8.2 TRIGONOMETRIC INTEGRALS

1. $\int \cos 2x\, dx = \frac{1}{2}\int \cos 2x \cdot 2dx = \frac{1}{2}\sin 2x + C$

2. $\int_0^{\pi} 3 \sin \frac{x}{3}\, dx = 9\int_0^{\pi} \sin \frac{x}{3} \cdot \frac{1}{3}dx = 9\left[-\cos\frac{x}{3}\right]_0^{\pi} = 9\left(-\cos\frac{\pi}{3} + \cos 0\right) = 9\left(-\frac{1}{2} + 1\right) = \frac{9}{2}$

3. $\int \cos^3 x \sin x\, dx = -\int \cos^3 x\, (-\sin x)dx = -\frac{1}{4}\cos^4 x + C$

4. $\int \sin^4 2x \cos 2x\, dx = \frac{1}{2}\int \sin^4 2x \cos 2x \cdot 2dx = \frac{1}{10}\sin^5 2x + C$

5. $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = \int \left(1 - \cos^2 x\right) \sin x\, dx = \int \sin x\, dx - \int \cos^2 x \sin x\, dx = -\cos x + \frac{1}{3}\cos^3 x + C$

6. $\int \cos^3 4x\, dx = \int \cos^2 4x \cos 4x\, dx = \frac{1}{4}\int \left(1 - \sin^2 4x\right) \cos 4x \cdot 4dx = \frac{1}{4}\int \cos 4x \cdot 4dx - \frac{1}{4}\int \sin^2 4x \cos 4x \cdot 4dx$
$= \frac{1}{4}\sin 4x - \frac{1}{12}\sin^3 4x + C$

7. $\int \sin^5 x\, dx = \int (\sin^2 x)^2 \sin x\, dx = \int (1 - \cos^2 x)^2 \sin x\, dx = \int (1 - 2\cos^2 x + \cos^4 x)\sin x\, dx$
$= \int \sin x\, dx - \int 2\cos^2 x \sin x\, dx + \int \cos^4 x \sin x\, dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$

8. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right) dx$ (using Exercise 7) $= \int_0^{\pi} \sin\left(\frac{x}{2}\right) dx - \int_0^{\pi} 2\cos^2\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) dx + \int_0^{\pi} \cos^4\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) dx$
$= \left[-2\cos\left(\frac{x}{2}\right) + \frac{4}{3}\cos^3\left(\frac{x}{2}\right) - \frac{2}{5}\cos^5\left(\frac{x}{2}\right)\right]_0^{\pi} = (0) - \left(-2 + \frac{4}{3} - \frac{2}{5}\right) = \frac{16}{15}$

9. $\int \cos^3 x\, dx = \int (\cos^2 x)\cos x\, dx = \int (1 - \sin^2 x)\cos x\, dx = \int \cos x\, dx - \int \sin^2 x \cos x\, dx = \sin x - \frac{1}{3}\sin^3 x + C$

10. $\int_0^{\pi/6} 3\cos^5 3x\, dx = \int_0^{\pi/6} (\cos^2 3x)^2 \cos 3x \cdot 3dx = \int_0^{\pi/6} (1 - \sin^2 3x)^2 \cos 3x \cdot 3dx = \int_0^{\pi/6} (1 - 2\sin^2 3x + \sin^4 3x)\cos 3x \cdot 3dx$
$= \int_0^{\pi/6} \cos 3x \cdot 3dx - 2\int_0^{\pi/6} \sin^2 3x \cos 3x \cdot 3dx + \int_0^{\pi/6} \sin^4 3x \cos 3x \cdot 3dx = \left[\sin 3x - 2\frac{\sin^3 3x}{3} + \frac{\sin^5 3x}{5}\right]_0^{\pi/6}$
$= \left(1 - \frac{2}{3} + \frac{1}{5}\right) - (0) = \frac{8}{15}$

11. $\int \sin^3 x \cos^3 x\, dx = \int \sin^3 x \cos^2 x \cos x dx = \int \sin^3 x\, (1 - \sin^2 x)\cos x\, dx = \int \sin^3 x \cos x\, dx - \int \sin^5 x \cos x\, dx$
$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$

12. $\int \cos^3 2x \sin^5 2x\, dx = \frac{1}{2}\int \cos^3 2x \sin^5 2x \cdot 2dx = \frac{1}{2}\int \cos 2x \cos^2 2x \sin^5 2x \cdot 2dx = \frac{1}{2}\int (1 - \sin^2 2x) \sin^5 2x \cos 2x \cdot 2dx$
$= \frac{1}{2}\int \sin^5 2x \cos 2x \cdot 2dx - \frac{1}{2}\int \sin^7 2x \cos 2x \cdot 2dx = \frac{1}{12}\sin^6 2x - \frac{1}{16}\sin^8 2x + C$

13. $\int \cos^2 x\, dx = \int \frac{1 + \cos 2x}{2}\, dx = \frac{1}{2}\int \left(1 + \cos 2x\right) dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\, dx = \frac{1}{2}\int dx + \frac{1}{4}\int \cos 2x \cdot 2dx$
$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

14. $\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \frac{1 - \cos 2x}{2}\, dx = \frac{1}{2}\int_0^{\pi/2} \left(1 - \cos 2x\right) dx = \frac{1}{2}\int_0^{\pi/2} dx - \frac{1}{2}\int_0^{\pi/2} \cos 2x\, dx = \frac{1}{2}\int_0^{\pi/2} dx - \frac{1}{4}\int_0^{\pi/2} \cos 2x \cdot 2dx$
$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \left(\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)\right) - \left(\frac{1}{2}(0) - \frac{1}{4}\sin 2(0)\right) = \left(\frac{\pi}{4} - 0\right) - (0 - 0) = \frac{\pi}{4}$

15. $\int_0^{\pi/2} \sin^7 y\, dy = \int_0^{\pi/2} \sin^6 y \sin y\, dy = \int_0^{\pi/2} (1 - \cos^2 y)^3 \sin y\, dy = \int_0^{\pi/2} \sin y\, dy - 3\int_0^{\pi/2} \cos^2 y \sin y\, dy$
$+ 3\int_0^{\pi/2} \cos^4 y \sin y\, dy - \int_0^{\pi/2} \cos^6 y \sin y\, dy = \left[-\cos y + 3\frac{\cos^3 y}{3} - 3\frac{\cos^5 y}{5} + \frac{\cos^7 y}{7}\right]_0^{\pi/2} = (0) - \left(-1 + 1 - \frac{3}{5} + \frac{1}{7}\right) = \frac{16}{35}$

16. $\int 7\cos^7 t\, dt$ (using Exercise 15) $= 7\left[\int \cos t\, dt - 3\int \sin^2 t \cos t\, dt + 3\int \sin^4 t \cos t\, dt - \int \sin^6 t \cos t\, dt\right]$
$= 7\left(\sin t - 3\frac{\sin^3 t}{3} + 3\frac{\sin^5 t}{5} - \frac{\sin^7 t}{7}\right) + C = 7\sin t - 7\sin^3 t + \frac{21}{5}\sin^5 t - \sin^7 t + C$

17. $\int_0^{\pi} 8\sin^4 x\, dx = 8\int_0^{\pi} \left(\frac{1 - \cos 2x}{2}\right)^2 dx = 2\int_0^{\pi} (1 - 2\cos 2x + \cos^2 2x) dx = 2\int_0^{\pi} dx - 2\int_0^{\pi} \cos 2x \cdot 2dx + 2\int_0^{\pi} \frac{1 + \cos 4x}{2}\, dx$
$= [2x - 2\sin 2x]_0^{\pi} + \int_0^{\pi} dx + \int_0^{\pi} \cos 4x\, dx = 2\pi + \left[x + \frac{1}{2}\sin 4x\right]_0^{\pi} = 2\pi + \pi = 3\pi$

18. $\int 8\cos^4 2\pi x\, dx = 8\int \left(\frac{1 + \cos 4\pi x}{2}\right)^2 dx = 2\int (1 + 2\cos 4\pi x + \cos^2 4\pi x) dx = 2\int dx + 4\int \cos 4\pi x\, dx + 2\int \frac{1 + \cos 8\pi x}{2}\, dx$
$= 3\int dx + 4\int \cos 4\pi x\, dx + \int \cos 8\pi x\, dx = 3x + \frac{1}{\pi}\sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C$

19. $\int 16 \sin^2 x \cos^2 x\, dx = 16\int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) dx = 4\int (1 - \cos^2 2x) dx = 4\int dx - 4\int \left(\frac{1 + \cos 4x}{2}\right) dx$
$= 4x - 2\int dx - 2\int \cos 4x\, dx = 4x - 2x - \frac{1}{2}\sin 4x + C = 2x - \frac{1}{2}\sin 4x + C = 2x - \sin 2x \cos 2x + C$
$= 2x - 2\sin x \cos x\, (2\cos^2 x - 1) + C = 2x - 4\sin x \cos^3 x + 2\sin x \cos x + C$

20. $\int_0^{\pi} 8\sin^4 y\cos^2 y\,dy = 8\int_0^{\pi}\left(\frac{1-\cos 2y}{2}\right)^2\left(\frac{1+\cos 2y}{2}\right)dy = \int_0^{\pi} dy - \int_0^{\pi}\cos 2y\,dy - \int_0^{\pi}\cos^2 2y\,dy + \int_0^{\pi}\cos^3 2y\,dy$
$= \left[y - \frac{1}{2}\sin 2y\right]_0^{\pi} - \int_0^{\pi}\left(\frac{1+\cos 4y}{2}\right)dy + \int_0^{\pi}(1-\sin^2 2y)\cos 2y\,dy = \pi - \frac{1}{2}\int_0^{\pi} dy - \frac{1}{2}\int_0^{\pi}\cos 4y\,dy + \int_0^{\pi}\cos 2y\,dy$
$-\int_0^{\pi}\sin^2 2y\cos 2y\,dy = \pi + \left[-\frac{1}{2}y - \frac{1}{8}\sin 4y + \frac{1}{2}\sin 2y - \frac{1}{2}\cdot\frac{\sin^3 2y}{3}\right]_0^{\pi} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

21. $\int 8\cos^3 2\theta\sin 2\theta\,d\theta = 8\left(-\frac{1}{2}\right)\frac{\cos^4 2\theta}{4} + C = -\cos^4 2\theta + C$

22. $\int_0^{\pi/2}\sin^2 2\theta\cos^3 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta(1-\sin^2 2\theta)\cos 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta\cos 2\theta\,d\theta - \int_0^{\pi/2}\sin^4 2\theta\cos 2\theta\,d\theta$
$= \left[\frac{1}{2}\cdot\frac{\sin^3 2\theta}{3} - \frac{1}{2}\cdot\frac{\sin^5 2\theta}{5}\right]_0^{\pi/2} = 0$

23. $\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}\,dx = \int_0^{2\pi}\left|\sin\frac{x}{2}\right|dx = \int_0^{2\pi}\sin\frac{x}{2}\,dx = \left[-2\cos\frac{x}{2}\right]_0^{2\pi} = 2+2 = 4$

24. $\int_0^{\pi}\sqrt{1-\cos 2x}\,dx = \int_0^{\pi}\sqrt{2}\,|\sin 2x|\,dx = \int_0^{\pi}\sqrt{2}\sin 2x\,dx = \left[-\sqrt{2}\cos 2x\right]_0^{\pi} = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

25. $\int_0^{\pi}\sqrt{1-\sin^2 t}\,dt = \int_0^{\pi}|\cos t|\,dt = \int_0^{\pi/2}\cos t\,dt - \int_{\pi/2}^{\pi}\cos t\,dt = [\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^{\pi} = 1 - 0 - 0 + 1 = 2$

26. $\int_0^{\pi}\sqrt{1-\cos^2\theta}\,d\theta = \int_0^{\pi}|\sin\theta|\,d\theta = \int_0^{\pi}\sin\theta\,d\theta = [-\cos\theta]_0^{\pi} = 1+1 = 2$

27. $\int_{\pi/3}^{\pi/2}\frac{\sin^2 x}{\sqrt{1-\cos x}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sin^2 x}{\sqrt{1-\cos x}}\frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sin^2 x\sqrt{1+\cos x}}{\sqrt{1-\cos^2 x}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sin^2 x\sqrt{1+\cos x}}{\sqrt{\sin^2 x}}\,dx$
$= \int_{\pi/3}^{\pi/2}\sin x\sqrt{1+\cos x}\,dx = \left[-\frac{2}{3}(1+\cos x)^{3/2}\right]_{\pi/3}^{\pi/2} = -\frac{2}{3}\left(1+\cos\left(\frac{\pi}{2}\right)\right)^{3/2} + \frac{2}{3}\left(1+\cos\left(\frac{\pi}{3}\right)\right)^{3/2} = -\frac{2}{3} + \frac{2}{3}\left(\frac{3}{2}\right)^{3/2}$
$= \sqrt{\frac{3}{2}} - \frac{2}{3}$

28. $\int_0^{\pi/6}\sqrt{1+\sin x}\,dx = \int_0^{\pi/6}\frac{\sqrt{1+\sin x}}{1}\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6}\frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6}\frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6}\frac{\cos x}{\sqrt{1-\sin x}}\,dx$
$= \left[-2(1-\sin x)^{1/2}\right]_0^{\pi/6} = -2\sqrt{1-\sin\left(\frac{\pi}{6}\right)} + 2\sqrt{1-\sin 0} = -2\sqrt{\frac{1}{2}} + 2\sqrt{1} = 2-\sqrt{2}$

29. $\int_{5\pi/6}^{\pi}\frac{\cos^4 x}{\sqrt{1-\sin x}}\,dx = \int_{5\pi/6}^{\pi}\frac{\cos^4 x}{\sqrt{1-\sin x}}\frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}\,dx = \int_{5\pi/6}^{\pi}\frac{\cos^4 x\sqrt{1+\sin x}}{\sqrt{1-\sin^2 x}}\,dx = \int_{5\pi/6}^{\pi}\frac{\cos^4 x\sqrt{1+\sin x}}{\sqrt{\cos^2 x}}\,dx$
$= \int_{5\pi/6}^{\pi}\frac{\cos^4 x\sqrt{1+\sin x}}{-\cos x}\,dx = -\int_{5\pi/6}^{\pi}\cos^3 x\sqrt{1+\sin x}\,dx = -\int_{5\pi/6}^{\pi}\cos x(1-\sin^2 x)\sqrt{1+\sin x}\,dx$
$= -\int_{5\pi/6}^{\pi}\cos x\sqrt{1+\sin x}\,dx + \int_{5\pi/6}^{\pi}\cos x\sin^2 x\sqrt{1+\sin x}\,dx;\ u^2\sqrt{u}\,du$
$\left[\text{Let } u = 1+\sin x \Rightarrow u-1 = \sin x \Rightarrow du = \cos x\,dx,\ x = \frac{5\pi}{6} \Rightarrow u = 1+\sin\left(\frac{5\pi}{6}\right) = \frac{3}{2},\ x = \pi \Rightarrow u = 1+\sin\pi = 1\right]$
$= \left[-\frac{2}{3}(1+\sin x)^{3/2}\right]_{5\pi/6}^{\pi} + \int_{3/2}^{1}(u-1)^2\sqrt{u}\,du = \left[-\frac{2}{3}(1+\sin x)^{3/2}\right]_{5\pi/6}^{\pi} + \int_{3/2}^{1}\left(u^{5/2} - 2u^{3/2} + \sqrt{u}\right)du$
$= \left(-\frac{2}{3}(1+\sin\pi)^{3/2} + \frac{2}{3}\left(1+\sin\left(\frac{5\pi}{6}\right)\right)^{3/2}\right) + \left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_{3/2}^{1}$
$= \left(-\frac{2}{3} + \frac{2}{3}\left(\frac{3}{2}\right)^{3/2}\right) + \left(\frac{2}{7} - \frac{4}{5} + \frac{2}{3}\right) - \left(\frac{2}{7}\left(\frac{3}{2}\right)^{7/2} - \frac{4}{5}\left(\frac{3}{2}\right)^{5/2} + \frac{2}{3}\left(\frac{3}{2}\right)^{3/2}\right) = \frac{4}{5}\left(\frac{3}{2}\right)^{5/2} - \frac{2}{7}\left(\frac{3}{2}\right)^{7/2} - \frac{18}{35}$

30. $\int_{\pi/2}^{7\pi/12}\sqrt{1-\sin 2x}\,dx = \int_{\pi/2}^{7\pi/12}\frac{\sqrt{1-\sin 2x}}{1}\frac{\sqrt{1+\sin 2x}}{\sqrt{1+\sin 2x}}\,dx = \int_{\pi/2}^{7\pi/12}\frac{\sqrt{1-\sin^2 2x}}{\sqrt{1+\sin 2x}}\,dx = \int_{\pi/2}^{7\pi/12}\frac{\sqrt{\cos^2 2x}}{\sqrt{1+\sin 2x}}\,dx$
$= \int_{\pi/2}^{7\pi/12}\frac{-\cos 2x}{\sqrt{1+\sin 2x}}\,dx = \left[-\sqrt{1+\sin 2x}\right]_{\pi/2}^{7\pi/12} = -\sqrt{1+\sin 2\left(\frac{7\pi}{12}\right)} + \sqrt{1+\sin 2\left(\frac{\pi}{2}\right)} = -\sqrt{\frac{1}{2}} + 1 = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$

31. $\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta}\, d\theta = \int_0^{\pi/2} \theta\sqrt{2}\,|\sin\theta|\, d\theta = \sqrt{2}\int_0^{\pi/2} \theta\sin\theta\, d\theta = \sqrt{2}\left[-\theta\cos\theta + \sin\theta\right]_0^{\pi/2} = \sqrt{2}(1) = \sqrt{2}$

32. $\int_{-\pi}^{\pi} (1-\cos^2 t)^{3/2}\, dt = \int_{-\pi}^{\pi} (\sin^2 t)^{3/2}\, dt = \int_{-\pi}^{\pi} |\sin^3 t|\, dt = -\int_{-\pi}^{0} \sin^3 t\, dt + \int_0^{\pi} \sin^3 t\, dt = -\int_{-\pi}^{0} (1-\cos^2 t)\sin t\, dt$
$+ \int_0^{\pi} (1-\cos^2 t)\sin t\, dt = -\int_{-\pi}^{0} \sin t\, dt + \int_{-\pi}^{0} \cos^2 t\sin t\, dt + \int_0^{\pi} \sin t\, dt - \int_0^{\pi} \cos^2 t\sin t\, dt = \left[\cos t - \frac{\cos^3 t}{3}\right]_{-\pi}^{0}$
$+ \left[-\cos t + \frac{\cos^3 t}{3}\right]_0^{\pi} = \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) + \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = \frac{8}{3}$

33. $\int \sec^2 x\tan x\, dx = \int \tan x\sec^2 x\, dx = \frac{1}{2}\tan^2 x + C$

34. $\int \sec x\tan^2 x\, dx = \int \sec x\tan x\tan x dx$; $u = \tan x$, $du = \sec^2 x\, dx$, $dv = \sec x\tan x\, dx$, $v = \sec x$;
$= \sec x\tan x - \int \sec^3 x\, dx = \sec x\tan x - \int \sec^2 x\sec x dx = \sec x\tan x - \int \left(\tan^2 x + 1\right)\sec x dx$
$= \sec x\tan x - \left(\int \tan^2 x\sec x dx + \int \sec x dx\right) = \sec x\tan x - \ln|\sec x + \tan x| - \int \tan^2 x\sec x dx$
$\Rightarrow \int \sec x\tan^2 x\, dx = \sec x\tan x - \ln|\sec x + \tan x| - \int \tan^2 x\sec x dx$
$\Rightarrow 2\int \tan^2 x\sec x dx = \sec x\tan x - \ln|\sec x + \tan x| \Rightarrow \int \tan^2 x\sec x dx = \frac{1}{2}\sec x\tan x - \frac{1}{2}\ln|\sec x + \tan x| + C$

35. $\int \sec^3 x\tan x\, dx = \int \sec^2 x\sec x\tan x\, dx = \frac{1}{3}\sec^3 x + C$

36. $\int \sec^3 x\tan^3 x\, dx = \int \sec^2 x\tan^2 x\sec x\tan x\, dx = \int \sec^2 x(\sec^2 x - 1)\sec x\tan x\, dx$
$= \int \sec^4 x\sec x\tan x\, dx - \int \sec^2 x\sec x\tan x\, dx = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$

37. $\int \sec^2 x\tan^2 x\, dx = \int \tan^2 x\sec^2 x\, dx = \frac{1}{3}\tan^3 x + C$

38. $\int \sec^4 x\tan^2 x\, dx = \int \sec^2 x\tan^2 x\sec^2 x\, dx = \int \left(\tan^2 x + 1\right)\tan^2 x\sec^2 x\, dx = \int \tan^4 x\sec^2 x\, dx + \int \tan^2 x\sec^2 x\, dx$
$= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$

39. $\int_{-\pi/3}^{0} 2\sec^3 x\, dx$; $u = \sec x$, $du = \sec x\tan x\, dx$, $dv = \sec^2 x\, dx$, $v = \tan x$;
$\int_{-\pi/3}^{0} 2\sec^3 x\, dx = \left[2\sec x\tan x\right]_{-\pi/3}^{0} - 2\int_{-\pi/3}^{0} \sec x\tan^2 x\, dx = 2\cdot 1\cdot 0 - 2\cdot 2\cdot\sqrt{3} - 2\int_{-\pi/3}^{0} \sec x\,(\sec^2 x - 1)dx$
$= 4\sqrt{3} - 2\int_{-\pi/3}^{0} \sec^3 x\, dx + 2\int_{-\pi/3}^{0} \sec x\, dx$; $2\int_{-\pi/3}^{0} 2\sec^3 x\, dx = 4\sqrt{3} + \left[2\ln|\sec x + \tan x|\right]_{-\pi/3}^{0}$
$2\int_{-\pi/3}^{0} 2\sec^3 x\, dx = 4\sqrt{3} + 2\ln|1+0| - 2\ln|2 - \sqrt{3}| = 4\sqrt{3} - 2\ln\left(2 - \sqrt{3}\right)$
$\int_{-\pi/3}^{0} 2\sec^3 x\, dx = 2\sqrt{3} - \ln\left(2 - \sqrt{3}\right)$

40. $\int e^x\sec^3(e^x)dx$; $u = \sec(e^x)$, $du = \sec(e^x)\tan(e^x)e^x dx$, $dv = \sec^2(e^x)e^x dx$, $v = \tan(e^x)$.
$\int e^x\sec^3(e^x)\, dx = \sec(e^x)\tan(e^x) - \int \sec(e^x)\tan^2(e^x)e^x dx$
$= \sec(e^x)\tan(e^x) - \int \sec(e^x)(\sec^2(e^x) - 1)e^x dx$
$= \sec(e^x)\tan(e^x) - \int \sec^3(e^x)e^x dx + \int \sec(e^x)e^x dx$
$2\int e^x\sec^3(e^x)\, dx = \sec(e^x)\tan(e^x) + \ln\left|\sec(e^x) + \tan(e^x)\right| + C$
$\int e^x\sec^3(e^x)\, dx = \frac{1}{2}\left(\sec(e^x)\tan(e^x) + \ln\left|\sec(e^x) + \tan(e^x)\right|\right) + C$

41. $\int \sec^4\theta\, d\theta = \int (1+\tan^2\theta)\sec^2\theta\, d\theta = \int \sec^2\theta\, d\theta + \int \tan^2\theta \sec^2\theta\, d\theta = \tan\theta + \frac{1}{3}\tan^3\theta + C$
$= \tan\theta + \frac{1}{3}\tan\theta(\sec^2\theta - 1) + C = \frac{1}{3}\tan\theta\sec^2\theta + \frac{2}{3}\tan\theta + C$

42. $\int 3\sec^4(3x)\, dx = \int (1+\tan^2(3x))\sec^2(3x)3dx = \int \sec^2(3x)3dx + \int \tan^2(3x)\sec^2(3x)3dx = \tan(3x) + \frac{1}{3}\tan^3(3x) + C$

43. $\int_{\pi/4}^{\pi/2} \csc^4\theta\, d\theta = \int_{\pi/4}^{\pi/2} (1+\cot^2\theta)\csc^2\theta\, d\theta = \int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} \cot^2\theta\csc^2\theta\, d\theta = \left[-\cot\theta - \frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2}$
$= (0) - \left(-1 - \frac{1}{3}\right) = \frac{4}{3}$

44. $\int \sec^6 x\, dx = \int \sec^4 x\sec^2 x\, dx = \int \left(\tan^2 x + 1\right)^2 \sec^2 x\, dx = \int \left(\tan^4 x + 2\tan^2 x + 1\right)\sec^2 x\, dx$
$= \int \tan^4 x\sec^2 x\, dx + 2\int \tan^2 x\sec^2 x\, dx + \int \sec^2 x\, dx = \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

45. $\int 4\tan^3 x\, dx = 4\int (\sec^2 x - 1)\tan x\, dx = 4\int \sec^2 x\tan x\, dx - 4\int \tan x\, dx = 4\frac{\tan^2 x}{2} - 4\ln|\sec x| + C$
$= 2\tan^2 x - 4\ln|\sec x| + C = 2\tan^2 x - 2\ln|\sec^2 x| + C = 2\tan^2 x - 2\ln(1+\tan^2 x) + C$

46. $\int_{-\pi/4}^{\pi/4} 6\tan^4 x\, dx = 6\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)\tan^2 x\, dx = 6\int_{-\pi/4}^{\pi/4} \sec^2 x\tan^2 x\, dx - 6\int_{-\pi/4}^{\pi/4} \tan^2 x\, dx$
$= 6\int_{-\pi/4}^{\pi/4} \sec^2 x\tan^2 x\, dx - 6\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)dx = \left[6\frac{\tan^3 x}{3}\right]_{-\pi/4}^{\pi/4} - 6\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx + 6\int_{-\pi/4}^{\pi/4} dx$
$= 2(1-(-1)) - [6\tan x]_{-\pi/4}^{\pi/4} + [6x]_{-\pi/4}^{\pi/4} = 4 - 6(1-(-1)) + \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi - 8$

47. $\int \tan^5 x\, dx = \int \tan^4 x\tan x\, dx = \int \left(\sec^2 x - 1\right)^2 \tan x\, dx = \int \left(\sec^4 x - 2\sec^2 x + 1\right)\tan x\, dx$
$= \int \sec^4 x\tan x\, dx - 2\int \sec^2 x\tan x\, dx + \int \tan x\, dx = \int \sec^3 x\sec x\tan x\, dx - 2\int \sec x\sec x\tan x\, dx + \int \tan x\, dx$
$= \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C = \frac{1}{4}\left(\tan^2 x + 1\right)^2 - \left(\tan^2 x + 1\right) + \ln|\sec x| + C = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C$

48. $\int \cot^6 2x\, dx = \int \cot^4 2x\cot^2 2x\, dx = \int \cot^4 2x\,(\csc^2 2x - 1)\, dx = \int \cot^4 2x\csc^2 2x\, dx - \int \cot^4 2x\, dx$
$= \int \cot^4 2x\csc^2 2x\, dx - \int \cot^2 2x\cot^2 2x\, dx = \int \cot^4 2x\csc^2 2x\, dx - \int \cot^2 2x(\csc^2 2x - 1)dx$
$= \int \cot^4 2x\csc^2 2x\, dx - \int \cot^2 2x\csc^2 2x\, dx + \int \cot^2 2x\, dx$
$= \int \cot^4 2x\csc^2 2x\, dx - \int \cot^2 2x\csc^2 2x dx + \int (\csc^2 2x - 1)dx$
$= \int \cot^4 2x\csc^2 2x\, dx - \int \cot^2 2x\csc^2 2x dx + \int \csc^2 2x\, dx - \int dx = -\frac{1}{10}\cot^5 2x + \frac{1}{6}\cot^3 2x - \frac{1}{2}\cot 2x - x + C$

49. $\int_{\pi/6}^{\pi/3} \cot^3 x\, dx = \int_{\pi/6}^{\pi/3} (\csc^2 x - 1)\cot x\, dx = \int_{\pi/6}^{\pi/3} \csc^2 x\cot x\, dx - \int_{\pi/6}^{\pi/3} \cot x\, dx = \left[-\frac{\cot^2 x}{2} + \ln|\csc x|\right]_{\pi/6}^{\pi/3}$
$= -\frac{1}{2}\left(\frac{1}{3} - 3\right) + \left(\ln\frac{2}{\sqrt{3}} - \ln 2\right) = \frac{4}{3} - \ln\sqrt{3}$

50. $\int 8\cot^4 t\, dt = 8\int (\csc^2 t - 1)\cot^2 t\, dt = 8\int \csc^2 t\cot^2 t\, dt - 8\int \cot^2 t\, dt = -\frac{8}{3}\cot^3 t - 8\int (\csc^2 t - 1)dt$
$= -\frac{8}{3}\cot^3 t + 8\cot t + 8t + C$

51. $\int \sin 3x\cos 2x\, dx = \frac{1}{2}\int (\sin x + \sin 5x)\, dx = -\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$

52. $\int \sin 2x\cos 3x\, dx = \frac{1}{2}\int (\sin(-x) + \sin 5x)\, dx = \frac{1}{2}\int (-\sin x + \sin 5x)\, dx = \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$

53. $\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 0 - \cos 6x) \, dx = \frac{1}{2}\int_{-\pi}^{\pi} dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos 6x \, dx = \frac{1}{2}\left[x - \frac{1}{12}\sin 6x\right]_{-\pi}^{\pi} = \frac{\pi}{2} + \frac{\pi}{2} - 0 = \pi$

54. $\int_{0}^{\pi/2} \sin x \cos x \, dx = \frac{1}{2}\int_{0}^{\pi/2} (\sin 0 + \sin 2x) \, dx = \frac{1}{2}\int_{0}^{\pi/2} \sin 2x \, dx = -\frac{1}{4}[\cos 2x]_{0}^{\pi/2} = -\frac{1}{4}(-1 - 1) = \frac{1}{2}$

55. $\int \cos 3x \cos 4x \, dx = \frac{1}{2}\int (\cos(-x) + \cos 7x) \, dx = \frac{1}{2}\int (\cos x + \cos 7x) \, dx = \frac{1}{2}\sin x + \frac{1}{14}\sin 7x + C$

56. $\int_{-\pi/2}^{\pi/2} \cos 7x \cos x \, dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 6x + \cos 8x) \, dx = \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x\right]_{-\pi/2}^{\pi/2} = 0$

57. $\int \sin^2 \theta \cos 3\theta \, d\theta = \int \frac{1 - \cos 2\theta}{2} \cos 3\theta \, d\theta = \frac{1}{2}\int \cos 3\theta \, d\theta - \frac{1}{2}\int \cos 2\theta \cos 3\theta \, d\theta$
$= \frac{1}{2}\int \cos 3\theta \, d\theta - \frac{1}{2}\int \frac{1}{2}(\cos(2-3)\theta + \cos(2+3)\theta) \, d\theta = \frac{1}{2}\int \cos 3\theta \, d\theta - \frac{1}{4}\int (\cos(-\theta) + \cos 5\theta) \, d\theta$
$= \frac{1}{2}\int \cos 3\theta \, d\theta - \frac{1}{4}\int \cos \theta \, d\theta - \frac{1}{4}\int \cos 5\theta \, d\theta = \frac{1}{6}\sin 3\theta - \frac{1}{4}\sin \theta - \frac{1}{20}\sin 5\theta + C$

58. $\int \cos^2 2\theta \sin \theta \, d\theta = \int (2\cos^2 \theta - 1)^2 \sin \theta \, d\theta = \int (4\cos^4 \theta - 4\cos^2 \theta + 1) \sin \theta \, d\theta$
$= \int 4\cos^4 \theta \sin \theta \, d\theta - \int 4\cos^2 \theta \sin \theta \, d\theta + \int \sin \theta \, d\theta = -\frac{4}{5}\cos^5 \theta + \frac{4}{3}\cos^3 \theta - \cos \theta + C$

59. $\int \cos^3 \theta \sin 2\theta \, d\theta = \int \cos^3 \theta \, (2\sin \theta \cos \theta) \, d\theta = 2\int \cos^4 \theta \sin \theta \, d\theta = -\frac{2}{5}\cos^5 \theta + C$

60. $\int \sin^3 \theta \cos 2\theta \, d\theta = \int \sin^2 \theta \cos 2\theta \sin \theta \, d\theta = \int \left(1 - \cos^2 \theta\right)(2\cos^2 \theta - 1)\sin \theta \, d\theta$
$= \int (-2\cos^4 \theta + 3\cos^2 \theta - 1)\sin \theta \, d\theta = -2\int \cos^4 \theta \sin \theta \, d\theta + 3\int \cos^2 \theta \sin \theta \, d\theta - \int \sin \theta \, d\theta$
$= \frac{2}{5}\cos^5 \theta - \cos^3 \theta + \cos \theta + C$

61. $\int \sin \theta \cos \theta \cos 3\theta \, d\theta = \frac{1}{2}\int 2\sin \theta \cos \theta \cos 3\theta \, d\theta = \frac{1}{2}\int \sin 2\theta \cos 3\theta \, d\theta = \frac{1}{2}\int \frac{1}{2}(\sin(2-3)\theta + \sin(2+3)\theta) \, d\theta$
$= \frac{1}{4}\int (\sin(-\theta) + \sin 5\theta) \, d\theta = \frac{1}{4}\int (-\sin \theta + \sin 5\theta) \, d\theta = \frac{1}{4}\cos \theta - \frac{1}{20}\cos 5\theta + C$

62. $\int \sin \theta \sin 2\theta \sin 3\theta \, d\theta = \int \frac{1}{2}\left(\cos(1-2)\theta - \cos(1+2)\theta\right) \sin 3\theta \, d\theta = \frac{1}{2}\int \left(\cos(-\theta) - \cos 3\theta\right) \sin 3\theta \, d\theta$
$= \frac{1}{2}\int \sin 3\theta \cos \theta \, d\theta - \frac{1}{2}\int \sin 3\theta \cos 3\theta \, d\theta = \frac{1}{2}\int \frac{1}{2}(\sin(3-1)\theta + \sin(3+1)\theta) d\theta - \frac{1}{4}\int 2\sin 3\theta \cos 3\theta \, d\theta$
$= \frac{1}{4}\int (\sin 2\theta + \sin 4\theta) d\theta - \frac{1}{4}\int \sin 6\theta \, d\theta = \frac{1}{4}\int (\sin 2\theta + \sin 4\theta) d\theta - \frac{1}{4}\int \sin 6\theta \, d\theta$
$= -\frac{1}{8}\cos 2\theta - \frac{1}{16}\cos 4\theta + \frac{1}{24}\cos 6\theta + C$

63. $\int \frac{\sec^3 x}{\tan x} \, dx = \int \frac{\sec^2 x \sec x}{\tan x} \, dx = \int \frac{\left(\tan^2 x + 1\right)\sec x}{\tan x} \, dx = \int \frac{\tan^2 x \sec x}{\tan x} \, dx + \int \frac{\sec x}{\tan x} \, dx = \int \tan x \sec x \, dx + \int \csc x \, dx$
$= \sec x - \ln|\csc x + \cot x| + C$

64. $\int \frac{\sin^3 x}{\cos^4 x} \, dx = \int \frac{\sin^2 x \sin x}{\cos^4 x} \, dx = \int \frac{\left(1 - \cos^2 x\right) \sin x}{\cos^4 x} \, dx = \int \frac{\sin x}{\cos^4 x} \, dx - \int \frac{\cos^2 x \sin x}{\cos^4 x} \, dx = \int \sec^3 x \tan x \, dx - \int \sec x \tan x \, dx$
$= \int \sec^2 x \sec x \tan x \, dx - \int \sec x \tan x \, dx = \frac{1}{3}\sec^3 x - \sec x + C$

65. $\int \frac{\tan^2 x}{\csc x} \, dx = \int \frac{\sin^2 x}{\cos^2 x} \sin x \, dx = \int \frac{\left(1 - \cos^2 x\right)}{\cos^2 x} \sin x \, dx = \int \frac{1}{\cos^2 x} \sin x \, dx - \int \frac{\cos^2 x}{\cos^2 x} \sin x \, dx = \int \sec x \tan x \, dx - \int \sin x \, dx$
$= \sec x + \cos x + C$

66. $\int \frac{\cot x}{\cos^2 x}\,dx = \int \frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x}\,dx = \int \frac{2}{2\sin x\cos x}\,dx = \int \frac{2}{\sin 2x}\,dx = \int \csc 2x\,2dx = -\ln|\csc 2x + \cot 2x| + C$

67. $\int x\sin^2 x\,dx = \int x\,\frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx\ \left[u = x,\ du = dx,\ dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x\right]$
$= \frac{1}{4}x^2 - \frac{1}{2}\left[\frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx\right] = \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$

68. $\int x\cos^3 x\,dx = \int x\cos^2 x\cos x\,dx = \int x(1-\sin^2 x)\cos x\,dx = \int x\cos x\,dx - \int x\sin^2 x\cos x\,dx;$
$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x;$
$\left[u = x,\ du = dx,\ dv = \cos x\,dx,\ v = \sin x\right]$
$\int x\sin^2 x\cos x\,dx = \frac{1}{3}x\sin^3 x - \int \frac{1}{3}\sin^3 x\,dx;$
$\left[u = x,\ du = dx,\ dv = \sin^2 x\cos x\,dx,\ v = \frac{1}{3}\sin^3 x\right]$
$= \frac{1}{3}x\sin^3 x - \frac{1}{3}\int (1-\cos^2 x)\sin x\,dx = \frac{1}{3}x\sin^3 x - \frac{1}{3}\int \sin x\,dx + \frac{1}{3}\int \cos^2 x\sin x\,dx = \frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x;$
$\Rightarrow \int x\cos x\,dx - \int x\sin^2 x\cos x\,dx = (x\sin x + \cos x) - \left(\frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x\right) + C$
$= x\sin x - \frac{1}{3}x\sin^3 x + \frac{2}{3}\cos x + \frac{1}{9}\cos^3 x + C$

69. $y = \ln(\sec x);\ y' = \frac{\sec x\tan x}{\sec x} = \tan x; (y')^2 = \tan^2 x;\ \int_0^{\pi/4}\sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4}|\sec x|\,dx = [\ln|\sec x + \tan x|]_0^{\pi/4}$
$= \ln\left(\sqrt{2}+1\right) - \ln(0+1) = \ln\left(\sqrt{2}+1\right)$

70. $M = \int_{-\pi/4}^{\pi/4}\sec x\,dx = [\ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln|\sqrt{2}-1| = \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$
$\overline{y} = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}\int_{-\pi/4}^{\pi/4}\frac{\sec^2 x}{2}\,dx = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}[\tan x]_{-\pi/4}^{\pi/4} = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}(1-(-1)) = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}$
$\Rightarrow (\overline{x},\overline{y}) = \left(0, \left(\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^{-1}\right)$

71. $V = \pi\int_0^{\pi}\sin^2 x\,dx = \pi\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx = \frac{\pi}{2}\int_0^{\pi}dx - \frac{\pi}{2}\int_0^{\pi}\cos 2x\,dx = \frac{\pi}{2}[x]_0^{\pi} - \frac{\pi}{4}[\sin 2x]_0^{\pi} = \frac{\pi}{2}(\pi-0) - \frac{\pi}{4}(0-0) = \frac{\pi^2}{2}$

72. $A = \int_0^{\pi}\sqrt{1+\cos 4x}\,dx = \int_0^{\pi}\sqrt{2}|\cos 2x|dx = \sqrt{2}\int_0^{\pi/4}\cos 2x\,dx - \sqrt{2}\int_{\pi/4}^{3\pi/4}\cos 2x\,dx + \sqrt{2}\int_{3\pi/4}^{\pi}\cos 2x\,dx$
$= \frac{\sqrt{2}}{2}[\sin 2x]_0^{\pi/4} - \frac{\sqrt{2}}{2}[\sin 2x]_{\pi/4}^{3\pi/4} + \frac{\sqrt{2}}{2}[\sin 2x]_{3\pi/4}^{\pi} = \frac{\sqrt{2}}{2}(1-0) - \frac{\sqrt{2}}{2}(-1-1) + \frac{\sqrt{2}}{2}(0+1) = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

73. $M = \int_0^{2\pi}(x+\cos x)dx = \left[\frac{1}{2}x^2 + \sin x\right]_0^{2\pi} = \left(\frac{1}{2}(2\pi)^2 + \sin(2\pi)\right) - \left(\frac{1}{2}(0)^2 + \sin(0)\right) = 2\pi^2;$
$\overline{x} = \frac{1}{2\pi^2}\int_0^{2\pi}x(x+\cos x)dx = \frac{1}{2\pi^2}\int_0^{2\pi}(x^2 + x\cos x)dx = \frac{1}{2\pi^2}\int_0^{2\pi}x^2dx + \frac{1}{2\pi^2}\int_0^{2\pi}x\cos x\,dx$
$\left[u = x,\ du = dx,\ dv = \cos x\,dx,\ v = \sin x\right]$
$= \frac{1}{6\pi^2}\left[x^3\right]_0^{2\pi} + \frac{1}{2\pi^2}\left(\left[x\sin x\right]_0^{2\pi} - \int_0^{2\pi}\sin x\,dx\right) = \frac{1}{6\pi^2}(8\pi^3 - 0) + \frac{1}{2\pi^2}\left(2\pi\sin 2\pi - 0 - \int_0^{2\pi}\sin x\,dx\right)$
$= \frac{4\pi}{3} + \frac{1}{2\pi^2}\left[\cos x\right]_0^{2\pi} = \frac{4\pi}{3} + \frac{1}{2\pi^2}(\cos 2\pi - \cos 0) = \frac{4\pi}{3} + 0 = \frac{4\pi}{3};\ \overline{y} = \frac{1}{2\pi^2}\int_0^{2\pi}\frac{1}{2}(x+\cos x)^2dx$
$= \frac{1}{4\pi^2}\int_0^{2\pi}(x^2 + 2x\cos x + \cos^2 x)dx = \frac{1}{4\pi^2}\int_0^{2\pi}x^2dx + \frac{1}{2\pi^2}\int_0^{2\pi}x\cos x\,dx + \frac{1}{4\pi^2}\int_0^{2\pi}\cos^2 x\,dx$

$= \frac{1}{12\pi^2}\left[x^3\right]_0^{2\pi} + \frac{1}{2\pi^2}\left[x\sin x + \cos x\right]_0^{2\pi} + \frac{1}{4\pi^2}\int_0^{2\pi}\frac{\cos 2x + 1}{2}\,dx = \frac{2\pi}{3} + 0 + \frac{1}{8\pi^2}\int_0^{2\pi}\cos 2x\,dx + \frac{1}{8\pi^2}\int_0^{2\pi}dx$

$= \frac{2\pi}{3} + \frac{1}{16\pi^2}\left[\sin 2x\right]_0^{2\pi} + \frac{1}{8\pi^2}\left[x\right]_0^{2\pi} = \frac{2\pi}{3} + 0 + \frac{1}{4\pi} = \frac{8\pi^2+3}{12\pi} \Rightarrow$ The centroid is $\left(\frac{4\pi}{3}, \frac{8\pi^2+3}{12\pi}\right)$.

74. $V = \int_0^{\pi/3}\pi(\sin x + \sec x)^2\,dx = \pi\int_0^{\pi/3}(\sin^2 x + 2\sin x\sec x + \sec^2 x)\,dx$

$= \pi\int_0^{\pi/3}\sin^2 x\,dx + \pi\int_0^{\pi/3}2\tan x\,dx + \pi\int_0^{\pi/3}\sec^2 x\,dx = \pi\int_0^{\pi/3}\frac{1-\cos 2x}{2}\,dx + 2\pi\left[\ln|\sec x|\right]_0^{\pi/3} + \pi\left[\tan x\right]_0^{\pi/3}$

$= \frac{\pi}{2}\int_0^{\pi/3}dx - \frac{\pi}{2}\int_0^{\pi/3}\cos 2x\,dx + 2\pi\left(\ln|\sec\frac{\pi}{3}| - \ln|\sec 0|\right) + \pi\left(\tan\frac{\pi}{3} - \tan 0\right)$

$= \frac{\pi}{2}\left[x\right]_0^{\pi/3} - \frac{\pi}{4}\left[\sin 2x\right]_0^{\pi/3} + 2\pi\ln 2 + \pi\sqrt{3} = \frac{\pi}{2}\left(\frac{\pi}{3} - 0\right) - \frac{\pi}{4}\left(\sin 2\left(\frac{\pi}{3}\right) - \sin 2(0)\right) + 2\pi\ln 2 + \pi\sqrt{3}$

$= \frac{\pi^2}{6} - \frac{\pi\sqrt{3}}{8} + 2\pi\ln 2 + \pi\sqrt{3} = \frac{\pi\left(4\pi + 21\sqrt{3} - 48\ln 2\right)}{24}$