8.3 TRIGONOMETRIC SUBSTITUTIONS

1. $x = 3\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{3\,d\theta}{\cos^2\theta},\ 9 + x^2 = 9(1 + \tan^2\theta) = 9\sec^2\theta \Rightarrow \frac{1}{\sqrt{9+x^2}} = \frac{1}{3|\sec\theta|} = \frac{|\cos\theta|}{3} = \frac{\cos\theta}{3}$;
$\left(\text{because } \cos\theta > 0 \text{ when } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$;
$\int \frac{dx}{\sqrt{9+x^2}} = 3\int \frac{\cos\theta\,d\theta}{3\cos^2\theta} = \int \frac{d\theta}{\cos\theta} = \ln|\sec\theta + \tan\theta| + C' = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C' = \ln\left|\sqrt{9+x^2} + x\right| + C$

2. $\int \frac{3\,dx}{\sqrt{1+9x^2}}$; $[3x = u] \rightarrow \int \frac{du}{\sqrt{1+u^2}}$; $u = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2},\ du = \frac{dt}{\cos^2 t},\ \sqrt{1+u^2} = |\sec t| = \sec t$;
$\int \frac{du}{\sqrt{1+u^2}} = \int \frac{dt}{\cos^2 t\,(\sec t)} = \int \sec t\,dt = \ln|\sec t + \tan t| + C = \ln\left|\sqrt{u^2+1} + u\right| + C = \ln\left|\sqrt{1+9x^2} + 3x\right| + C$

3. $\int_{-2}^{2} \frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2} = \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}(-1) = \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) - \left(\frac{1}{2}\right)\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$

4. $\int_{0}^{2} \frac{dx}{8+2x^2} = \frac{1}{2}\int_{0}^{2} \frac{dx}{4+x^2} = \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{0}^{2} = \frac{1}{2}\left(\frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{16}$

5. $\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_{0}^{3/2} = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}$

6. $\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$; $[t = 2x] \rightarrow \int_{0}^{1/2\sqrt{2}} \frac{dt}{\sqrt{1-t^2}} = [\sin^{-1} t]_{0}^{1/\sqrt{2}} = \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

7. $t = 5\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = 5\cos\theta\,d\theta,\ \sqrt{25-t^2} = 5\cos\theta$;
$\int \sqrt{25-t^2}\,dt = \int (5\cos\theta)(5\cos\theta)\,d\theta = 25\int \cos^2\theta\,d\theta = 25\int \frac{1+\cos 2\theta}{2}\,d\theta = 25\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C$
$= \frac{25}{2}(\theta + \sin\theta\cos\theta) + C = \frac{25}{2}\left[\sin^{-1}\left(\frac{t}{5}\right) + \left(\frac{t}{5}\right)\left(\frac{\sqrt{25-t^2}}{5}\right)\right] + C = \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25-t^2}}{2} + C$

8. $t = \frac{1}{3}\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\cos\theta\,d\theta,\ \sqrt{1-9t^2} = \cos\theta$;
$\int \sqrt{1-9t^2}\,dt = \frac{1}{3}\int (\cos\theta)(\cos\theta)\,d\theta = \frac{1}{3}\int \cos^2\theta\,d\theta = \frac{1}{6}(\theta + \sin\theta\cos\theta) + C = \frac{1}{6}\left[\sin^{-1}(3t) + 3t\sqrt{1-9t^2}\right] + C$

9. $x = \frac{7}{2}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{7}{2}\sec\theta\tan\theta\,d\theta,\ \sqrt{4x^2-49} = \sqrt{49\sec^2\theta - 49} = 7\tan\theta$;
$\int \frac{dx}{\sqrt{4x^2-49}} = \int \frac{\left(\frac{7}{2}\sec\theta\tan\theta\right)d\theta}{7\tan\theta} = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right| + C$

10. $x = \frac{3}{5}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{3}{5}\sec\theta\tan\theta\,d\theta,\ \sqrt{25x^2 - 9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta;$
$\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta\right)d\theta}{3\tan\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$

11. $y = 7\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dy = 7\sec\theta\tan\theta\,d\theta,\ \sqrt{y^2 - 49} = 7\tan\theta;$
$\int \frac{\sqrt{y^2-49}}{y}\,dy = \int \frac{(7\tan\theta)(7\sec\theta\tan\theta)\,d\theta}{7\sec\theta} = 7\int \tan^2\theta\,d\theta = 7\int(\sec^2\theta - 1)\,d\theta = 7(\tan\theta - \theta) + C$
$= 7\left[\frac{\sqrt{y^2-49}}{7} - \sec^{-1}\left(\frac{y}{7}\right)\right] + C$

12. $y = 5\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dy = 5\sec\theta\tan\theta\,d\theta,\ \sqrt{y^2 - 25} = 5\tan\theta;$
$\int \frac{\sqrt{y^2-25}}{y^3}\,dy = \int \frac{(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta}{125\sec^3\theta} = \frac{1}{5}\int \tan^2\theta\cos^2\theta\,d\theta = \frac{1}{5}\int \sin^2\theta\,d\theta = \frac{1}{10}\int(1 - \cos 2\theta)\,d\theta$
$= \frac{1}{10}(\theta - \sin\theta\cos\theta) + C = \frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right) - \left(\frac{\sqrt{y^2-25}}{y}\right)\left(\frac{5}{y}\right)\right] + C = \left[\frac{\sec^{-1}\left(\frac{y}{5}\right)}{10} - \frac{\sqrt{y^2-25}}{2y^2}\right] + C$

13. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ \sqrt{x^2 - 1} = \tan\theta;$
$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\tan\theta} = \int \frac{d\theta}{\sec\theta} = \sin\theta + C = \frac{\sqrt{x^2-1}}{x} + C$

14. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ \sqrt{x^2 - 1} = \tan\theta;$
$\int \frac{2\,dx}{x^3\sqrt{x^2-1}} = \int \frac{2\tan\theta\sec\theta\,d\theta}{\sec^3\theta\tan\theta} = 2\int \cos^2\theta\,d\theta = 2\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \theta + \sin\theta\cos\theta + C$
$= \theta + \tan\theta\cos^2\theta + C = \sec^{-1}x + \sqrt{x^2-1}\left(\frac{1}{x}\right)^2 + C = \sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2} + C$

15. $u = 9 - x^2 \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx;$
$\int \frac{x\,dx}{\sqrt{9-x^2}} = -\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -\sqrt{u} + C = -\sqrt{9-x^2} + C$

16. $x = 2\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = 2\sec^2\theta\,d\theta,\ 4 + x^2 = 4\sec^2\theta$
$\int \frac{x^2\,dx}{4+x^2} = \int \frac{(4\tan^2\theta)(2\sec^2\theta)d\theta}{4\sec^2\theta} = \int 2\tan^2\theta\,d\theta = 2\int(\sec^2\theta - 1)d\theta = 2\int\sec^2\theta\,d\theta - 2\int d\theta = 2\tan\theta - 2\theta + C$
$= x - 2\tan^{-1}\left(\frac{x}{2}\right) + C$

17. $x = 2\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{2\,d\theta}{\cos^2\theta},\ \sqrt{x^2+4} = \frac{2}{\cos\theta};$
$\int \frac{x^3\,dx}{\sqrt{x^2+4}} = \int \frac{(8\tan^3\theta)(\cos\theta)\,d\theta}{\cos^2\theta} = 8\int \frac{\sin^3\theta\,d\theta}{\cos^4\theta} = 8\int \frac{(\cos^2\theta - 1)(-\sin\theta)\,d\theta}{\cos^4\theta};$
$[t = \cos\theta] \rightarrow 8\int \frac{t^2-1}{t^4}\,dt = 8\int\left(\frac{1}{t^2} - \frac{1}{t^4}\right)dt = 8\left(-\frac{1}{t} + \frac{1}{3t^3}\right) + C = 8\left(-\sec\theta + \frac{\sec^3\theta}{3}\right) + C$
$= 8\left(-\frac{\sqrt{x^2+4}}{2} + \frac{(x^2+4)^{3/2}}{8\cdot 3}\right) + C = \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C = \frac{1}{3}(x^2 - 8)\sqrt{x^2+4} + C$

18. $x = \tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \sec^2\theta\,d\theta,\ \sqrt{x^2+1} = \sec\theta;$
$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sec\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \frac{-\sqrt{x^2+1}}{x} + C$

19. $w = 2\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 2\cos\theta\,d\theta,\ \sqrt{4 - w^2} = 2\cos\theta;$
$\int \frac{8\,dw}{w^2\sqrt{4-w^2}} = \int \frac{8\cdot 2\cos\theta\,d\theta}{4\sin^2\theta\cdot 2\cos\theta} = 2\int \frac{d\theta}{\sin^2\theta} = -2\cot\theta + C = \frac{-2\sqrt{4-w^2}}{w} + C$

20. $w = 3\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 3\cos\theta\, d\theta,\ \sqrt{9 - w^2} = 3\cos\theta;$
$$\int \frac{\sqrt{9-w^2}}{w^2}\, dw = \int \frac{3\cos\theta \cdot 3\cos\theta\, d\theta}{9\sin^2\theta} = \int \cot^2\theta\, d\theta = \int \left(\frac{1-\sin^2\theta}{\sin^2\theta}\right) d\theta = \int (\csc^2\theta - 1)\, d\theta$$
$$= -\cot\theta - \theta + C = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$$

21. $u = 5x \Rightarrow du = 5dx,\ a = 6$
$$\int \frac{100}{36+25x^2} dx = 20\int \frac{1}{(6)^2+(5x)^2} 5dx = 20\int \frac{1}{a^2+u^2} du = 20 \cdot \frac{1}{6}\tan^{-1}\left(\frac{u}{6}\right) + C = \frac{10}{3}\tan^{-1}\left(\frac{5x}{6}\right) + C$$

22. $u = x^2 - 4 \Rightarrow du = 2x\, dx \Rightarrow \frac{1}{2}du = x\, dx$
$$\int x\sqrt{x^2-4}\, dx = \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2-4)^{3/2} + C$$

23. $x = \sin\theta,\ 0 \le \theta \le \frac{\pi}{3},\ dx = \cos\theta\, d\theta,\ (1-x^2)^{3/2} = \cos^3\theta;$
$$\int_0^{\sqrt{3}/2} \frac{4x^2\, dx}{(1-x^2)^{3/2}} = \int_0^{\pi/3} \frac{4\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta} = 4\int_0^{\pi/3} \left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) d\theta = 4\int_0^{\pi/3} (\sec^2\theta - 1)\, d\theta$$
$$= 4\left[\tan\theta - \theta\right]_0^{\pi/3} = 4\sqrt{3} - \frac{4\pi}{3}$$

24. $x = 2\sin\theta,\ 0 \le \theta \le \frac{\pi}{6},\ dx = 2\cos\theta\, d\theta,\ (4-x^2)^{3/2} = 8\cos^3\theta;$
$$\int_0^1 \frac{dx}{(4-x^2)^{3/2}} = \int_0^{\pi/6} \frac{2\cos\theta\, d\theta}{8\cos^3\theta} = \frac{1}{4}\int_0^{\pi/6} \frac{d\theta}{\cos^2\theta} = \frac{1}{4}\left[\tan\theta\right]_0^{\pi/6} = \frac{\sqrt{3}}{12} = \frac{1}{4\sqrt{3}}$$

25. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ (x^2-1)^{3/2} = \tan^3\theta;$
$$\int \frac{dx}{(x^2-1)^{3/2}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\tan^3\theta} = \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = -\frac{x}{\sqrt{x^2-1}} + C$$

26. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ (x^2-1)^{5/2} = \tan^5\theta;$
$$\int \frac{x^2\, dx}{(x^2-1)^{5/2}} = \int \frac{\sec^2\theta \cdot \sec\theta\tan\theta\, d\theta}{\tan^5\theta} = \int \frac{\cos\theta}{\sin^4\theta}\, d\theta = -\frac{1}{3\sin^3\theta} + C = -\frac{x^3}{3(x^2-1)^{3/2}} + C$$

27. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\, d\theta,\ (1-x^2)^{3/2} = \cos^3\theta;$
$$\int \frac{(1-x^2)^{3/2}\, dx}{x^6} = \int \frac{\cos^3\theta \cdot \cos\theta\, d\theta}{\sin^6\theta} = \int \cot^4\theta\csc^2\theta\, d\theta = -\frac{\cot^5\theta}{5} + C = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$$

28. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\, d\theta,\ (1-x^2)^{1/2} = \cos\theta;$
$$\int \frac{(1-x^2)^{1/2}\, dx}{x^4} = \int \frac{\cos\theta \cdot \cos\theta\, d\theta}{\sin^4\theta} = \int \cot^2\theta\csc^2\theta\, d\theta = -\frac{\cot^3\theta}{3} + C = -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C$$

29. $x = \frac{1}{2}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{1}{2}\sec^2\theta\, d\theta,\ (4x^2+1)^2 = \sec^4\theta;$
$$\int \frac{8\, dx}{(4x^2+1)^2} = \int \frac{8\left(\frac{1}{2}\sec^2\theta\right) d\theta}{\sec^4\theta} = 4\int \cos^2\theta\, d\theta = 2(\theta + \sin\theta\cos\theta) + C = 2\tan^{-1}2x + \frac{4x}{(4x^2+1)} + C$$

30. $t = \frac{1}{3}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\sec^2\theta\, d\theta,\ 9t^2 + 1 = \sec^2\theta;$
$$\int \frac{6\, dt}{(9t^2+1)^2} = \int \frac{6\left(\frac{1}{3}\sec^2\theta\right) d\theta}{\sec^4\theta} = 2\int \cos^2\theta\, d\theta = \theta + \sin\theta\cos\theta + C = \tan^{-1}3t + \frac{3t}{(9t^2+1)} + C$$

31. $u = x^2 - 1 \Rightarrow du = 2x\, dx \Rightarrow \frac{1}{2}du = x\, dx$
$$\int \frac{x^3}{x^2-1} dx = \int \left(x + \frac{x}{x^2-1}\right) dx = \int x\, dx + \int \frac{x}{x^2-1} dx = \frac{1}{2}x^2 + \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}x^2 + \frac{1}{2}\ln|u| + C = \frac{1}{2}x^2 + \frac{1}{2}\ln|x^2-1| + C$$

32. $u = 25 + 4x^2 \Rightarrow du = 8x\,dx \Rightarrow \frac{1}{8}du = x\,dx$
$\int \frac{x}{25+4x^2}dx = \frac{1}{8}\int \frac{1}{u}du = \frac{1}{8}\ln|u| + C = \frac{1}{8}\ln(25 + 4x^2) + C$

33. $v = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dv = \cos\theta\,d\theta,\ (1-v^2)^{5/2} = \cos^5\theta;$
$\int \frac{v^2\,dv}{(1-v^2)^{5/2}} = \int \frac{\sin^2\theta\cos\theta\,d\theta}{\cos^5\theta} = \int \tan^2\theta\sec^2\theta\,d\theta = \frac{\tan^3\theta}{3} + C = \frac{1}{3}\left(\frac{v}{\sqrt{1-v^2}}\right)^3 + C$

34. $r = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2};$
$\int \frac{(1-r^2)^{5/2}\,dr}{r^8} = \int \frac{\cos^5\theta\cdot\cos\theta\,d\theta}{\sin^8\theta} = \int \cot^6\theta\csc^2\theta\,d\theta = -\frac{\cot^7\theta}{7} + C = -\frac{1}{7}\left[\frac{\sqrt{1-r^2}}{r}\right]^7 + C$

35. Let $e^t = 3\tan\theta,\ t = \ln(3\tan\theta),\ \tan^{-1}\left(\frac{1}{3}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right),\ dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta,\ \sqrt{e^{2t}+9} = \sqrt{9\tan^2\theta+9} = 3\sec\theta;$
$\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3\tan\theta\cdot\sec^2\theta\,d\theta}{\tan\theta\cdot 3\sec\theta} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec\theta\,d\theta = [\ln|\sec\theta + \tan\theta|]_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)}$
$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3} + \frac{1}{3}\right) = \ln 9 - \ln\left(1 + \sqrt{10}\right)$

36. Let $e^t = \tan\theta,\ t = \ln(\tan\theta),\ \tan^{-1}\left(\frac{3}{4}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right),\ dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta,\ 1 + e^{2t} = 1 + \tan^2\theta = \sec^2\theta;$
$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{(\tan\theta)\left(\frac{\sec^2\theta}{\tan\theta}\right)d\theta}{\sec^3\theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta\,d\theta = [\sin\theta]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$

37. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t} + 4t\sqrt{t}};\ \left[u = 2\sqrt{t},\ du = \frac{1}{\sqrt{t}}\,dt\right] \to \int_{1/\sqrt{3}}^{1} \frac{2\,du}{1+u^2};\ u = \tan\theta,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{4},\ du = \sec^2\theta\,d\theta,\ 1 + u^2 = \sec^2\theta;$
$\int_{1/\sqrt{3}}^{1} \frac{2\,du}{1+u^2} = \int_{\pi/6}^{\pi/4} \frac{2\sec^2\theta\,d\theta}{\sec^2\theta} = [2\theta]_{\pi/6}^{\pi/4} = 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi}{6}$

38. $y = e^{\tan\theta},\ 0 \le \theta \le \frac{\pi}{4},\ dy = e^{\tan\theta}\sec^2\theta\,d\theta,\ \sqrt{1 + (\ln y)^2} = \sqrt{1 + \tan^2\theta} = \sec\theta;$
$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}\,d\theta = \int_0^{\pi/4} \sec\theta\,d\theta = [\ln|\sec\theta + \tan\theta|]_0^{\pi/4} = \ln\left(1 + \sqrt{2}\right)$

39. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta;$
$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec\theta\tan\theta} = \theta + C = \sec^{-1}x + C$

40. $x = \tan\theta,\ dx = \sec^2\theta\,d\theta,\ 1 + x^2 = \sec^2\theta;$
$\int \frac{dx}{x^2+1} = \int \frac{\sec^2\theta\,d\theta}{\sec^2\theta} = \theta + C = \tan^{-1}x + C$

41. $x = \sec\theta,\ dx = \sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta;$
$\int \frac{x\,dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\cdot\sec\theta\tan\theta\,d\theta}{\tan\theta} = \int \sec^2\theta\,d\theta = \tan\theta + C = \sqrt{x^2-1} + C$

42. $x = \sin\theta,\ dx = \cos\theta\,d\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2};$
$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\,d\theta}{\cos\theta} = \theta + C = \sin^{-1}x + C$

43. Let $x^2 = \tan\theta,\ 0 \le \theta < \frac{\pi}{2},\ 2x\,dx = \sec^2\theta\,d\theta \Rightarrow x\,dx = \frac{1}{2}\sec^2\theta\,d\theta;\ \sqrt{1+x^4} = \sqrt{1+\tan^2\theta} = \sec\theta$
$\int \frac{x}{\sqrt{1+x^4}}dx = \frac{1}{2}\int \frac{\sec^2\theta}{\sec\theta}d\theta = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln|\sqrt{1+x^4} + x^2| + C$

44. Let $\ln x = \sin\theta,\ -\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$, $\frac{1}{x}dx = \cos\theta\, d\theta$, $\sqrt{1-(\ln x)^2} = \cos\theta$

$$\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx = \int \frac{\cos^2\theta}{\sin\theta}d\theta = \int \frac{1-\sin^2\theta}{\sin\theta}d\theta = \int \csc\theta\, d\theta - \int \sin\theta\, d\theta = -\ln|\csc\theta + \cot\theta| + \cos\theta + C$$
$$= -\ln\left|\frac{1}{\ln x} + \frac{\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C = -\ln\left|\frac{1+\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C$$

45. Let $u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u\, du \Rightarrow \int \sqrt{\frac{4-x}{x}}\, dx = \int \sqrt{\frac{4-u^2}{u^2}}\, 2u\, du = 2\int \sqrt{4-u^2}\, du$;

$u = 2\sin\theta$, $du = 2\cos\theta\, d\theta$, $0 < \theta \le \frac{\pi}{2}$, $\sqrt{4-u^2} = 2\cos\theta$

$$2\int \sqrt{4-u^2}\, du = 2\int (2\cos\theta)(2\cos\theta)\, d\theta = 8\int \cos^2\theta\, d\theta = 8\int \frac{1+\cos 2\theta}{2}\, d\theta = 4\int d\theta + 4\int \cos 2\theta\, d\theta$$
$$= 4\theta + 2\sin 2\theta + C = 4\theta + 4\sin\theta\cos\theta + C = 4\sin^{-1}\left(\frac{u}{2}\right) + 4\left(\frac{u}{2}\right)\left(\frac{\sqrt{4-u^2}}{2}\right) + C = 4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sqrt{x}\sqrt{4-x} + C$$
$$= 4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sqrt{4x-x^2} + C$$

46. Let $u = x^{3/2} \Rightarrow x = u^{2/3} \Rightarrow dx = \frac{2}{3}u^{-1/3}du$

$$\int \sqrt{\frac{x}{1-x^3}}dx = \int \sqrt{\frac{u^{2/3}}{1-(u^{2/3})^3}}\left(\frac{2}{3}u^{-1/3}\right)du = \int \frac{u^{1/3}}{\sqrt{1-u^2}}\left(\frac{2}{3u^{1/3}}\right)du = \frac{2}{3}\int \frac{1}{\sqrt{1-u^2}}du = \frac{2}{3}\sin^{-1}u + C = \frac{2}{3}\sin^{-1}\left(x^{3/2}\right) + C$$

47. Let $u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u\, du \Rightarrow \int \sqrt{x}\sqrt{1-x}\, dx = \int u\sqrt{1-u^2}\, 2u\, du = 2\int u^2\sqrt{1-u^2}\, du$;

$u = \sin\theta$, $du = \cos\theta\, d\theta$, $-\frac{\pi}{2} < \theta \le \frac{\pi}{2}$, $\sqrt{1-u^2} = \cos\theta$

$$2\int u^2\sqrt{1-u^2}\, du = 2\int \sin^2\theta\cos\theta\cos\theta\, d\theta = 2\int \sin^2\theta\cos^2\theta\, d\theta = \frac{1}{2}\int \sin^2 2\theta\, d\theta = \frac{1}{2}\int \frac{1-\cos 4\theta}{2}\, d\theta$$
$$= \frac{1}{4}\int d\theta - \frac{1}{4}\int \cos 4\theta\, d\theta = \frac{1}{4}\theta - \frac{1}{16}\sin 4\theta + C = \frac{1}{4}\theta - \frac{1}{8}\sin 2\theta\cos 2\theta + C = \frac{1}{4}\theta - \frac{1}{4}\sin\theta\cos\theta\,(2\cos^2\theta - 1) + C$$
$$= \frac{1}{4}\theta - \frac{1}{2}\sin\theta\cos^3\theta + \frac{1}{4}\sin\theta\cos\theta + C = \frac{1}{4}\sin^{-1}u - \frac{1}{2}u(1-u^2)^{3/2} - \frac{1}{4}u\sqrt{1-u^2} + C$$
$$= \frac{1}{4}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x}(1-x)^{3/2} - \frac{1}{4}\sqrt{x}\sqrt{1-x} + C$$

48. Let $w = \sqrt{x-1} \Rightarrow w^2 = x-1 \Rightarrow 2w\, dw = dx \Rightarrow \int \frac{\sqrt{x-2}}{\sqrt{x-1}}dx = \int \frac{\sqrt{w^2-1}}{w}2w\, dw = 2\int \sqrt{w^2-1}\, dw$

$w = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$, $0 < \theta < \frac{\pi}{2}$, $\sqrt{w^2-1} = \tan\theta$

$2\int \sqrt{w^2-1}\, dw = 2\int \tan\theta\sec\theta\tan\theta\, d\theta$; $u = \tan\theta$, $du = \sec^2\theta\, d\theta$, $dv = \sec\theta\tan\theta\, d\theta$, $v = \sec\theta$

$$2\int \tan\theta\sec\theta\tan\theta\, d\theta = 2\sec\theta\tan\theta - 2\int \sec^3\theta\, d\theta = 2\sec\theta\tan\theta - 2\int \sec^2\theta\sec\theta d\theta$$
$$= 2\sec\theta\tan\theta - 2\int \left(\tan^2\theta + 1\right)\sec\theta\, d\theta = 2\sec\theta\tan\theta - 2\left(\int \tan^2\theta\sec\theta\, d\theta + \int \sec\theta\, d\theta\right)$$
$$= 2\sec\theta\tan\theta - 2\ln|\sec\theta + \tan\theta| - 2\int \tan^2\theta\sec\theta\, d\theta \Rightarrow 2\int \tan^2\theta\sec\theta\, d\theta = \sec\theta\tan\theta - \ln|\sec\theta + \tan\theta| + C$$
$$= w\sqrt{w^2-1} - \ln|w + \sqrt{w^2-1}| + C = \sqrt{x-1}\sqrt{x-2} - \ln|\sqrt{x-1} + \sqrt{x-2}| + C$$

49. $x\frac{dy}{dx} = \sqrt{x^2-4}$; $dy = \sqrt{x^2-4}\,\frac{dx}{x}$; $y = \int \frac{\sqrt{x^2-4}}{x}\, dx$; $\begin{bmatrix} x = 2\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 2\sec\theta\tan\theta\, d\theta \\ \sqrt{x^2-4} = 2\tan\theta \end{bmatrix}$

$$\to\ y = \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)\, d\theta}{2\sec\theta} = 2\int \tan^2\theta\, d\theta = 2\int (\sec^2\theta - 1)\, d\theta = 2(\tan\theta - \theta) + C$$
$$= 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right] + C;\ x = 2 \text{ and } y = 0 \Rightarrow 0 = 0 + C \Rightarrow C = 0 \Rightarrow y = 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\frac{x}{2}\right]$$

50. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1$, $dy = \frac{dx}{\sqrt{x^2-9}}$; $y = \int \frac{dx}{\sqrt{x^2-9}}$; $\begin{bmatrix} x = 3\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 3\sec\theta\tan\theta\,d\theta \\ \sqrt{x^2-9} = 3\tan\theta \end{bmatrix} \to y = \int \frac{3\sec\theta\tan\theta\,d\theta}{3\tan\theta}$

$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$; $x = 5$ and $y = \ln 3 \Rightarrow \ln 3 = \ln 3 + C \Rightarrow C = 0$

$\Rightarrow y = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|$

51. $(x^2+4)\frac{dy}{dx} = 3$, $dy = \frac{3\,dx}{x^2+4}$; $y = 3\int \frac{dx}{x^2+4} = \frac{3}{2}\tan^{-1}\frac{x}{2} + C$; $x = 2$ and $y = 0 \Rightarrow 0 = \frac{3}{2}\tan^{-1} 1 + C$

$\Rightarrow C = -\frac{3\pi}{8} \Rightarrow y = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{3\pi}{8}$

52. $(x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}$, $dy = \frac{dx}{(x^2+1)^{3/2}}$; $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $(x^2+1)^{3/2} = \sec^3\theta$;

$y = \int \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \int \cos\theta\,d\theta = \sin\theta + C = \tan\theta\cos\theta + C = \frac{\tan\theta}{\sec\theta} + C = \frac{x}{\sqrt{x^2+1}} + C$; $x = 0$ and $y = 1$

$\Rightarrow 1 = 0 + C \Rightarrow y = \frac{x}{\sqrt{x^2+1}} + 1$

53. $A = \int_0^3 \frac{\sqrt{9-x^2}}{3}\,dx$; $x = 3\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$, $dx = 3\cos\theta\,d\theta$, $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\cos\theta$;

$A = \int_0^{\pi/2} \frac{3\cos\theta \cdot 3\cos\theta\,d\theta}{3} = 3\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{3}{2}\left[\theta + \sin\theta\cos\theta\right]_0^{\pi/2} = \frac{3\pi}{4}$

54. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \pm b\sqrt{1-\frac{x^2}{a^2}}$; $A = 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = 4b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx$

$\left[x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ dx = a\cos\theta\,d\theta,\ \sqrt{1-\frac{x^2}{a^2}} = \cos\theta,\ x = 0 = a\sin\theta \Rightarrow \theta = 0,\ x = a = a\sin\theta \Rightarrow \theta = \frac{\pi}{2}\right]$

$4b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx = 4b\int_0^{\pi/2}\cos\theta\,(a\cos\theta)\,d\theta = 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}d\theta$

$= 2ab\int_0^{\pi/2} d\theta + 2ab\int_0^{\pi/2}\cos 2\theta\,d\theta = 2ab\Big[\theta\Big]_0^{\pi/2} + ab\Big[\sin 2\theta\Big]_0^{\pi/2} = 2ab\left(\frac{\pi}{2} - 0\right) + ab(\sin\pi - \sin 0) = \pi ab$

55. (a) $A = \int_0^{1/2} \sin^{-1}x\,dx$ $\left[u = \sin^{-1}x,\ du = \frac{1}{\sqrt{1-x^2}}dx,\ dv = dx,\ v = x\right]$

$= \Big[x\sin^{-1}x\Big]_0^{1/2} - \int_0^{1/2}\frac{x}{\sqrt{1-x^2}}\,dx = = \left(\frac{1}{2}\sin^{-1}\frac{1}{2} - 0\right) + \left[\sqrt{1-x^2}\right]_0^{1/2} = \frac{\pi + 6\sqrt{3} - 12}{12}$

(b) $M = \int_0^{1/2}\sin^{-1}x\,dx = \frac{\pi+6\sqrt{3}-12}{12}$; $\overline{x} = \frac{1}{\frac{\pi+6\sqrt{3}-12}{12}}\int_0^{1/2} x\sin^{-1}x\,dx = \frac{12}{\pi+6\sqrt{3}-12}\int_0^{1/2} x\sin^{-1}x\,dx$

$\left[u = \sin^{-1}x,\ du = \frac{1}{\sqrt{1-x^2}}dx,\ dv = x\,dx,\ v = \frac{1}{2}x^2\right]$

$= \frac{12}{\pi+6\sqrt{3}-12}\left(\left[\frac{1}{2}x^2\sin^{-1}x\right]_0^{1/2} - \frac{1}{2}\int_0^{1/2}\frac{x^2}{\sqrt{1-x^2}}\,dx\right)$

$\left[x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ \sqrt{1-x^2} = \cos\theta,\ x = 0 = \sin\theta \Rightarrow \theta = 0,\ x = \frac{1}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{6}\right]$

$= \frac{12}{\pi+6\sqrt{3}-12}\left(\left(\frac{1}{2}\left(\frac{1}{2}\right)^2\sin^{-1}\left(\frac{1}{2}\right) - 0\right) - \frac{1}{2}\int_0^{\pi/6}\frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta\right) = \frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48} - \frac{1}{2}\int_0^{\pi/6}\sin^2\theta\,d\theta\right)$

$= \frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48} - \frac{1}{2}\int_0^{\pi/6}\frac{1-\cos 2\theta}{2}\,d\theta\right) = \frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48} - \frac{1}{4}\int_0^{\pi/6}d\theta + \frac{1}{4}\int_0^{\pi/6}\cos 2\theta\,d\theta\right)$

$= \frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48} + \left[-\frac{\theta}{4} + \frac{1}{8}\sin 2\theta\right]_0^{\pi/6}\right) = \frac{3\sqrt{3}-\pi}{4\left(\pi+6\sqrt{3}-12\right)}$; $\overline{y} = \frac{1}{\frac{\pi+6\sqrt{3}-12}{12}}\int_0^{1/2}\frac{1}{2}\left(\sin^{-1}x\right)^2dx$

$\left[u = \left(\sin^{-1}x\right)^2,\ du = \frac{2\sin^{-1}x}{\sqrt{1-x^2}}dx,\ dv = dx,\ v = x\right]$

$= \frac{6}{\pi+6\sqrt{3}-12}\left(\left[x(\sin^{-1}x\,dx)^2\right]_0^{1/2} - \int_0^{1/2}\frac{2x\sin^{-1}x}{\sqrt{1-x^2}}dx\right)$

$\left[u = \sin^{-1}x,\ du = \frac{1}{\sqrt{1-x^2}}dx,\ dv = \frac{2x}{\sqrt{1-x^2}}dx,\ v = -2\sqrt{1-x^2}\right]$

$= \frac{6}{\pi+6\sqrt{3}-12}\left(\left(\frac{1}{2}\left(\sin^{-1}\left(\frac{1}{2}\right)\right)^2 - 0\right) + \left[2\sqrt{1-x^2}\sin^{-1}x\right]_0^{1/2} - \int_0^{1/2}\frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}}dx\right)$

$= \frac{6}{\pi+6\sqrt{3}-12}\left(\frac{\pi^2}{72} + \left(2\sqrt{1-\left(\frac{1}{2}\right)^2}\sin^{-1}\left(\frac{1}{2}\right) - 0\right) - \left[2x\right]_0^{1/2}\right) = \frac{6}{\pi+6\sqrt{3}-12}\left(\frac{\pi^2}{72} + \frac{\pi\sqrt{3}}{6} - 1\right) = \frac{\pi^2+12\pi\sqrt{3}-72}{12\left(\pi+6\sqrt{3}-12\right)}$

56. $V = \int_0^1 \pi\left(\sqrt{x\tan^{-1}x}\right)^2 dx = \pi\int_0^1 x\tan^{-1}x\,dx \qquad \left[u = \tan^{-1}x,\ du = \frac{1}{1+x^2}dx,\ dv = x\,dx,\ v = \frac{1}{2}x^2\right]$

$= \pi\left(\left[\frac{1}{2}x^2\tan^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1\frac{x^2}{1+x^2}dx\right) = \pi\left(\left(\frac{1}{2}\tan^{-1}1 - 0\right) - \frac{1}{2}\int_0^1\left(1 - \frac{1}{1+x^2}\right)dx\right) = \pi\left(\frac{\pi}{8} - \frac{1}{2}\int_0^1\left(1 - \frac{1}{1+x^2}\right)dx\right)$

$= \pi\left(\frac{\pi}{8} - \frac{1}{2}\int_0^1 dx + \frac{1}{2}\int_0^1\frac{1}{1+x^2}dx\right) = \pi\left(\frac{\pi}{8} + \left[-\frac{1}{2}x + \frac{1}{2}\tan^{-1}x\right]_0^1\right) = \pi\left(\frac{\pi}{8} + \left(-\frac{1}{2} + \frac{1}{2}\tan^{-1}1 + 0 - 0\right)\right) = \frac{\pi(\pi-2)}{4}$

57. (a) Integration by parts: $u = x^2,\ du = 2x\,dx,\ dv = x\sqrt{1-x^2}\,dx,\ v = -\frac{1}{3}(1-x^2)^{3/2}$

$\int x^3\sqrt{1-x^2}\,dx = -\frac{1}{3}x^2(1-x^2)^{3/2} + \frac{1}{3}\int(1-x^2)^{3/2}2x\,dx = -\frac{1}{3}x^2(1-x^2)^{3/2} - \frac{2}{15}(1-x^2)^{5/2} + C$

(b) Substitution: $u = 1-x^2 \Rightarrow x^2 = 1-u \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx$

$\int x^3\sqrt{1-x^2}\,dx = \int x^2\sqrt{1-x^2}\,x\,dx = -\frac{1}{2}\int(1-u)\sqrt{u}\,du = -\frac{1}{2}\int\left(\sqrt{u} - u^{3/2}\right)du = -\frac{1}{3}u^{3/2} + \frac{1}{5}u^{5/2} + C$

$= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C$

(c) Trig substitution: $x = \sin\theta,\ \frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ \sqrt{1-x^2} = \cos\theta$

$\int x^3\sqrt{1-x^2}\,dx = \int\sin^3\theta\cos\theta\cos\theta\,d\theta = \int\sin^2\theta\cos^2\theta\sin\theta\,d\theta = \int(1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta$

$= \int\cos^2\theta\sin\theta\,d\theta - \int\cos^4\theta\sin\theta\,d\theta = -\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C = -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C$

58. (a) The slope of the line tangent to $y = f(x)$ is given by $f'(x)$. Consider the triangle whose hypotenuse is the 30 ft rope, the length of the base is x and the height $h = \sqrt{900-x^2}$. The slope of the tangent line is also $-\frac{\sqrt{900-x^2}}{x}$, thus $f'(x) = -\frac{\sqrt{900-x^2}}{x}$.

(b) $f(x) = \int -\frac{\sqrt{900-x^2}}{x}dx \qquad \left[x = 30\sin\theta,\ 0 < \theta \le \frac{\pi}{2},\ dx = 30\cos\theta\,d\theta,\ \sqrt{900-x^2} = 30\cos\theta\right]$

$= -\int\frac{30\cos\theta}{30\sin\theta}30\cos\theta\,d\theta = -30\int\frac{\cos^2\theta}{\sin\theta}d\theta = -30\int\frac{(1-\sin^2\theta)}{\sin\theta}d\theta = -30\int\csc\theta\,d\theta + 30\int\sin\theta\,d\theta$

$= 30\ln|\csc\theta + \cot\theta| - 30\cos\theta + C = 30\ln\left|\frac{30}{x} + \frac{\sqrt{900-x^2}}{x}\right| - \sqrt{900-x^2} + C;\ f(30) = 0$

$\Rightarrow 0 = 30\ln\left|\frac{30}{30} + \frac{\sqrt{900-30^2}}{30}\right| - \sqrt{900-30^2} + C = C \Rightarrow f(x) = 30\ln\left|\frac{30}{x} + \frac{\sqrt{900-x^2}}{x}\right| - \sqrt{900-x^2}$