CHAPTER 8 TECHNIQUES OF INTEGRATION

8.1 INTEGRATION BY PARTS

1. $u = x,\ du = dx;\ dv = \sin\frac{x}{2}\,dx,\ v = -2\cos\frac{x}{2}$;
$\int x \sin\frac{x}{2}\,dx = -2x\cos\frac{x}{2} - \int\left(-2\cos\frac{x}{2}\right)dx = -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C$

2. $u = \theta,\ du = d\theta;\ dv = \cos\pi\theta\,d\theta,\ v = \frac{1}{\pi}\sin\pi\theta$;
$\int \theta\cos\pi\theta\,d\theta = \frac{\theta}{\pi}\sin\pi\theta - \int\frac{1}{\pi}\sin\pi\theta\,d\theta = \frac{\theta}{\pi}\sin\pi\theta + \frac{1}{\pi^2}\cos\pi\theta + C$

3.
$$\begin{array}{lcl} & & \cos t \\ t^2 & \xrightarrow{(+)} & \sin t \\ 2t & \xrightarrow{(-)} & -\cos t \\ 2 & \xrightarrow{(+)} & -\sin t \\ 0 & & \end{array}$$
$\int t^2\cos t\,dt = t^2\sin t + 2t\cos t - 2\sin t + C$

4.
$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$
$\int x^2\sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$

5. $u = \ln x,\ du = \frac{dx}{x};\ dv = x\,dx,\ v = \frac{x^2}{2}$;
$\int_1^2 x\ln x\,dx = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2\frac{x^2}{2}\,\frac{dx}{x} = 2\ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2\ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$

6. $u = \ln x,\ du = \frac{dx}{x};\ dv = x^3\,dx,\ v = \frac{x^4}{4}$;
$\int_1^e x^3\ln x\,dx = \left[\frac{x^4}{4}\ln x\right]_1^e - \int_1^e\frac{x^4}{4}\,\frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4+1}{16}$

7. $u = x,\ du = dx;\ dv = e^x dx,\ v = e^x$;
$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C$

8. $u = x,\ du = dx;\ dv = e^{3x}dx,\ v = \frac{1}{3}e^{3x}$;
$\int x e^{3x}dx = \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x}dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C$

9.

		e^{-x}
x^2	$\xrightarrow{(+)}$	$-e^{-x}$
$2x$	$\xrightarrow{(-)}$	e^{-x}
2	$\xrightarrow{(+)}$	$-e^{-x}$
0		

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} - 2x\,e^{-x} - 2\,e^{-x} + C$$

10.

		e^{2x}
$x^2 - 2x + 1$	$\xrightarrow{(+)}$	$\frac{1}{2}e^{2x}$
$2x - 2$	$\xrightarrow{(-)}$	$\frac{1}{4}e^{2x}$
2	$\xrightarrow{(+)}$	$\frac{1}{8}e^{2x}$
0		

$$\int (x^2 - 2x + 1)e^{2x}\,dx = \tfrac{1}{2}(x^2 - 2x + 1)e^{2x} - \tfrac{1}{4}(2x - 2)e^{2x} + \tfrac{1}{4}e^{2x} + C$$
$$= \left(\tfrac{1}{2}x^2 - \tfrac{3}{2}x + \tfrac{5}{4}\right)e^{2x} + C$$

11. $u = \tan^{-1} y$, $du = \frac{dy}{1+y^2}$; $dv = dy$, $v = y$;
$$\int \tan^{-1} y\,dy = y\tan^{-1} y - \int \frac{y\,dy}{(1+y^2)} = y\tan^{-1} y - \tfrac{1}{2}\ln(1+y^2) + C = y\tan^{-1} y - \ln\sqrt{1+y^2} + C$$

12. $u = \sin^{-1} y$, $du = \frac{dy}{\sqrt{1-y^2}}$; $dv = dy$, $v = y$;
$$\int \sin^{-1} y\,dy = y\sin^{-1} y - \int \frac{y\,dy}{\sqrt{1-y^2}} = y\sin^{-1} y + \sqrt{1-y^2} + C$$

13. $u = x$, $du = dx$; $dv = \sec^2 x\,dx$, $v = \tan x$;
$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C$$

14. $\int 4x\sec^2 2x\,dx$; $[y = 2x] \rightarrow \int y\sec^2 y\,dy = y\tan y - \int \tan y\,dy = y\tan y - \ln|\sec y| + C$
$= 2x\tan 2x - \ln|\sec 2x| + C$

15.

		e^x
x^3	$\xrightarrow{(+)}$	e^x
$3x^2$	$\xrightarrow{(-)}$	e^x
$6x$	$\xrightarrow{(+)}$	e^x
6	$\xrightarrow{(-)}$	e^x
0		

$$\int x^3 e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = (x^3 - 3x^2 + 6x - 6)\,e^x + C$$

16.

$$\begin{array}{lcl} & & e^{-p} \\ p^4 & \xrightarrow{(+)} & -e^{-p} \\ 4p^3 & \xrightarrow{(-)} & e^{-p} \\ 12p^2 & \xrightarrow{(+)} & -e^{-p} \\ 24p & \xrightarrow{(-)} & e^{-p} \\ 24 & \xrightarrow{(+)} & -e^{-p} \\ 0 & & \end{array}$$

$$\int p^4 e^{-p}\,dp = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24pe^{-p} - 24e^{-p} + C$$
$$= \left(-p^4 - 4p^3 - 12p^2 - 24p - 24\right) e^{-p} + C$$

17.

$$\begin{array}{lcl} & & e^{x} \\ x^2 - 5x & \xrightarrow{(+)} & e^{x} \\ 2x - 5 & \xrightarrow{(-)} & e^{x} \\ 2 & \xrightarrow{(+)} & e^{x} \\ 0 & & \end{array}$$

$$\int \left(x^2 - 5x\right) e^x\,dx = \left(x^2 - 5x\right) e^x - (2x - 5)e^x + 2e^x + C = x^2 e^x - 7xe^x + 7e^x + C$$
$$= \left(x^2 - 7x + 7\right) e^x + C$$

18.

$$\begin{array}{lcl} & & e^{r} \\ r^2 + r + 1 & \xrightarrow{(+)} & e^{r} \\ 2r + 1 & \xrightarrow{(-)} & e^{r} \\ 2 & \xrightarrow{(+)} & e^{r} \\ 0 & & \end{array}$$

$$\int \left(r^2 + r + 1\right) e^r\,dr = \left(r^2 + r + 1\right) e^r - (2r + 1)\,e^r + 2e^r + C$$
$$= \left[\left(r^2 + r + 1\right) - (2r + 1) + 2\right] e^r + C = \left(r^2 - r + 2\right) e^r + C$$

19.

$$\begin{array}{lcl} & & e^{x} \\ x^5 & \xrightarrow{(+)} & e^{x} \\ 5x^4 & \xrightarrow{(-)} & e^{x} \\ 20x^3 & \xrightarrow{(+)} & e^{x} \\ 60x^2 & \xrightarrow{(-)} & e^{x} \\ 120x & \xrightarrow{(+)} & e^{x} \\ 120 & \xrightarrow{(-)} & e^{x} \\ 0 & & \end{array}$$

$$\int x^5 e^x\,dx = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120xe^x - 120e^x + C$$
$$= \left(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120\right) e^x + C$$

20.

		e^{4t}
t^2	$\xrightarrow{(+)}$	$\frac{1}{4}e^{4t}$
$2t$	$\xrightarrow{(-)}$	$\frac{1}{16}e^{4t}$
2	$\xrightarrow{(+)}$	$\frac{1}{64}e^{4t}$
0		

$$\int t^2 e^{4t}\,dt = \frac{t^2}{4}e^{4t} - \frac{2t}{16}e^{4t} + \frac{2}{64}e^{4t} + C = \frac{t^2}{4}e^{4t} - \frac{t}{8}e^{4t} + \frac{1}{32}e^{4t} + C$$
$$= \left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32}\right)e^{4t} + C$$

21. $I = \int e^{\theta}\sin\theta\,d\theta$; $[u = \sin\theta,\ du = \cos\theta\,d\theta;\ dv = e^{\theta}\,d\theta,\ v = e^{\theta}] \Rightarrow I = e^{\theta}\sin\theta - \int e^{\theta}\cos\theta\,d\theta$; $[u = \cos\theta,\ du = -\sin\theta\,d\theta;\ dv = e^{\theta}\,d\theta,\ v = e^{\theta}] \Rightarrow I = e^{\theta}\sin\theta - \left(e^{\theta}\cos\theta + \int e^{\theta}\sin\theta\,d\theta\right)$ $= e^{\theta}\sin\theta - e^{\theta}\cos\theta - I + C' \Rightarrow 2I = (e^{\theta}\sin\theta - e^{\theta}\cos\theta) + C' \Rightarrow I = \frac{1}{2}(e^{\theta}\sin\theta - e^{\theta}\cos\theta) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

22. $I = \int e^{-y}\cos y\,dy$; $[u = \cos y,\ du = -\sin y\,dy;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$ $\Rightarrow I = -e^{-y}\cos y - \int(-e^{-y})(-\sin y)\,dy = -e^{-y}\cos y - \int e^{-y}\sin y\,dy$; $[u = \sin y,\ du = \cos y\,dy;$ $dv = e^{-y}\,dy,\ v = -e^{-y}] \Rightarrow I = -e^{-y}\cos y - \left(-e^{-y}\sin y - \int(-e^{y})\cos y\,dy\right) = -e^{-y}\cos y + e^{-y}\sin y - I + C'$ $\Rightarrow 2I = e^{-y}(\sin y - \cos y) + C' \Rightarrow I = \frac{1}{2}(e^{-y}\sin y - e^{-y}\cos y) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

23. $I = \int e^{2x}\cos 3x\,dx$; $\left[u = \cos 3x;\ du = -3\sin 3x\,dx,\ dv = e^{2x}\,dx;\ v = \frac{1}{2}e^{2x}\right]$ $\Rightarrow I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx$; $\left[u = \sin 3x,\ du = 3\cos 3x,\ dv = e^{2x}\,dx;\ v = \frac{1}{2}e^{2x}\right]$ $\Rightarrow I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left(\frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx\right) = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}I + C'$ $\Rightarrow \frac{13}{4}I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C' \Rightarrow \frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$, where $C = \frac{4}{13}C'$

24. $\int e^{-2x}\sin 2x\,dx$; $[y = 2x] \rightarrow \frac{1}{2}\int e^{-y}\sin y\,dy = I$; $[u = \sin y,\ du = \cos y\,dy;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$ $\Rightarrow I = \frac{1}{2}\left(-e^{-y}\sin y + \int e^{-y}\cos y\,dy\right)$ $[u = \cos y,\ du = -\sin y;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$ $\Rightarrow I = -\frac{1}{2}e^{-y}\sin y + \frac{1}{2}\left(-e^{-y}\cos y - \int(-e^{-y})(-\sin y)\,dy\right) = -\frac{1}{2}e^{-y}(\sin y + \cos y) - I + C'$ $\Rightarrow 2I = -\frac{1}{2}e^{-y}(\sin y + \cos y) + C' \Rightarrow I = -\frac{1}{4}e^{-y}(\sin y + \cos y) + C = -\frac{e^{-2x}}{4}(\sin 2x + \cos 2x) + C$, where $C = \frac{C'}{2}$

25. $\int e^{\sqrt{3s+9}}\,ds$; $\begin{bmatrix} 3s + 9 = x^2 \\ ds = \frac{2}{3}x\,dx \end{bmatrix} \rightarrow \int e^x \cdot \frac{2}{3}x\,dx = \frac{2}{3}\int xe^x\,dx$; $[u = x,\ du = dx;\ dv = e^x\,dx,\ v = e^x]$; $\frac{2}{3}\int xe^x\,dx = \frac{2}{3}\left(xe^x - \int e^x\,dx\right) = \frac{2}{3}(xe^x - e^x) + C = \frac{2}{3}\left(\sqrt{3s+9}\,e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$

26. $u = x,\ du = dx;\ dv = \sqrt{1-x}\,dx,\ v = -\frac{2}{3}\sqrt{(1-x)^3}$; $\int_0^1 x\sqrt{1-x}\,dx = \left[-\frac{2}{3}\sqrt{(1-x)^3}\,x\right]_0^1 + \frac{2}{3}\int_0^1 \sqrt{(1-x)^3}\,dx = \frac{2}{3}\left[-\frac{2}{5}(1-x)^{5/2}\right]_0^1 = \frac{4}{15}$

27. $u = x,\ du = dx;\ dv = \tan^2 x\,dx,\ v = \int \tan^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \frac{dx}{\cos^2 x} - \int dx$ $= \tan x - x$; $\int_0^{\pi/3} x\tan^2 x\,dx = [x(\tan x - x)]_0^{\pi/3} - \int_0^{\pi/3}(\tan x - x)\,dx = \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \left[\ln|\cos x| + \frac{x^2}{2}\right]_0^{\pi/3}$ $= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \ln\frac{1}{2} + \frac{\pi^2}{18} = \frac{\pi\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18}$

28. $u = \ln(x + x^2)$, $du = \frac{(2x+1)\,dx}{x+x^2}$; $dv = dx$, $v = x$; $\int \ln(x + x^2)\,dx = x\ln(x + x^2) - \int \frac{2x+1}{x(x+1)} \cdot x\,dx$
$= x\ln(x + x^2) - \int \frac{(2x+1)\,dx}{x+1} = x\ln(x + x^2) - \int \frac{2(x+1)-1}{x+1}\,dx = x\ln(x + x^2) - 2x + \ln|x + 1| + C$

29. $\int \sin(\ln x)\,dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ dx = e^u\,du \end{bmatrix} \rightarrow \int (\sin u)\,e^u\,du$. From Exercise 21, $\int (\sin u)\,e^u\,du = e^u\left(\frac{\sin u - \cos u}{2}\right) + C$
$= \frac{1}{2}\left[-x\cos(\ln x) + x\sin(\ln x)\right] + C$

30. $\int z(\ln z)^2\,dz$; $\begin{bmatrix} u = \ln z \\ du = \frac{1}{z}\,dz \\ dz = e^u\,du \end{bmatrix} \rightarrow \int e^u \cdot u^2 \cdot e^u\,du = \int e^{2u} \cdot u^2\,du$;

		e^{2u}
u^2	$\xrightarrow{(+)}$	$\frac{1}{2}e^{2u}$
$2u$	$\xrightarrow{(-)}$	$\frac{1}{4}e^{2u}$
2	$\xrightarrow{(+)}$	$\frac{1}{8}e^{2u}$
0		

$\int u^2 e^{2u}\,du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}\left[2u^2 - 2u + 1\right] + C$
$= \frac{z^2}{4}\left[2(\ln z)^2 - 2\ln z + 1\right] + C$

31. $\int x\sec x^2\,dx$ $\left[\text{Let } u = x^2,\ du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx\right] \rightarrow \int x\sec x^2\,dx = \frac{1}{2}\int \sec u\,du = \frac{1}{2}\ln|\sec u + \tan u| + C$
$= \frac{1}{2}\ln|\sec x^2 + \tan x^2| + C$

32. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ $\left[\text{Let } u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2du = \frac{1}{\sqrt{x}}\,dx\right] \rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$

33. $\int x(\ln x)^2\,dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ dx = e^u\,du \end{bmatrix} \rightarrow \int e^u \cdot u^2 \cdot e^u\,du = \int e^{2u} \cdot u^2\,du$;

		e^{2u}
u^2	$\xrightarrow{(+)}$	$\frac{1}{2}e^{2u}$
$2u$	$\xrightarrow{(-)}$	$\frac{1}{4}e^{2u}$
2	$\xrightarrow{(+)}$	$\frac{1}{8}e^{2u}$
0		

$\int u^2 e^{2u}\,du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}\left[2u^2 - 2u + 1\right] + C$
$= \frac{x^2}{4}\left[2(\ln x)^2 - 2\ln x + 1\right] + C = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C$

34. $\int \frac{1}{x(\ln x)^2}\,dx$ $\left[\text{Let } u = \ln x,\ du = \frac{1}{x}\,dx\right] \rightarrow \int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$

35. $u = \ln x$, $du = \frac{1}{x}\,dx$; $dv = \frac{1}{x^2}\,dx$, $v = -\frac{1}{x}$;
$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C$

36. $\int \frac{(\ln x)^3}{x}\,dx$ $\left[\text{Let } u = \ln x,\ du = \frac{1}{x}\,dx\right] \rightarrow \int \frac{(\ln x)^3}{x}\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}(\ln x)^4 + C$

37. $\int x^3 e^{x^4}\,dx$ $\left[\text{Let } u = x^4,\ du = 4x^3\,dx \Rightarrow \frac{1}{4}du = x^3\,dx\right] \to \int x^3 e^{x^4}\,dx = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C = \frac{1}{4}e^{x^4} + C$

38. $u = x^3,\ du = 3x^2\,dx;\ dv = x^2 e^{x^3}\,dx,\ v = \frac{1}{3}e^{x^3}$;
$\int x^5 e^{x^3}\,dx = \int x^3 e^{x^3}\,x^2 dx = \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}\int e^{x^3}\,3x^2 dx = \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3} + C$

39. $u = x^2,\ du = 2x\,dx;\ dv = \sqrt{x^2+1}\,x\,dx,\ v = \frac{1}{3}(x^2+1)^{3/2}$;
$\int x^3\sqrt{x^2+1}\,dx = \frac{1}{3}x^2(x^2+1)^{3/2} - \frac{1}{3}\int (x^2+1)^{3/2} 2x\,dx = \frac{1}{3}x^2(x^2+1)^{3/2} - \frac{2}{15}(x^2+1)^{5/2} + C$

40. $\int x^2 \sin x^3\,dx$ $\left[\text{Let } u = x^3,\ du = 3x^2\,dx \Rightarrow \frac{1}{3}du = x^2\,dx\right] \to \int x^2 \sin x^3\,dx = \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C$
$= -\frac{1}{3}\cos x^3 + C$

41. $u = \sin 3x,\ du = 3\cos 3x\,dx;\ dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x$;
$\int \sin 3x \cos 2x\,dx = \frac{1}{2}\sin 3x \sin 2x - \frac{3}{2}\int \cos 3x \sin 2x\,dx$
$u = \cos 3x,\ du = -3\sin 3x\,dx;\ dv = \sin 2x\,dx,\ v = -\frac{1}{2}\cos 2x$;
$\int \sin 3x \cos 2x\,dx = \frac{1}{2}\sin 3x \sin 2x - \frac{3}{2}\left[-\frac{1}{2}\cos 3x \cos 2x - \frac{3}{2}\int \sin 3x \cos 2x\,dx\right]$
$= \frac{1}{2}\sin 3x \sin 2x + \frac{3}{4}\cos 3x \cos 2x + \frac{9}{4}\int \sin 3x \cos 2x\,dx \Rightarrow -\frac{5}{4}\int \sin 3x \cos 2x\,dx = \frac{1}{2}\sin 3x \sin 2x + \frac{3}{4}\cos 3x \cos 2x$
$\Rightarrow \int \sin 3x \cos 2x\,dx = -\frac{2}{5}\sin 3x \sin 2x - \frac{3}{5}\cos 3x \cos 2x + C$

42. $u = \sin 2x,\ du = 2\cos 2x\,dx;\ dv = \cos 4x\,dx,\ v = \frac{1}{4}\sin 4x$;
$\int \sin 2x \cos 4x\,dx = \frac{1}{4}\sin 2x \sin 4x - \frac{1}{2}\int \cos 2x \sin 4x\,dx$
$u = \cos 2x,\ du = -2\sin 2x\,dx;\ dv = \sin 4x\,dx,\ v = -\frac{1}{4}\cos 4x$;
$\int \sin 2x \cos 4x\,dx = \frac{1}{4}\sin 2x \sin 4x - \frac{1}{2}\left[-\frac{1}{4}\cos 2x \cos 4x - \frac{1}{2}\int \sin 2x \cos 4x\,dx\right]$
$= \frac{1}{4}\sin 2x \sin 4x + \frac{1}{8}\cos 2x \cos 4x + \frac{1}{4}\int \sin 2x \cos 4x\,dx \Rightarrow \frac{3}{4}\int \sin 2x \cos 4x\,dx = \frac{1}{4}\sin 2x \sin 4x + \frac{1}{8}\cos 2x \cos 4x$
$\Rightarrow \int \sin 2x \cos 4x\,dx = \frac{1}{3}\sin 2x \sin 4x + \frac{1}{6}\cos 2x \cos 4x + C$

43. $\int e^x \sin e^x\,dx$ $\left[\text{Let } u = e^x,\ du = e^x\,dx\right] \to \int e^x \sin e^x\,dx = \int \sin u\,du = -\cos u + C = -\cos e^x + C$

44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ $\left[\text{Let } u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2du = \frac{1}{\sqrt{x}}\,dx\right] \to \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$

45. $\int \cos\sqrt{x}\,dx;$ $\begin{bmatrix} y = \sqrt{x} \\ dy = \frac{1}{2\sqrt{x}}\,dx \\ dx = 2y\,dy \end{bmatrix} \to \int \cos y\, 2y\,dy = \int 2y \cos y\,dy;$
$u = 2y,\ du = 2\,dy;\ dv = \cos y\,dy,\ v = \sin y$;
$\int 2y \cos y\,dy = 2y \sin y - \int 2\sin y\,dy = 2y \sin y + 2\cos y + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$

46. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$; $\begin{bmatrix} y = \sqrt{x} \\ dy = \frac{1}{2\sqrt{x}}\, dx \\ dx = 2y\, dy \end{bmatrix} \rightarrow \int y\, e^y\, 2y\, dy = \int 2y^2\, e^y\, dy$;

$$\begin{array}{lcl} & & e^y \\ 2y^2 & \xrightarrow{(+)} & e^y \\ 4y & \xrightarrow{(-)} & e^y \\ 4 & \xrightarrow{(+)} & e^y \\ 0 & & \end{array}$$

$$\int 2y^2\, e^y\, dy = 2y^2 e^y - 4y\, e^y + 4\, e^y + C = 2x\, e^{\sqrt{x}} - 4\sqrt{x}\, e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$$

47.
$$\begin{array}{lcl} & & \sin 2\theta \\ \theta^2 & \xrightarrow{(+)} & -\frac{1}{2}\cos 2\theta \\ 2\theta & \xrightarrow{(-)} & -\frac{1}{4}\sin 2\theta \\ 2 & \xrightarrow{(+)} & \frac{1}{8}\cos 2\theta \\ 0 & & \end{array}$$

$$\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta = \left[-\frac{\theta^2}{2}\cos 2\theta + \frac{\theta}{2}\sin 2\theta + \frac{1}{4}\cos 2\theta\right]_0^{\pi/2}$$
$$= \left[-\frac{\pi^2}{8}\cdot(-1) + \frac{\pi}{4}\cdot 0 + \frac{1}{4}\cdot(-1)\right] - \left[0 + 0 + \frac{1}{4}\cdot 1\right] = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2 - 4}{8}$$

48.
$$\begin{array}{lcl} & & \cos 2x \\ x^3 & \xrightarrow{(+)} & \frac{1}{2}\sin 2x \\ 3x^2 & \xrightarrow{(-)} & -\frac{1}{4}\cos 2x \\ 6x & \xrightarrow{(+)} & -\frac{1}{8}\sin 2x \\ 6 & \xrightarrow{(-)} & \frac{1}{16}\cos 2x \\ 0 & & \end{array}$$

$$\int_0^{\pi/2} x^3 \cos 2x\, dx = \left[\frac{x^3}{2}\sin 2x + \frac{3x^2}{4}\cos 2x - \frac{3x}{4}\sin 2x - \frac{3}{8}\cos 2x\right]_0^{\pi/2}$$
$$= \left[\frac{\pi^3}{16}\cdot 0 + \frac{3\pi^2}{16}\cdot(-1) - \frac{3\pi}{8}\cdot 0 - \frac{3}{8}\cdot(-1)\right] - \left[0 + 0 - 0 - \frac{3}{8}\cdot 1\right] = -\frac{3\pi^2}{16} + \frac{3}{4} = \frac{3\,(4 - \pi^2)}{16}$$

49. $u = \sec^{-1} t$, $du = \frac{dt}{t\sqrt{t^2-1}}$; $dv = t\, dt$, $v = \frac{t^2}{2}$;
$$\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\, dt = \left[\frac{t^2}{2}\sec^{-1} t\right]_{2/\sqrt{3}}^{2} - \int_{2/\sqrt{3}}^{2}\left(\frac{t^2}{2}\right)\frac{dt}{t\sqrt{t^2-1}} = \left(2\cdot\frac{\pi}{3} - \frac{2}{3}\cdot\frac{\pi}{6}\right) - \int_{2/\sqrt{3}}^{2}\frac{t\, dt}{2\sqrt{t^2-1}}$$
$$= \frac{5\pi}{9} - \left[\frac{1}{2}\sqrt{t^2-1}\right]_{2/\sqrt{3}}^{2} = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \sqrt{\frac{4}{3} - 1}\right) = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = \frac{5\pi}{9} - \frac{\sqrt{3}}{3} = \frac{5\pi - 3\sqrt{3}}{9}$$

50. $u = \sin^{-1}(x^2)$, $du = \frac{2x\, dx}{\sqrt{1-x^4}}$; $dv = 2x\, dx$, $v = x^2$;
$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx = \left[x^2 \sin^{-1}(x^2)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \cdot \frac{2x\, dx}{\sqrt{1-x^4}} = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) + \int_0^{1/\sqrt{2}} \frac{d\,(1-x^4)}{2\sqrt{1-x^4}}$$
$$= \frac{\pi}{12} + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

51. (a) $u = x$, $du = dx$; $dv = \sin x\, dx$, $v = -\cos x$;
$$S_1 = \int_0^{\pi} x \sin x\, dx = [-x\cos x]_0^{\pi} + \int_0^{\pi} \cos x\, dx = \pi + [\sin x]_0^{\pi} = \pi$$

(b) $S_2 = -\int_{\pi}^{2\pi} x \sin x\, dx = -\left[[-x \cos x]_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos x\, dx\right] = -[-3\pi + [\sin x]_{\pi}^{2\pi}] = 3\pi$

(c) $S_3 = \int_{2\pi}^{3\pi} x \sin x\, dx = [-x \cos x]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} \cos x\, dx = 5\pi + [\sin x]_{2\pi}^{3\pi} = 5\pi$

(d) $S_{n+1} = (-1)^{n+1} \int_{n\pi}^{(n+1)\pi} x \sin x\, dx = (-1)^{n+1} \left[[-x \cos x]_{n\pi}^{(n+1)\pi} + [\sin x]_{n\pi}^{(n+1)\pi}\right]$
$= (-1)^{n+1} \left[-(n+1)\pi(-1)^n + n\pi(-1)^{n+1}\right] + 0 = (2n+1)\pi$

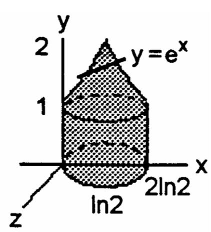
52. (a) $u = x$, $du = dx$; $dv = \cos x\, dx$, $v = \sin x$;
$S_1 = -\int_{\pi/2}^{3\pi/2} x \cos x\, dx = -\left[[x \sin x]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \sin x\, dx\right] = -\left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) - [\cos x]_{\pi/2}^{3\pi/2} = 2\pi$

(b) $S_2 = \int_{3\pi/2}^{5\pi/2} x \cos x\, dx = [x \sin x]_{3\pi/2}^{5\pi/2} - \int_{3\pi/2}^{5\pi/2} \sin x\, dx = \left[\frac{5\pi}{2} - \left(-\frac{3\pi}{2}\right)\right] - [\cos x]_{3\pi/2}^{5\pi/2} = 4\pi$

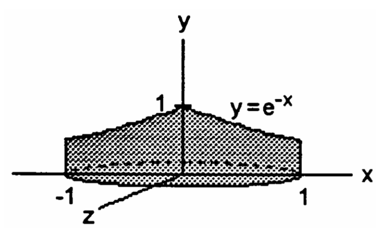
(c) $S_3 = -\int_{5\pi/2}^{7\pi/2} x \cos x\, dx = -\left[[x \sin x]_{5\pi/2}^{7\pi/2} - \int_{5\pi/2}^{7\pi/2} \sin x\, dx\right] = -\left(-\frac{7\pi}{2} - \frac{5\pi}{2}\right) - [\cos x]_{5\pi/2}^{7\pi/2} = 6\pi$

(d) $S_n = (-1)^n \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} x \cos x\, dx = (-1)^n \left[[x \sin x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} - {}^n \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} \sin x\, dx\right]$
$= (-1)^n \left[\frac{(2n+1)\pi}{2}(-1)^n - \frac{(2n-1)\pi}{2}(-1)^{n-1}\right] - [\cos x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} = \frac{1}{2}(2n\pi + \pi + 2n\pi - \pi) = 2n\pi$

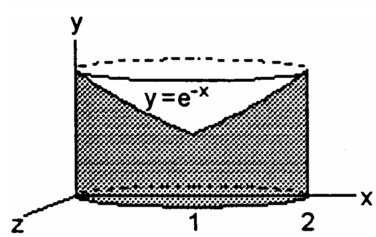
53. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x) e^x\, dx = 2\pi \ln 2 \int_0^{\ln 2} e^x\, dx - 2\pi \int_0^{\ln 2} xe^x\, dx$
$= (2\pi \ln 2)\left[e^x\right]_0^{\ln 2} - 2\pi\left(\left[xe^x\right]_0^{\ln 2} - \int_0^{\ln 2} e^x\, dx\right)$
$= 2\pi \ln 2 - 2\pi\left(2 \ln 2 - \left[e^x\right]_0^{\ln 2}\right) = -2\pi \ln 2 + 2\pi = 2\pi(1 - \ln 2)$



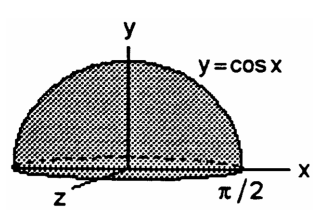
54. (a) $V = \int_0^1 2\pi xe^{-x}\, dx = 2\pi\left(\left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\, dx\right)$
$= 2\pi\left(-\frac{1}{e} + \left[-e^{-x}\right]_0^1\right) = 2\pi\left(-\frac{1}{e} - \frac{1}{e} + 1\right)$
$= 2\pi - \frac{4\pi}{e}$



(b) $V = \int_0^1 2\pi(1 - x)e^{-x}\, dx$; $u = 1 - x$, $du = -dx$; $dv = e^{-x}\, dx$,
$v = -e^{-x}$; $V = 2\pi\left[\left[(1 - x)\left(-e^{-x}\right)\right]_0^1 - \int_0^1 e^{-x}\, dx\right]$
$= 2\pi\left[[0 - 1(-1)] + \left[e^{-x}\right]_0^1\right] = 2\pi\left(1 + \frac{1}{e} - 1\right) = \frac{2\pi}{e}$



55. (a) $V = \int_0^{\pi/2} 2\pi x \cos x\, dx = 2\pi\left([x \sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x\, dx\right)$
$= 2\pi\left(\frac{\pi}{2} + [\cos x]_0^{\pi/2}\right) = 2\pi\left(\frac{\pi}{2} + 0 - 1\right) = \pi(\pi - 2)$



(b) $V = \int_0^{\pi/2} 2\pi\left(\frac{\pi}{2} - x\right) \cos x\, dx$; $u = \frac{\pi}{2} - x$, $du = -dx$; $dv = \cos x\, dx$, $v = \sin x$;
$V = 2\pi\left[\left(\frac{\pi}{2} - x\right) \sin x\right]_0^{\pi/2} + 2\pi \int_0^{\pi/2} \sin x\, dx = 0 + 2\pi[-\cos x]_0^{\pi/2} = 2\pi(0 + 1) = 2\pi$

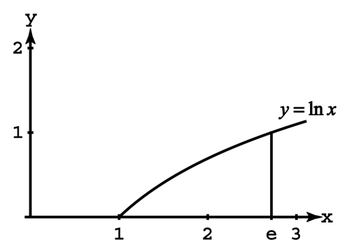
56. (a) $V = \int_0^{\pi} 2\pi x(x \sin x)\,dx;$

$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$\Rightarrow V = 2\pi \int_0^{\pi} x^2 \sin x\,dx = 2\pi\left[-x^2 \cos x + 2x \sin x + 2 \cos x\right]_0^{\pi} = 2\pi\left(\pi^2 - 4\right)$

(b) $V = \int_0^{\pi} 2\pi(\pi - x)x \sin x\,dx = 2\pi^2 \int_0^{\pi} x \sin x\,dx - 2\pi \int_0^{\pi} x^2 \sin x\,dx = 2\pi^2\left[-x \cos x + \sin x\right]_0^{\pi} - \left(2\pi^3 - 8\pi\right)$ $= 8\pi$

57. (a) $A = \int_1^e \ln x\,dx = \Big[x \ln x\Big]_1^e - \int_1^e dx$ $= (e \ln e - 1 \ln 1) - \Big[x\Big]_1^e = e - (e - 1) = 1$



(b) $V = \int_1^e \pi(\ln x)^2\,dx = \pi\left(\Big[x(\ln x)^2\Big]_1^e - \int_1^e 2 \ln x\,dx\right)$

$= \pi\left[\left(e(\ln e)^2 - 1(\ln 1)^2\right) - \left(\Big[2x \ln x\Big]_1^e - \int_1^e 2\,dx\right)\right]$

$= \pi\left[e - \left((2e \ln e - 2(1) \ln 1) - \Big[2x\Big]_1^e\right)\right]$

$= \pi\Big[e - (2e - (2e - 2))\Big] = \pi(e - 2)$

(c) $V = \int_1^e 2\pi(x + 2) \ln x\,dx = 2\pi \int_1^e (x + 2) \ln x\,dx = 2\pi\left(\left[\left(\tfrac{1}{2}x^2 + 2x\right)\ln x\right]_1^e - \int_1^e \left(\tfrac{1}{2}x + 2\right) dx\right)$

$= 2\pi\left(\left(\tfrac{1}{2}e^2 + 2e\right)\ln e - \left(\tfrac{1}{2} + 2\right)\ln 1 - \left[\left(\tfrac{1}{4}x^2 + 2x\right)\right]_1^e\right) = 2\pi\left(\left(\tfrac{1}{2}e^2 + 2e\right) - \left(\left(\tfrac{1}{4}e^2 + 2e\right) - \tfrac{9}{4}\right)\right) = \tfrac{\pi}{2}\left(e^2 + 9\right)$

(d) $M = \int_1^e \ln x\,dx = 1$ (from part (a)); $\overline{x} = \tfrac{1}{1}\int_1^e x \ln x\,dx = \left[\tfrac{1}{2}x^2 \ln x\right]_1^e - \int_1^e \tfrac{1}{2}x\,dx = \left(\tfrac{1}{2}e^2 \ln e - \tfrac{1}{2}(1)^2 \ln 1\right) - \left[\tfrac{1}{4}x^2\right]_1^e$

$= \tfrac{1}{2}e^2 - \left(\tfrac{1}{4}e^2 - \tfrac{1}{4}(1)^2\right) = \tfrac{1}{4}\left(e^2 + 1\right)$; $\overline{y} = \tfrac{1}{1}\int_1^e \tfrac{1}{2}(\ln x)^2\,dx = \tfrac{1}{2}\left(\Big[x(\ln x)^2\Big]_1^e - \int_1^e 2 \ln x\,dx\right)$

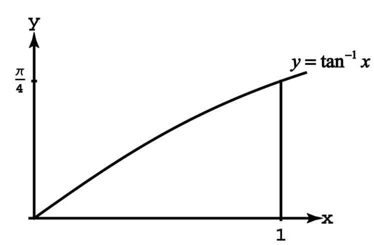
$= \tfrac{1}{2}\left(\left(e(\ln e)^2 - 1 \cdot (\ln 1)^2\right) - \left(\Big[2x \ln x\Big]_1^e - \int_1^e 2\,dx\right)\right) = \tfrac{1}{2}\left(e - \left((2e \ln e - 2(1) \ln 1) - \Big[2x\Big]_1^e\right)\right)$

$= \tfrac{1}{2}(e - 2e + 2e - 2) = \tfrac{1}{2}(e - 2) \Rightarrow (\overline{x}, \overline{y}) = \left(\tfrac{e^2 + 1}{4}, \tfrac{e - 2}{2}\right)$ is the centroid.

58. (a) $A = \int_0^1 \tan^{-1} x\,dx = \Big[x \tan^{-1} x\Big]_0^1 - \int_0^1 \tfrac{x}{1 + x^2}\,dx$

$= \left(\tan^{-1} 1 - 0\right) - \tfrac{1}{2}\Big[\ln\left(1 + x^2\right)\Big]_0^1$

$= \tfrac{\pi}{4} - \tfrac{1}{2}(\ln 2 - \ln 1) = \tfrac{\pi}{4} - \tfrac{1}{2}\ln 2$



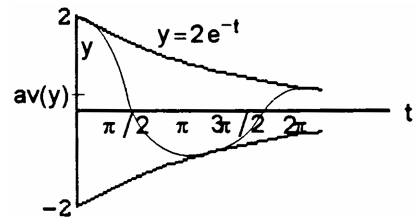
(b) $V = \int_0^1 2\pi x \tan^{-1} x\,dx$

$= 2\pi\left(\left[\tfrac{x^2}{2} \tan^{-1} x\right]_0^1 - \tfrac{1}{2}\int_0^1 \tfrac{x^2}{1 + x^2}\,dx\right)$

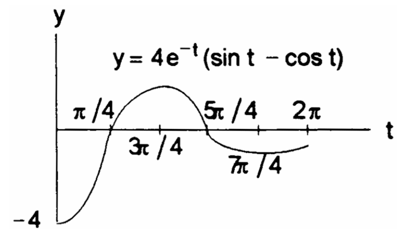
$= 2\pi\left(\tfrac{1}{2} \tan^{-1} 1 - 0 - \tfrac{1}{2}\int_0^1 \left(1 - \tfrac{1}{1 + x^2}\right)dx\right) = 2\pi\left(\tfrac{\pi}{8} - \tfrac{1}{2}\Big[x - \tan^{-1} x\Big]_0^1\right) = 2\pi\left(\tfrac{\pi}{8} - \tfrac{1}{2}\left(1 - \tan^{-1} 1 - (0 - 0)\right)\right)$

$= 2\pi\left(\tfrac{\pi}{8} - \tfrac{1}{2}\left(1 - \tfrac{\pi}{4}\right)\right) = \tfrac{\pi(\pi - 2)}{2}$

59. $\text{av}(y) = \frac{1}{2\pi}\int_0^{2\pi} 2e^{-t}\cos t\,dt$
$= \frac{1}{\pi}\left[e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi}$
(see Exercise 22) $\Rightarrow \text{av}(y) = \frac{1}{2\pi}\left(1 - e^{-2\pi}\right)$



60. $\text{av}(y) = \frac{1}{2\pi}\int_0^{2\pi} 4e^{-t}(\sin t - \cos t)\,dt$
$= \frac{2}{\pi}\int_0^{2\pi} e^{-t}\sin t\,dt - \frac{2}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt$
$= \frac{2}{\pi}\left[e^{-t}\left(\frac{-\sin t - \cos t}{2}\right) - e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi}$
$= \frac{2}{\pi}\left[-e^{-t}\sin t\right]_0^{2\pi} = 0$



61. $I = \int x^n \cos x\,dx$; $[u = x^n,\ du = nx^{n-1}\,dx;\ dv = \cos x\,dx,\ v = \sin x]$
$\Rightarrow I = x^n \sin x - \int nx^{n-1}\sin x\,dx$

62. $I = \int x^n \sin x\,dx$; $[u = x^n,\ du = nx^{n-1}\,dx;\ dv = \sin x\,dx,\ v = -\cos x]$
$\Rightarrow I = -x^n \cos x + \int nx^{n-1}\cos x\,dx$

63. $I = \int x^n e^{ax}\,dx$; $\left[u = x^n,\ du = nx^{n-1}\,dx;\ dv = e^{ax}\,dx,\ v = \frac{1}{a}e^{ax}\right]$
$\Rightarrow I = \frac{x^n e^{ax}}{a}e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx,\ a \neq 0$

64. $I = \int (\ln x)^n\,dx$; $\left[u = (\ln x)^n,\ du = \frac{n(\ln x)^{n-1}}{x}\,dx;\ dv = 1\,dx,\ v = x\right]$
$\Rightarrow I = x(\ln x)^n - \int n(\ln x)^{n-1}\,dx$

65. $\int_a^b (x-a)\,f(x)\,dx$; $\left[u = x - a,\ du = dx;\ dv = f(x)\,dx,\ v = \int_b^x f(t)\,dt = -\int_x^b f(t)\,dt\right]$
$= \left[(x-a)\int_b^x f(t)\,dt\right]_a^b - \int_a^b\left(\int_b^x f(t)\,dt\right)dx = \left((b-a)\int_b^b f(t)\,dt - (a-a)\int_b^a f(t)\,dt\right) - \int_a^b\left(-\int_x^b f(t)\,dt\right)dx$
$= 0 + \int_a^b\left(\int_x^b f(t)\,dt\right)dx = \int_a^b\left(\int_x^b f(t)\,dt\right)dx$

66. $\int \sqrt{1-x^2}\,dx$; $\left[u = \sqrt{1-x^2},\ du = \frac{-x}{\sqrt{1-x^2}}dx;\ dv = dx,\ v = x\right]$
$= x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\,dx = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx = x\sqrt{1-x^2} - \left(\int \frac{1-x^2}{\sqrt{1-x^2}}\,dx - \int \frac{1}{\sqrt{1-x^2}}\,dx\right)$
$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$
$\Rightarrow \int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \sqrt{1-x^2}\,dx \Rightarrow 2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx$
$\Rightarrow \int \sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx + C$

67. $\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \sin y\,dy = x\sin^{-1} x + \cos y + C = x\sin^{-1} x + \cos\left(\sin^{-1} x\right) + C$

68. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \tan y\,dy = x\tan^{-1} x + \ln|\cos y| + C = x\tan^{-1} x + \ln\left|\cos\left(\tan^{-1} x\right)\right| + C$