) $f^{(3)}(x) = \sin(x+\pi) \Rightarrow f^{(4)}(x) = \cos(x+\pi) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \leq \frac{2}{180} \left(\frac{2}{n}\right)^4 (1) = \frac{32}{180n^4} < 10^{-4} \Rightarrow n^4 > \frac{32(10^4)}{180} \Rightarrow n > \sqrt[4]{\frac{32(10^4)}{180}} \Rightarrow n > 6.49$, so let $n = 8$ (n must be even)

23. $\frac{5}{2}(6.0 + 2(8.2) + 2(9.1) \dots + 2(12.7) + 13.0)(30) = 15,990 \text{ ft}^3$.

24. Use the conversion 30 mph = 44 fps (ft per sec) since time is measured in seconds. The distance traveled as the car accelerates from, say, 40 mph = 58.67 fps to 50 mph = 73.33 fps in $(4.5 - 3.2) = 1.3$ sec is the area of the trapezoid (see figure) associated with that time interval: $\frac{1}{2}(58.67 + 73.33)(1.3) = 85.8$ ft. The total distance traveled by the Ford Mustang Cobra is the sum of all these eleven trapezoids (using $\frac{\Delta t}{2}$ and the table below):



v (mph)	0	30	40	50	60	70	80	90	100	110	120	130
v (fps)	0	44	58.67	73.33	88	102.67	117.33	132	146.67	161.33	176	190.67
t (sec)	0	2.2	3.2	4.5	5.9	7.8	10.2	12.7	16	20.6	26.2	37.1
$\Delta t/2$	0	1.1	0.5	0.65	0.7	0.95	1.2	1.25	1.65	2.3	2.8	5.45

$$s = (44)(1.1) + (102.67)(0.5) + (132)(0.65) + (161.33)(0.7) + (190.67)(0.95) + (220)(1.2) + (249.33)(1.25) + (278.67)(1.65) + (308)(2.3) + (337.33)(2.8) + (366.67)(5.45) = 5166.346 \text{ ft} \approx 0.9785 \text{ mi}$$

25. Using Simpson's Rule, $\Delta x = 1 \Rightarrow \frac{\Delta x}{3} = \frac{1}{3}$;
 $\sum my_i = 33.6 \Rightarrow$ Cross Section Area $\approx \frac{1}{3}(33.6)$
 $= 11.2 \text{ ft}^2$. Let x be the length of the tank. Then the
 Volume $V = (\text{Cross Sectional Area})x = 11.2x$.
 Now 5000 lb of gasoline at 42 lb/ft³
 $\Rightarrow V = \frac{5000}{42} = 119.05 \text{ ft}^3$
 $\Rightarrow 119.05 = 11.2x \Rightarrow x \approx 10.63 \text{ ft}$

	x_i	y_i	m	my_i
x_0	0	1.5	1	1.5
x_1	1	1.6	4	6.4
x_2	2	1.8	2	3.6
x_3	3	1.9	4	7.6
x_4	4	2.0	2	4.0
x_5	5	2.1	4	8.4
x_6	6	2.1	1	2.1

26. $\frac{24}{2}[0.019 + 2(0.020) + 2(0.021) + \dots + 2(0.031) + 0.035] = 4.2 \text{ L}$

27. (a) $|E_s| \leq \frac{b-a}{180} (\Delta x^4) M$; $n = 4 \Rightarrow \Delta x = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$; $|f^{(4)}| \leq 1 \Rightarrow M = 1 \Rightarrow |E_s| \leq \frac{(\frac{\pi}{2}-0)}{180} (\frac{\pi}{8})^4 (1) \approx 0.00021$

(b) $\Delta x = \frac{\pi}{8} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{24}$;
 $\sum mf(x_i) = 10.47208705$
 $\Rightarrow S = \frac{\pi}{24} (10.47208705) \approx 1.37079$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	1	1	1
x_1	$\pi/8$	0.974495358	4	3.897981432
x_2	$\pi/4$	0.900316316	2	1.800632632
x_3	$3\pi/8$	0.784213303	4	3.136853212
x_4	$\pi/2$	0.636619772	1	0.636619772

(c) $\approx (\frac{0.00021}{1.37079}) \times 100 \approx 0.015\%$

28. (a) $\Delta x = \frac{b-a}{n} = \frac{1-0}{10} = 0.1 \Rightarrow \text{erf}(1) = \frac{2}{\sqrt{\pi}} (\frac{0.1}{3})(y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 4y_9 + y_{10})$
 $\frac{2}{30\sqrt{\pi}}(e^0 + 4e^{-0.01} + 2e^{-0.04} + 4e^{-0.09} + \dots + 4e^{-0.81} + e^{-1}) \approx 0.843$

(b) $|E_s| \leq \frac{1-0}{180} (0.1)^4 (12) \approx 6.7 \times 10^{-6}$

29. $T = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \dots + 2y_{n-1} + y_n)$ where $\Delta x = \frac{b-a}{n}$ and f is continuous on $[a, b]$. So

$$T = \frac{b-a}{n} \frac{(y_0 + y_1 + y_1 + y_2 + y_2 + \dots + y_{n-1} + y_{n-1} + y_n)}{2} = \frac{b-a}{n} \left(\frac{f(x_0) + f(x_1)}{2} + \frac{f(x_1) + f(x_2)}{2} + \dots + \frac{f(x_{n-1}) + f(x_n)}{2} \right).$$

Since f is continuous on each interval $[x_{k-1}, x_k]$, and $\frac{f(x_{k-1}) + f(x_k)}{2}$ is always between $f(x_{k-1})$ and $f(x_k)$, there is a point c_k in $[x_{k-1}, x_k]$ with $f(c_k) = \frac{f(x_{k-1}) + f(x_k)}{2}$; this is a consequence of the Intermediate Value Theorem. Thus our sum is

$$\sum_{k=1}^n \left(\frac{b-a}{n} \right) f(c_k) \text{ which has the form } \sum_{k=1}^n \Delta x_k f(c_k) \text{ with } \Delta x_k = \frac{b-a}{n} \text{ for all } k. \text{ This is a Riemann Sum for } f \text{ on } [a, b].$$

30. $S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 2y_{n-2} + 4y_{n-1} + y_n)$ where n is even, $\Delta x = \frac{b-a}{n}$ and f is continuous on $[a, b]$. So

$$S = \frac{b-a}{n} \left(\frac{y_0 + 4y_1 + y_2}{3} + \frac{y_2 + 4y_3 + y_4}{3} + \frac{y_4 + 4y_5 + y_6}{3} + \dots + \frac{y_{n-2} + 4y_{n-1} + y_n}{3} \right)$$

$$= \frac{b-a}{\frac{n}{2}} \left(\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} + \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} + \frac{f(x_4) + 4f(x_5) + f(x_6)}{6} + \dots + \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right)$$

$\frac{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})}{6}$ is the average of the six values of the continuous function on the interval $[x_{2k}, x_{2k+2}]$, so it is between the minimum and maximum of f on this interval. By the Extreme Value Theorem for continuous functions, f takes on its maximum and minimum in this interval, so there are x_a and x_b with $x_{2k} \leq x_a, x_b \leq x_{2k+2}$ and

$$f(x_a) \leq \frac{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})}{6} \leq f(x_b). \text{ By the Intermediate Value Theorem, there is } c_k \text{ in } [x_{2k}, x_{2k+2}] \text{ with}$$

$$f(c_k) = \frac{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})}{6}. \text{ So our sum has the form } \sum_{k=1}^{n/2} \Delta x_k f(c_k) \text{ with } \Delta x_k = \frac{b-a}{(n/2)}, \text{ a Riemann sum for } f \text{ on } [a, b].$$