8.4 INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

1. $\frac{5x-13}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow 5x - 13 = A(x-2) + B(x-3) = (A+B)x - (2A+3B)$
$\Rightarrow \left.\begin{array}{c} A + B = 5 \\ 2A + 3B = 13 \end{array}\right\} \Rightarrow -B = (10-13) \Rightarrow B = 3 \Rightarrow A = 2$; thus, $\frac{5x-13}{(x-3)(x-2)} = \frac{2}{x-3} + \frac{3}{x-2}$

2. $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 5x - 7 = A(x-1) + B(x-2) = (A+B)x - (A+2B)$
$\Rightarrow \left.\begin{array}{c} A + B = 5 \\ A + 2B = 7 \end{array}\right\} \Rightarrow B = 2 \Rightarrow A = 3$; thus, $\frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$

3. $\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x+4 = A(x+1) + B = Ax + (A+B) \Rightarrow \left.\begin{array}{c} A = 1 \\ A+B = 4 \end{array}\right\} \Rightarrow A = 1$ and $B = 3$; thus, $\frac{x+4}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}$

4. $\frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 2x+2 = A(x-1) + B = Ax + (-A+B) \Rightarrow \left.\begin{array}{c} A = 2 \\ -A+B = 2 \end{array}\right\}$ $\Rightarrow A = 2$ and $B = 4$; thus, $\frac{2x+2}{x^2-2x+1} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$

5. $\frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} \Rightarrow z+1 = Az(z-1) + B(z-1) + Cz^2 \Rightarrow z+1 = (A+C)z^2 + (-A+B)z - B$ $\Rightarrow \left.\begin{array}{c} A+C = 0 \\ -A+B = 1 \\ -B = 1 \end{array}\right\} \Rightarrow B = -1 \Rightarrow A = -2 \Rightarrow C = 2$; thus, $\frac{z+1}{z^2(z-1)} = \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1}$

6. $\frac{z}{z^3-z^2-6z} = \frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2} \Rightarrow 1 = A(z+2) + B(z-3) = (A+B)z + (2A-3B)$ $\Rightarrow \left.\begin{array}{c} A+B = 0 \\ 2A-3B = 1 \end{array}\right\} \Rightarrow -5B = 1 \Rightarrow B = -\frac{1}{5} \Rightarrow A = \frac{1}{5}$; thus, $\frac{z}{z^3-z^2-6z} = \frac{\frac{1}{5}}{z-3} + \frac{-\frac{1}{5}}{z+2}$

7. $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2}$ $\Rightarrow 5t+2 = A(t-2) + B(t-3) = (A+B)t + (-2A-3B) \Rightarrow \left.\begin{array}{c} A+B = 5 \\ -2A-3B = 2 \end{array}\right\} \Rightarrow -B = (10+2) = 12$ $\Rightarrow B = -12 \Rightarrow A = 17$; thus, $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$

8. $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^2(t^2+9)}$ (after long division); $\frac{-9t^2+9}{t^2(t^2+9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}$ $\Rightarrow -9t^2+9 = At(t^2+9) + B(t^2+9) + (Ct+D)t^2 = (A+C)t^3 + (B+D)t^2 + 9At + 9B$ $\Rightarrow \left.\begin{array}{c} A+C = 0 \\ B+D = -9 \\ 9A = 0 \\ 9B = 9 \end{array}\right\} \Rightarrow A = 0 \Rightarrow C = 0; B = 1 \Rightarrow D = -10$; thus, $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{1}{t^2} + \frac{-10}{t^2+9}$

9. $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x) + B(1-x); x = 1 \Rightarrow A = \frac{1}{2}; x = -1 \Rightarrow B = \frac{1}{2}$; $\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x} = \frac{1}{2}\left[\ln|1+x| - \ln|1-x|\right] + C$

10. $\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + Bx; x = 0 \Rightarrow A = \frac{1}{2}; x = -2 \Rightarrow B = -\frac{1}{2}$; $\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}\left[\ln|x| - \ln|x+2|\right] + C$

11. $\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1) + B(x+6); x = 1 \Rightarrow B = \frac{5}{7}; x = -6 \Rightarrow A = \frac{-2}{-7} = \frac{2}{7}$; $\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C = \frac{1}{7}\ln|(x+6)^2(x-1)^5| + C$

12. $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3) + B(x-4); x = 3 \Rightarrow B = \frac{7}{-1} = -7; x = 4 \Rightarrow A = \frac{9}{1} = 9$; $\int \frac{2x+1}{x^2-7x+12}\,dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3} = 9\ln|x-4| - 7\ln|x-3| + C = \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$

13. $\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1) + B(y-3); y = -1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4}; y = 3 \Rightarrow A = \frac{3}{4}$; $\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8 = \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right)$ $= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 = \frac{\ln 15}{2}$

14. $\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By;\ y=0 \Rightarrow A=4;\ y=-1 \Rightarrow B = \frac{3}{-1} = -3;$
$\int_{1/2}^{1} \frac{y+4}{y^2+y}\,dy = 4\int_{1/2}^{1} \frac{dy}{y} - 3\int_{1/2}^{1} \frac{dy}{y+1} = \left[4\ln|y| - 3\ln|y+1|\right]_{1/2}^{1} = (4\ln 1 - 3\ln 2) - \left(4\ln\frac{1}{2} - 3\ln\frac{3}{2}\right)$
$= \ln\frac{1}{8} - \ln\frac{1}{16} + \ln\frac{27}{8} = \ln\left(\frac{27}{8}\cdot\frac{1}{8}\cdot 16\right) = \ln\frac{27}{4}$

15. $\frac{1}{t^3+t^2-2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Rightarrow 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2);\ t=0 \Rightarrow A = -\frac{1}{2};\ t=-2$
$\Rightarrow B = \frac{1}{6};\ t=1 \Rightarrow C = \frac{1}{3};\ \int \frac{dt}{t^3+t^2-2t} = -\frac{1}{2}\int\frac{dt}{t} + \frac{1}{6}\int\frac{dt}{t+2} + \frac{1}{3}\int\frac{dt}{t-1}$
$= -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| + C$

16. $\frac{x+3}{2x^3-8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \Rightarrow \frac{1}{2}(x+3) = A(x+2)(x-2) + Bx(x-2) + Cx(x+2);\ x=0 \Rightarrow A = \frac{3}{-8};\ x=-2$
$\Rightarrow B = \frac{1}{16};\ x=2 \Rightarrow C = \frac{5}{16};\ \int \frac{x+3}{2x^3-8x}\,dx = -\frac{3}{8}\int\frac{dx}{x} + \frac{1}{16}\int\frac{dx}{x+2} + \frac{5}{16}\int\frac{dx}{x-2}$
$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + C = \frac{1}{16}\ln\left|\frac{(x-2)^5(x+2)}{x^6}\right| + C$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1) + B$
$= Ax + (A+B) \Rightarrow A=3,\ A+B=2 \Rightarrow A=3,\ B=-1;\ \int_0^1 \frac{x^3\,dx}{x^2+2x+1}$
$= \int_0^1 (x-2)\,dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$
$= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$

18. $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{(x-1)^2}$ (after long division); $\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1) + B$
$= Ax + (-A+B) \Rightarrow A=3,\ -A+B=-2 \Rightarrow A=3,\ B=1;\ \int_{-1}^{0} \frac{x^3\,dx}{x^2-2x+1}$
$= \int_{-1}^{0} (x+2)\,dx + 3\int_{-1}^{0} \frac{dx}{x-1} + \int_{-1}^{0} \frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}\right]_{-1}^{0}$
$= \left(0 + 0 + 3\ln 1 - \frac{1}{(-1)}\right) - \left(\frac{1}{2} - 2 + 3\ln 2 - \frac{1}{(-2)}\right) = 2 - 3\ln 2$

19. $\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$
$x=-1 \Rightarrow C = \frac{1}{4};\ x=1 \Rightarrow D = \frac{1}{4}$; coefficient of $x^3 = A+B \Rightarrow A+B=0$; constant $= A-B+C+D$
$\Rightarrow A-B+C+D = 1 \Rightarrow A-B = \frac{1}{2}$; thus, $A = \frac{1}{4} \Rightarrow B = -\frac{1}{4};\ \int \frac{dx}{(x^2-1)^2}$
$= \frac{1}{4}\int\frac{dx}{x+1} - \frac{1}{4}\int\frac{dx}{x-1} + \frac{1}{4}\int\frac{dx}{(x+1)^2} + \frac{1}{4}\int\frac{dx}{(x-1)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C$

20. $\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1);\ x=-1$
$\Rightarrow C = -\frac{1}{2};\ x=1 \Rightarrow A = \frac{1}{4}$; coefficient of $x^2 = A+B \Rightarrow A+B=1 \Rightarrow B = \frac{3}{4};\ \int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$
$= \frac{1}{4}\int\frac{dx}{x-1} + \frac{3}{4}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{dx}{(x+1)^2} = \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C = \frac{\ln|(x-1)(x+1)^3|}{4} + \frac{1}{2(x+1)} + C$

21. $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x+1);\ x=-1 \Rightarrow A = \frac{1}{2}$; coefficient of x^2
$= A+B \Rightarrow A+B=0 \Rightarrow B = -\frac{1}{2}$; constant $= A+C \Rightarrow A+C=1 \Rightarrow C = \frac{1}{2};\ \int_0^1 \frac{dx}{(x+1)(x^2+1)}$
$= \frac{1}{2}\int_0^1 \frac{dx}{x+1} + \frac{1}{2}\int_0^1 \frac{(-x+1)}{x^2+1}\,dx = \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x\right]_0^1$
$= \left(\frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\tan^{-1}1\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\ln 1 + \frac{1}{2}\tan^{-1}0\right) = \frac{1}{4}\ln 2 + \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{(\pi + 2\ln 2)}{8}$

22. $\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} \Rightarrow 3t^2+t+4 = A(t^2+1) + (Bt+C)t;\ t=0 \Rightarrow A=4$; coefficient of t^2
$= A+B \Rightarrow A+B=3 \Rightarrow B=-1$; coefficient of $t = C \Rightarrow C=1;\ \int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+1}\,dt$

$= 4\int_1^{\sqrt{3}} \frac{dt}{t} + \int_1^{\sqrt{3}} \frac{(-t+1)}{t^2+1}\,dt = \left[4\ln|t| - \frac{1}{2}\ln(t^2+1) + \tan^{-1} t\right]_1^{\sqrt{3}}$
$= \left(4\ln\sqrt{3} - \frac{1}{2}\ln 4 + \tan^{-1}\sqrt{3}\right) - \left(4\ln 1 - \frac{1}{2}\ln 2 + \tan^{-1} 1\right) = 2\ln 3 - \ln 2 + \frac{\pi}{3} + \frac{1}{2}\ln 2 - \frac{\pi}{4}$
$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12} = \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}$

23. $\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2} \Rightarrow y^2+2y+1 = (Ay+B)(y^2+1) + Cy + D$
$= Ay^3 + By^2 + (A+C)y + (B+D) \Rightarrow A = 0, B = 1; A + C = 2 \Rightarrow C = 2; B + D = 1 \Rightarrow D = 0;$
$\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy = \int \frac{1}{y^2+1}\,dy + 2\int \frac{y}{(y^2+1)^2}\,dy = \tan^{-1} y - \frac{1}{y^2+1} + C$

24. $\frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2} \Rightarrow 8x^2+8x+2 = (Ax+B)(4x^2+1) + Cx + D$
$= 4Ax^3 + 4Bx^2 + (A+C)x + (B+D); A = 0, B = 2; A + C = 8 \Rightarrow C = 8; B + D = 2 \Rightarrow D = 0;$
$\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx = 2\int \frac{dx}{4x^2+1} + 8\int \frac{x\,dx}{(4x^2+1)^2} = \tan^{-1} 2x - \frac{1}{4x^2+1} + C$

25. $\frac{2s+2}{(s^2+1)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3} \Rightarrow 2s+2$
$= (As+B)(s-1)^3 + C(s^2+1)(s-1)^2 + D(s^2+1)(s-1) + E(s^2+1)$
$= [As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B] + C(s^4 - 2s^3 + 2s^2 - 2s + 1) + D(s^3 - s^2 + s - 1)$
$+ E(s^2+1)$
$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$

$$\Rightarrow \left.\begin{array}{l} A + C = 0 \\ -3A + B - 2C + D = 0 \\ 3A - 3B + 2C - D + E = 0 \\ -A + 3B - 2C + D = 2 \\ -B + C - D + E = 2 \end{array}\right\} \text{summing all equations} \Rightarrow 2E = 4 \Rightarrow E = 2;$$

summing eqs (2) and (3) $\Rightarrow -2B + 2 = 0 \Rightarrow B = 1$; summing eqs (3) and (4) $\Rightarrow 2A + 2 = 2 \Rightarrow A = 0; C = 0$ from eq (1); then $-1 + 0 - D + 2 = 2$ from eq (5) $\Rightarrow D = -1$;
$\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1} s + C$

26. $\frac{s^4+81}{s(s^2+9)^2} = \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{(s^2+9)^2} \Rightarrow s^4 + 81 = A(s^2+9)^2 + (Bs+C)s(s^2+9) + (Ds+E)s$
$= A(s^4 + 18s^2 + 81) + (Bs^4 + Cs^3 + 9Bs^2 + 9Cs) + Ds^2 + Es$
$= (A+B)s^4 + Cs^3 + (18A + 9B + D)s^2 + (9C + E)s + 81A \Rightarrow 81A = 81$ or $A = 1; A + B = 1 \Rightarrow B = 0;$
$C = 0; 9C + E = 0 \Rightarrow E = 0; 18A + 9B + D = 0 \Rightarrow D = -18; \int \frac{s^4+81}{s(s^2+9)^2}\,ds = \int \frac{ds}{s} - 18\int \frac{s\,ds}{(s^2+9)^2}$
$= \ln|s| + \frac{9}{(s^2+9)} + C$

27. $\frac{x^2-x+2}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \Rightarrow x^2 - x + 2 = A(x^2+x+1) + (Bx+C)(x-1) = (A+B)x^2 + (A-B+C)x + (A-C)$
$\Rightarrow A + B = 1, A - B + C = -1, A - C = 2 \Rightarrow$ adding eq(2) and eq(3) $\Rightarrow 2A - B = 1$, add this equation to eq(1)
$\Rightarrow 3A = 2 \Rightarrow A = \frac{2}{3} \Rightarrow B = 1 - A = \frac{1}{3} \Rightarrow C = -1 - A + B = -\frac{4}{3}; \int \frac{x^2-x+2}{x^3-1}dx = \int\left(\frac{2/3}{x-1} + \frac{(1/3)x - 4/3}{x^2+x+1}\right)dx$
$= \frac{2}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{x-4}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}dx \quad \left[u = x + \frac{1}{2} \Rightarrow u - \frac{1}{2} = x \Rightarrow du = dx\right]$
$= \frac{2}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{u - \frac{9}{2}}{u^2 + \frac{3}{4}}du = \frac{2}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}du - \frac{3}{2}\int \frac{1}{u^2+\frac{3}{4}}du$
$= \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln\left|\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right| - \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + C = \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| - \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

28. $\frac{1}{x^4+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1} \Rightarrow 1 = A(x+1)(x^2-x+1) + Bx(x^2-x+1) + (Cx+D)x(x+1)$
$= (A+B+C)x^3 + (-B+C+D)x^2 + (B+D)x + A \Rightarrow A = 1, B+D = 0 \Rightarrow D = -B, -B+C+D = 0$
$\Rightarrow -2B+C = 0 \Rightarrow C = 2B, A+B+C = 0 \Rightarrow 1+B+2B = 0 \Rightarrow B = -\frac{1}{3} \Rightarrow C = -\frac{2}{3} \Rightarrow D = \frac{1}{3};$
$\int \frac{1}{x^4+x}dx = \int \left(\frac{1}{x} - \frac{1/3}{x+1} + \frac{(-2/3)x+1/3}{x^2-x+1}\right)dx = \int \frac{1}{x}dx - \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{2x-1}{x^2-x+1}dx$
$= \ln|x| - \frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x^2-x+1| + C$

29. $\frac{x^2}{x^4-1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \Rightarrow x^2 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x-1)(x+1)$
$= (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x - A + B - D \Rightarrow A+B+C = 0, -A+B+D = 1,$
$A+B-C = 0, -A+B-D = 0 \Rightarrow$ adding eq(1) to eq (3) gives $2A+2B = 0$, adding eq(2) to eq(4) gives
$-2A+2B = 1$, adding these two equations gives $4B = 1 \Rightarrow B = \frac{1}{4}$, using $2A+2B = 0 \Rightarrow A = -\frac{1}{4}$, using
$-A+B-D = 0 \Rightarrow D = \frac{1}{2}$, and using $A+B-C = 0 \Rightarrow C = 0$; $\int \frac{x^2}{x^4-1}dx = \int \left(\frac{-1/4}{x+1} + \frac{1/4}{x-1} + \frac{1/2}{x^2+1}\right)dx$
$= -\frac{1}{4}\int \frac{1}{x+1}dx + \frac{1}{4}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx = -\frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + \frac{1}{2}\tan^{-1}x + C = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\tan^{-1}x + C$

30. $\frac{x^2+x}{x^4-3x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1} \Rightarrow x^2 + x = A(x+2)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x-2)(x+2)$
$= (A+B+C)x^3 + (2A-2B+D)x^2 + (A+B-4C)x + 2A - 2B - 4D \Rightarrow A+B+C = 0, 2A-2B+D = 1,$
$A+B-4C = 1, 2A-2B-4D = 0 \Rightarrow$ subtractin eq(1) from eq (3) gives $-5C = 1 \Rightarrow C = -\frac{1}{5}$, subtacting eq(2) from eq(4) gives $-5D = -1 \Rightarrow D = \frac{1}{5}$, substituting for C in eq(1) gives $A+B = \frac{1}{5}$, and substituting for D in eq(4) gives $2A-2B = \frac{4}{5} \Rightarrow A-B = \frac{2}{5}$, adding this equation to the previous equatin gives $2A = \frac{3}{5} \Rightarrow A = \frac{3}{10} \Rightarrow B = -\frac{1}{10}$;
$\int \frac{x^2+x}{x^4-3x^2-4}dx = \int \left(\frac{3/10}{x-2} - \frac{1/10}{x+2} + \frac{(-1/5)x+1/5}{x^2+1}\right)dx = \frac{3}{10}\int \frac{1}{x-2}dx - \frac{1}{10}\int \frac{1}{x+2}dx - \frac{1}{5}\int \frac{x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx$
$\frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln|x^2+1| + \frac{1}{5}\tan^{-1}x + C$

31. $\frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} = \frac{A\theta+B}{\theta^2+2\theta+2} + \frac{C\theta+D}{(\theta^2+2\theta+2)^2} \Rightarrow 2\theta^3+5\theta^2+8\theta+4 = (A\theta+B)(\theta^2+2\theta+2) + C\theta + D$
$= A\theta^3 + (2A+B)\theta^2 + (2A+2B+C)\theta + (2B+D) \Rightarrow A = 2; 2A+B = 5 \Rightarrow B = 1; 2A+2B+C = 8 \Rightarrow C = 2;$
$2B+D = 4 \Rightarrow D = 2; \int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta = \int \frac{2\theta+1}{(\theta^2+2\theta+2)}\,d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2}\,d\theta$
$= \int \frac{2\theta+2}{\theta^2+2\theta+2}\,d\theta - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{d(\theta^2+2\theta+2)}{(\theta^2+2\theta+2)^2} = \int \frac{d(\theta^2+2\theta+2)}{\theta^2+2\theta+2} - \int \frac{d\theta}{(\theta+1)^2+1} - \frac{1}{\theta^2+2\theta+2}$
$= \frac{-1}{\theta^2+2\theta+2} + \ln(\theta^2+2\theta+2) - \tan^{-1}(\theta+1) + C$

32. $\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{(\theta^2+1)^2} + \frac{E\theta+F}{(\theta^2+1)^3} \Rightarrow \theta^4-4\theta^3+2\theta^2-3\theta+1$
$= (A\theta+B)(\theta^2+1)^2 + (C\theta+D)(\theta^2+1) + E\theta + F = (A\theta+B)(\theta^4+2\theta^2+1) + (C\theta^3+D\theta^2+C\theta+D) + E\theta + F$
$= (A\theta^5+B\theta^4+2A\theta^3+2B\theta^2+A\theta+B) + (C\theta^3+D\theta^2+C\theta+D) + E\theta + F$
$= A\theta^5 + B\theta^4 + (2A+C)\theta^3 + (2B+D)\theta^2 + (A+C+E)\theta + (B+D+F) \Rightarrow A = 0; B = 1; 2A+C = -4$
$\Rightarrow C = -4; 2B+D = 2 \Rightarrow D = 0; A+C+E = -3 \Rightarrow E = 1; B+D+F = 1 \Rightarrow F = 0;$
$\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta = \int \frac{d\theta}{\theta^2+1} - 4\int \frac{\theta\,d\theta}{(\theta^2+1)^2} + \int \frac{\theta\,d\theta}{(\theta^2+1)^3} = \tan^{-1}\theta + 2(\theta^2+1)^{-1} - \frac{1}{4}(\theta^2+1)^{-2} + C$

33. $\frac{2x^3-2x^2+1}{x^2-x} = 2x + \frac{1}{x^2-x} = 2x + \frac{1}{x(x-1)}; \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + Bx; x = 0 \Rightarrow A = -1;$
$x = 1 \Rightarrow B = 1; \int \frac{2x^3-2x^2+1}{x^2-x} = \int 2x\,dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C = x^2 + \ln\left|\frac{x-1}{x}\right| + C$

34. $\frac{x^4}{x^2-1} = (x^2+1) + \frac{1}{x^2-1} = (x^2+1) + \frac{1}{(x+1)(x-1)}; \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x+1);$
$x = -1 \Rightarrow A = -\frac{1}{2}; x = 1 \Rightarrow B = \frac{1}{2}; \int \frac{x^4}{x^2-1}\,dx = \int (x^2+1)\,dx - \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$
$= \frac{1}{3}x^3 + x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C = \frac{x^3}{3} + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

35. $\frac{9x^3-3x+1}{x^3-x^2} = 9 + \frac{9x^2-3x+1}{x^2(x-1)}$ (after long division); $\frac{9x^2-3x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$
$\Rightarrow 9x^2 - 3x + 1 = Ax(x-1) + B(x-1) + Cx^2$; $x = 1 \Rightarrow C = 7$; $x = 0 \Rightarrow B = -1$; $A + C = 9 \Rightarrow A = 2$;
$\int \frac{9x^3-3x+1}{x^3-x^2}\,dx = \int 9\,dx + 2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 7\int \frac{dx}{x-1} = 9x + 2\ln|x| + \frac{1}{x} + 7\ln|x-1| + C$

36. $\frac{16x^3}{4x^2-4x+1} = (4x+4) + \frac{12x-4}{4x^2-4x+1}$; $\frac{12x-4}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} \Rightarrow 12x - 4 = A(2x-1) + B$
$\Rightarrow A = 6$; $-A + B = -4 \Rightarrow B = 2$; $\int \frac{16x^3}{4x^2-4x+1}\,dx = 4\int (x+1)\,dx + 6\int \frac{dx}{2x-1} + 2\int \frac{dx}{(2x-1)^2}$
$= 2(x+1)^2 + 3\ln|2x-1| - \frac{1}{2x-1} + C_1 = 2x^2 + 4x + 3\ln|2x-1| - (2x-1)^{-1} + C$, where $C = 2 + C_1$

37. $\frac{y^4+y^2-1}{y^3+y} = y - \frac{1}{y(y^2+1)}$; $\frac{1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1} \Rightarrow 1 = A(y^2+1) + (By+C)y = (A+B)y^2 + Cy + A$
$7 \Rightarrow A = 1$; $A + B = 0 \Rightarrow B = -1$; $C = 0$; $\int \frac{y^4+y^2-1}{y^3+y}\,dy = \int y\,dy - \int \frac{dy}{y} + \int \frac{y\,dy}{y^2+1}$
$= \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln(1+y^2) + C$

38. $\frac{2y^4}{y^3-y^2+y-1} = 2y + 2 + \frac{2}{y^3-y^2+y-1}$; $\frac{2}{y^3-y^2+y-1} = \frac{2}{(y^2+1)(y-1)} = \frac{A}{y-1} + \frac{By+C}{y^2+1}$
$\Rightarrow 2 = A(y^2+1) + (By+C)(y-1) = (Ay^2+A) + (By^2+Cy-By-C) = (A+B)y^2 + (-B+C)y + (A-C)$
$\Rightarrow A + B = 0$, $-B + C = 0$ or $C = B$, $A - C = A - B = 2 \Rightarrow A = 1$, $B = -1$, $C = -1$;
$\int \frac{2y^4}{y^3-y^2+y-1}\,dy = 2\int (y+1)\,dy + \int \frac{dy}{y-1} - \int \frac{y}{y^2+1}\,dy - \int \frac{dy}{y^2+1}$
$= (y+1)^2 + \ln|y-1| - \frac{1}{2}\ln(y^2+1) - \tan^{-1} y + C_1 = y^2 + 2y + \ln|y-1| - \frac{1}{2}\ln(y^2+1) - \tan^{-1} y + C$,
where $C = C_1 + 1$

39. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2} = [e^t = y]\int \frac{dy}{y^2+3y+2} = \int \frac{dy}{y+1} - \int \frac{dy}{y+2} = \ln\left|\frac{y+1}{y+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

40. $\int \frac{e^{4t}+2e^{2t}-e^t}{e^{2t}+1}\,dt = \int \frac{e^{3t}+2e^t-1}{e^{2t}+1}e^t dt$; $\begin{bmatrix} y = e^t \\ dy = e^t\,dt \end{bmatrix} \to \int \frac{y^3+2y-1}{y^2+1}\,dy = \int \left(y + \frac{y-1}{y^2+1}\right) dy = \frac{y^2}{2} + \int \frac{y}{y^2+1}\,dy - \int \frac{dy}{y^2+1}$
$= \frac{y^2}{2} + \frac{1}{2}\ln(y^2+1) - \tan^{-1} y + C = \frac{1}{2}e^{2t} + \frac{1}{2}\ln(e^{2t}+1) - \tan^{-1}(e^t) + C$

41. $\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$; $[\sin y = t, \cos y\,dy = dt] \to \int \frac{dy}{t^2+t-6} = \frac{1}{5}\int \left(\frac{1}{t-2} - \frac{1}{t+3}\right) dt = \frac{1}{5}\ln\left|\frac{t-2}{t+3}\right| + C$
$= \frac{1}{5}\ln\left|\frac{\sin y - 2}{\sin y + 3}\right| + C$

42. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$; $[\cos\theta = y] \to -\int \frac{dy}{y^2+y-2} = \frac{1}{3}\int \frac{dy}{y+2} - \frac{1}{3}\int \frac{dy}{y-1} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C = \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C$
$= \frac{1}{3}\ln\left|\frac{2+\cos\theta}{1-\cos\theta}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

43. $\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx = \int \frac{\tan^{-1}(2x)}{4x^2+1}\,dx - 3\int \frac{x}{(x-2)^2}\,dx$
$= \frac{1}{2}\int \tan^{-1}(2x)\,d(\tan^{-1}(2x)) - 3\int \frac{dx}{x-2} - 6\int \frac{dx}{(x-2)^2} = \frac{(\tan^{-1} 2x)^2}{4} - 3\ln|x-2| + \frac{6}{x-2} + C$

44. $\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\,dx = \int \frac{\tan^{-1}(3x)}{9x^2+1}\,dx + \int \frac{x}{(x+1)^2}\,dx$
$= \frac{1}{3}\int \tan^{-1}(3x)\,d(\tan^{-1}(3x)) + \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \frac{(\tan^{-1} 3x)^2}{6} + \ln|x+1| + \frac{1}{x+1} + C$

45. $\int \frac{1}{x^{3/2}-\sqrt{x}}dx = \int \frac{1}{\sqrt{x}(x-1)}dx \left[\text{Let } u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}dx\right] \to \int \frac{2}{u^2-1}du$;
$\frac{2}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A + B = 0, -A + B = 2$

$\Rightarrow B = 1 \Rightarrow A = -1; \int \frac{2}{u^2-1}du = \int \left(\frac{-1}{u+1} + \frac{1}{u-1}\right)du = -\int \frac{1}{u+1}du + \int \frac{1}{u-1}du = -\ln|u+1| + \ln|u-1| + C$
$= \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$

46. $\int \frac{1}{(x^{1/3}-1)\sqrt{x}}dx \left[\text{Let } x = u^6 \Rightarrow dx = 6u^5\,du\right] \to \int \frac{1}{(u^2-1)u^3}6u^5\,du = \int \frac{6u^2}{u^2-1}du = \int \left(6 + \frac{6}{u^2-1}\right)du$
$= 6\int du + \int \frac{6}{u^2-1}du;\ \frac{6}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 6 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A + B = 0,$
$-A + B = 6 \Rightarrow B = 3 \Rightarrow A = -3;\ 6\int du + \int \frac{6}{u^2-1}du = 6u + \int \left(\frac{-3}{u+1} + \frac{3}{u-1}\right)du = 6u - 3\int \frac{1}{u+1}du + 3\int \frac{1}{u-1}du$
$= 6u - 3\ln|u+1| + 3\ln|u-1| + C = 6x^{1/6} + 3\ln\left|\frac{x^{1/6}-1}{x^{1/6}+1}\right| + C$

47. $\int \frac{\sqrt{x+1}}{x}\,dx \left[\text{Let } x + 1 = u^2 \Rightarrow dx = 2u\,du\right] \to \int \frac{u}{u^2-1}\,2u\,du = \int \frac{2u^2}{u^2-1}\,du = \int \left(2 + \frac{2}{u^2-1}\right)du$
$= 2\int du + \int \frac{2}{u^2-1}du;\ \frac{2}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A + B = 0,$
$-A + B = 2 \Rightarrow B = 1 \Rightarrow A = -1;\ 2\int du + \int \frac{2}{u^2-1}du = 2u + \int \left(\frac{-1}{u+1} + \frac{1}{u-1}\right)du = 2u - \int \frac{1}{u+1}du + \int \frac{1}{u-1}du$
$= 2u - \ln|u+1| + \ln|u-1| + C = 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

48. $\int \frac{1}{x\sqrt{x+9}}\,dx \left[\text{Let } x + 9 = u^2 \Rightarrow dx = 2u\,du\right] \to \int \frac{1}{(u^2-9)u}\,2u\,du = \int \frac{2}{u^2-9}\,du;\ \frac{2}{u^2-9} = \frac{A}{u-3} + \frac{B}{u+3}$
$\Rightarrow 2 = A(u+3) + B(u-3) = (A+B)u + 3A - 3B \Rightarrow A + B = 0,\ 3A - 3B = 2 \Rightarrow A = \frac{1}{3} \Rightarrow B = -\frac{1}{3};$
$\int \frac{2}{u^2-9}\,du = \int \left(\frac{1/3}{u-3} - \frac{1/3}{u+3}\right)du = \frac{1}{3}\int \frac{1}{u-3}\,du - \frac{1}{3}\int \frac{1}{u+3}\,du = \frac{1}{3}\ln|u-3| - \frac{1}{3}\ln|u+3| + C = \frac{1}{3}\ln\left|\frac{\sqrt{x+9}-3}{\sqrt{x+9}+3}\right| + C$

49. $\int \frac{1}{x(x^4+1)}dx = \int \frac{x^3}{x^4(x^4+1)}dx \left[\text{Let } u = x^4 \Rightarrow du = 4x^3\,dx\right] \to \frac{1}{4}\int \frac{1}{u(u+1)}du;\ \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$
$\Rightarrow 1 = A(u+1) + Bu = (A+B)u + A \Rightarrow A = 1 \Rightarrow B = -1;\ \frac{1}{4}\int \frac{1}{u(u+1)}du = \frac{1}{4}\int \left(\frac{1}{u} - \frac{1}{u+1}\right)du$
$= \frac{1}{4}\int \frac{1}{u}du - \frac{1}{4}\int \frac{1}{u+1}du = \frac{1}{4}\ln|u| - \frac{1}{4}\ln|u+1| + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C$

50. $\int \frac{1}{x^6(x^5+4)}dx = \int \frac{x^4}{x^{10}(x^5+4)}dx = \left[\text{Let } u = x^5 \Rightarrow du = 5x^4\,dx\right] \to \frac{1}{5}\int \frac{1}{u^2(u+4)}du;\ \frac{1}{u^2(u+4)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+4}$
$\Rightarrow 1 = Au(u+4) + B(u+4) + Cu^2 = (A+C)u^2 + (4A+B)u + 4B \Rightarrow A + C = 0,\ 4A + B = 0,\ 4B = 1 \Rightarrow B = \frac{1}{4}$
$\Rightarrow A = -\frac{1}{16} \Rightarrow C = \frac{1}{16};\ \frac{1}{5}\int \frac{1}{u^2(u+4)}du = \frac{1}{5}\int \left(-\frac{1/16}{u} + \frac{1/4}{u^2} + \frac{1/16}{u+4}\right)du = -\frac{1}{80}\int \frac{1}{u}du + \frac{1}{20}\int \frac{1}{u^2}du + \frac{1}{80}\int \frac{1}{u+4}du$
$= -\frac{1}{80}\ln|u| - \frac{1}{20u} + \frac{1}{80}\ln|u+4| + C = -\frac{1}{80}\ln|x^5| - \frac{1}{20x^5} + \frac{1}{80}\ln|x^5+4| + C = \frac{1}{80}\ln\left|\frac{x^5+4}{x^5}\right| - \frac{1}{20x^5} + C$

51. $(t^2 - 3t + 2)\,\frac{dx}{dt} = 1;\ x = \int \frac{dt}{t^2-3t+2} = \int \frac{dt}{t-2} - \int \frac{dt}{t-1} = \ln\left|\frac{t-2}{t-1}\right| + C;\ \frac{t-2}{t-1} = Ce^x;\ t = 3 \text{ and } x = 0$
$\Rightarrow\ \frac{1}{2} = C\ \Rightarrow\ \frac{t-2}{t-1} = \frac{1}{2}e^x\ \Rightarrow\ x = \ln\left|2\left(\frac{t-2}{t-1}\right)\right| = \ln|t-2| - \ln|t-1| + \ln 2$

52. $(3t^4 + 4t^2 + 1)\,\frac{dx}{dt} = 2\sqrt{3};\ x = 2\sqrt{3}\int \frac{dt}{3t^4+4t^2+1} = \sqrt{3}\int \frac{dt}{t^2+\frac{1}{3}} - \sqrt{3}\int \frac{dt}{t^2+1}$
$= 3\tan^{-1}\left(\sqrt{3}t\right) - \sqrt{3}\tan^{-1}t + C;\ t = 1 \text{ and } x = \frac{-\pi\sqrt{3}}{4}\ \Rightarrow\ -\frac{\sqrt{3}\pi}{4} = \pi - \frac{\sqrt{3}}{4}\pi + C\ \Rightarrow\ C = -\pi$
$\Rightarrow\ x = 3\tan^{-1}\left(\sqrt{3}t\right) - \sqrt{3}\tan^{-1}t - \pi$

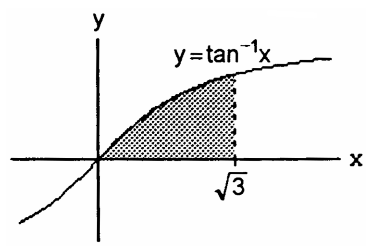
53. $(t^2+2t)\frac{dx}{dt}=2x+2;\ \frac{1}{2}\int\frac{dx}{x+1}=\int\frac{dt}{t^2+2t}\ \Rightarrow\ \frac{1}{2}\ln|x+1|=\frac{1}{2}\int\frac{dt}{t}-\frac{1}{2}\int\frac{dt}{t+2}\ \Rightarrow\ \ln|x+1|=\ln\left|\frac{t}{t+2}\right|+C;$
$t=1$ and $x=1\ \Rightarrow\ \ln 2=\ln\frac{1}{3}+C\ \Rightarrow\ C=\ln 2+\ln 3=\ln 6\ \Rightarrow\ \ln|x+1|=\ln 6\left|\frac{t}{t+2}\right|\ \Rightarrow\ x+1=\frac{6t}{t+2}$
$\Rightarrow\ x=\frac{6t}{t+2}-1,\ t>0$

54. $(t+1)\frac{dx}{dt}=x^2+1\ \Rightarrow\ \int\frac{dx}{x^2+1}=\int\frac{dt}{t+1}\ \Rightarrow\ \tan^{-1}x=\ln|t+1|+C;$ $t=0$ and $x=0\ \Rightarrow\ \tan^{-1}0=\ln|1|+C$
$\Rightarrow\ C=\tan^{-1}0=0\ \Rightarrow\ \tan^{-1}x=\ln|t+1|\Rightarrow\ x=\tan(\ln(t+1)),\ t>-1$

55. $V=\pi\int_{0.5}^{2.5}y^2\,dx=\pi\int_{0.5}^{2.5}\frac{9}{3x-x^2}\,dx=3\pi\left(\int_{0.5}^{2.5}\left(-\frac{1}{x-3}+\frac{1}{x}\right)\right)dx=\left[3\pi\ln\left|\frac{x}{x-3}\right|\right]_{0.5}^{2.5}=3\pi\ln 25$

56. $V=2\pi\int_0^1 xy\,dx=2\pi\int_0^1\frac{2x}{(x+1)(2-x)}\,dx=4\pi\int_0^1\left(-\frac{1}{3}\left(\frac{1}{x+1}\right)+\frac{2}{3}\left(\frac{1}{2-x}\right)\right)dx$
$=\left[-\frac{4\pi}{3}(\ln|x+1|+2\ln|2-x|)\right]_0^1=\frac{4\pi}{3}(\ln 2)$

57. $A=\int_0^{\sqrt{3}}\tan^{-1}x\,dx=\left[x\tan^{-1}x\right]_0^{\sqrt{3}}-\int_0^{\sqrt{3}}\frac{x}{1+x^2}\,dx$
$=\frac{\pi\sqrt{3}}{3}-\left[\frac{1}{2}\ln(x^2+1)\right]_0^{\sqrt{3}}=\frac{\pi\sqrt{3}}{3}-\ln 2;$
$\overline{x}=\frac{1}{A}\int_0^{\sqrt{3}}x\tan^{-1}x\,dx$
$=\frac{1}{A}\left(\left[\frac{1}{2}x^2\tan^{-1}x\right]_0^{\sqrt{3}}-\frac{1}{2}\int_0^{\sqrt{3}}\frac{x^2}{1+x^2}\,dx\right)$
$=\frac{1}{A}\left[\frac{\pi}{2}-\left[\frac{1}{2}(x-\tan^{-1}x)\right]_0^{\sqrt{3}}\right]$
$=\frac{1}{A}\left(\frac{\pi}{2}-\frac{\sqrt{3}}{2}+\frac{\pi}{6}\right)=\frac{1}{A}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)\cong 1.10$



58. $A=\int_3^5\frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx=3\int_3^5\frac{dx}{x}-\int_3^5\frac{dx}{x+3}+2\int_3^5\frac{dx}{x-1}=\left[3\ln|x|-\ln|x+3|+2\ln|x-1|\right]_3^5=\ln\frac{125}{9};$
$\overline{x}=\frac{1}{A}\int_3^5\frac{x(4x^2+13x-9)}{x^3+2x^2-3x}\,dx=\frac{1}{A}\left(\left[4x\right]_3^5+3\int_3^5\frac{dx}{x+3}+2\int_3^5\frac{dx}{x-1}\right)=\frac{1}{A}(8+11\ln 2-3\ln 6)\cong 3.90$

59. (a) $\frac{dx}{dt}=kx(N-x)\ \Rightarrow\int\frac{dx}{x(N-x)}=\int k\,dt\ \Rightarrow\ \frac{1}{N}\int\frac{dx}{x}+\frac{1}{N}\int\frac{dx}{N-x}=\int k\,dt\ \Rightarrow\ \frac{1}{N}\ln\left|\frac{x}{N-x}\right|=kt+C;$
$k=\frac{1}{250},\ N=1000,\ t=0$ and $x=2\ \Rightarrow\ \frac{1}{1000}\ln\left|\frac{2}{998}\right|=C\ \Rightarrow\ \frac{1}{1000}\ln\left|\frac{x}{1000-x}\right|=\frac{t}{250}+\frac{1}{1000}\ln\left(\frac{1}{499}\right)$
$\Rightarrow\ \ln\left|\frac{499x}{1000-x}\right|=4t\ \Rightarrow\ \frac{499x}{1000-x}=e^{4t}\ \Rightarrow\ 499x=e^{4t}(1000-x)\ \Rightarrow\ (499+e^{4t})\,x=1000e^{4t}\ \Rightarrow\ x=\frac{1000e^{4t}}{499+e^{4t}}$
(b) $x=\frac{1}{2}N=500\ \Rightarrow\ 500=\frac{1000e^{4t}}{499+e^{4t}}\ \Rightarrow\ 500\cdot 499+500e^{4t}=1000e^{4t}\ \Rightarrow\ e^{4t}=499\ \Rightarrow\ t=\frac{1}{4}\ln 499\approx 1.55$ days

60. $\frac{dx}{dt}=k(a-x)(b-x)\ \Rightarrow\ \frac{dx}{(a-x)(b-x)}=k\,dt$
(a) $a=b$: $\int\frac{dx}{(a-x)^2}=\int k\,dt\ \Rightarrow\ \frac{1}{a-x}=kt+C;$ $t=0$ and $x=0\ \Rightarrow\ \frac{1}{a}=C\ \Rightarrow\ \frac{1}{a-x}=kt+\frac{1}{a}$
$\Rightarrow\ \frac{1}{a-x}=\frac{akt+1}{a}\ \Rightarrow\ a-x=\frac{a}{akt+1}\ \Rightarrow\ x=a-\frac{a}{akt+1}=\frac{a^2kt}{akt+1}$
(b) $a\neq b$: $\int\frac{dx}{(a-x)(b-x)}=\int k\,dt\ \Rightarrow\ \frac{1}{b-a}\int\frac{dx}{a-x}-\frac{1}{b-a}\int\frac{dx}{b-x}=\int k\,dt\ \Rightarrow\ \frac{1}{b-a}\ln\left|\frac{b-x}{a-x}\right|=kt+C;$
$t=0$ and $x=0\ \Rightarrow\ \frac{1}{b-a}\ln\frac{b}{a}=C\ \Rightarrow\ \ln\left|\frac{b-x}{a-x}\right|=(b-a)kt+\ln\left(\frac{b}{a}\right)\ \Rightarrow\ \frac{b-x}{a-x}=\frac{b}{a}e^{(b-a)kt}$
$\Rightarrow\ x=\frac{ab\left[1-e^{(b-a)kt}\right]}{a-be^{(b-a)kt}}$