CHAPTER 7 INTEGRALS AND TRANSCENDENTAL FUNCTIONS

7.1 THE LOGARITHM DEFINED AS AN INTEGRAL

1. $\int_{-3}^{-2} \frac{1}{x}\, dx = \left[\ln|x|\right]_{-3}^{-2} = \ln 2 - \ln 3 = \ln \frac{2}{3}$

2. $\int_{-1}^{0} \frac{3}{3x-2}\, dx = \left[\ln|3x-2|\right]_{-1}^{0} = \ln 2 - \ln 5 = \ln \frac{2}{5}$

3. $\int \frac{2y}{y^2-25}\, dy = \ln\left|y^2 - 25\right| + C$

4. $\int \frac{8r}{4r^2-5}\, dr = \ln\left|4r^2 - 5\right| + C$

5. Let $u = 6 + 3\tan t \Rightarrow du = 3\sec^2 t\, dt$; $\int \frac{3\sec^2 t}{6+3\tan t}\, dt = \int \frac{du}{u} = \ln|u| + C = \ln|6 + 3\tan t| + C$

6. Let $u = 2 + \sec y \Rightarrow du = \sec y \tan y\, dy$; $\int \frac{\sec y \tan y}{2+\sec y}\, dy = \int \frac{du}{u} = \ln|u| + C = \ln|2 + \sec y| + C$

7. $\int \frac{dx}{2\sqrt{x}+2x} = \int \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)}$; let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\, dx$;
   $\int \frac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)} = \int \frac{du}{u} = \ln|u + C| = \ln\left|1 + \sqrt{x}\right| + C = \ln\left(1 + \sqrt{x}\right) + C$

8. Let $u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x)dx = (\sec x)(\tan x + \sec x)dx \Rightarrow \sec x\, dx = \frac{du}{u}$;
   $\int \frac{\sec x\, dx}{\sqrt{\ln(\sec x + \tan x)}} = \int \frac{du}{u\sqrt{\ln u}} = \int (\ln u)^{-1/2} \cdot \frac{1}{u}\, du = 2(\ln u)^{1/2} + C = 2\sqrt{\ln(\sec x + \tan x)} + C$

9. $\int_{\ln 2}^{\ln 3} e^x\, dx = \left[e^x\right]_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2} = 3 - 2 = 1$

10. $\int 8e^{(x+1)}\, dx = 8e^{(x+1)} + C$

11. $\int_1^4 \frac{(\ln x)^3}{2x}\, dx = \frac{1}{2}\int_1^4 (\ln x)^3 \left(\frac{1}{x}\right) dx = \left[\frac{(\ln x)^4}{8}\right]_1^4 = \frac{(\ln 4)^4}{8} - \frac{(\ln 1)^4}{8} = \frac{(\ln 4)^4}{8}$

12. Let $u = \ln(\ln x) \Rightarrow du = \frac{1}{\ln x} \cdot \frac{1}{x}\, dx = \frac{1}{x \ln x}\, dx$; $\int \frac{\ln(\ln x)}{x \ln x}\, dx = \int \ln(\ln x) \frac{1}{x \ln x}\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln(\ln x))^2}{2} + C$

13. $\int_{\ln 4}^{\ln 9} e^{x/2}\, dx = \left[2e^{x/2}\right]_{\ln 4}^{\ln 9} = 2\left[e^{(\ln 9)/2} - e^{(\ln 4)/2}\right] = 2(e^{\ln 3} - e^{\ln 2}) = 2(3-2) = 2$

14. Let $u = \ln(\cos x) \Rightarrow du = \frac{1}{\cos x}(-\sin x)dx = -\tan x\, dx \Rightarrow -du = \tan x\, dx$;
    $\int \tan x \ln(\cos x)dx = -\int u\, du = -\frac{u^2}{2} + C = -\frac{(\ln(\cos x))^2}{2} + C$

15. Let $u = r^{1/2} \Rightarrow du = \frac{1}{2} r^{-1/2} dr \Rightarrow 2\, du = r^{-1/2} dr$;
    $\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\, dr = \int e^{r^{1/2}} \cdot r^{-1/2} dr = 2\int e^u\, du = 2e^u + C = 2e^{r^{1/2}} + C = 2e^{\sqrt{r}} + C$

16. Let $u=-r^{1/2} \Rightarrow du = -\frac{1}{2}r^{-1/2}dr \Rightarrow -2\,du = r^{-1/2}dr$;
$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}dr = \int e^{-r^{1/2}} \cdot r^{-1/2}dr = -2\int e^{u}du = -2e^{-r^{1/2}} + C = -2e^{-\sqrt{r}} + C$

17. Let $u=-t^2 \Rightarrow du = -2t\,dt \Rightarrow -du = 2t\,dt$; $\int 2te^{-t^2}dt = -\int e^u du = -e^u + C = -e^{-t^2} + C$

18. Let $u = \ln^2 x + 1 \Rightarrow du = 2\ln x \cdot \frac{1}{x}dx = \frac{2\ln x}{x}dx \Rightarrow \frac{1}{2}du = \frac{\ln x}{x}dx$;
$\int \frac{\ln x}{x\sqrt{\ln^2 x+1}}dx = \int \frac{1}{2\sqrt{u}}du = \sqrt{u} + C = \sqrt{\ln^2 x + 1} + C$

19. Let $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2}dx \Rightarrow -du = \frac{1}{x^2}dx$; $\int \frac{e^{1/x}}{x^2}dx = \int -e^u du = -e^u + C = -e^{1/x} + C$

20. Let $u = -x^{-2} \Rightarrow du = 2x^{-3}dx \Rightarrow \frac{1}{2}du = x^{-3}dx$;
$\int \frac{e^{-1/x^2}}{x^3}dx = \int e^{-x^{-2}} \cdot x^{-3}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{-x^{-2}} + C = \frac{1}{2}e^{-1/x^2} + C$

21. Let $u = \sec \pi t \Rightarrow du = \pi \sec \pi t \tan \pi t\, dt \Rightarrow \frac{du}{\pi} = \sec \pi t \tan \pi t\, dt$;
$\int e^{\sec(\pi t)}\sec(\pi t)\tan(\pi t)dt = \frac{1}{\pi}\int e^u du = \frac{e^u}{\pi} + C = \frac{e^{\sec(\pi t)}}{\pi} + C$

22. Let $u = \csc(\pi + t) \Rightarrow du = -\csc(\pi + t)\cot(\pi + t)dt$;
$\int e^{\csc(\pi+t)}\csc(\pi+t)\cot(\pi+t)dt = -\int e^u du = -e^u + C = -e^{\csc(\pi+t)} + C$

23. Let $u = e^v \Rightarrow du = e^v dv \Rightarrow 2\,du = 2e^v dv$; $v = \ln\frac{\pi}{6} \Rightarrow u = \frac{\pi}{6}$, $v = \ln\frac{\pi}{2} \Rightarrow u = \frac{\pi}{2}$;
$\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v dv = 2\int_{\pi/6}^{\pi/2}\cos u\, du = \left[2\sin u\right]_{\pi/6}^{\pi/2} = 2\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)\right] = 2\left(1 - \frac{1}{2}\right) = 1$

24. Let $u = e^{x^2} \Rightarrow du = 2xe^{x^2}dx$; $x = 0 \Rightarrow u = 1$, $x = \sqrt{\ln \pi} \Rightarrow u = e^{\ln \pi} = \pi$;
$\int_0^{\sqrt{\ln \pi}} 2xe^{x^2}\cos\left(e^{x^2}\right)dx = \int_1^{\pi}\cos u\, du = \left[\sin u\right]_1^{\pi} = \sin(\pi) - \sin(1) = -\sin(1) \approx -0.84147$

25. Let $u = 1 + e^r \Rightarrow du = e^r dr$; $\int \frac{e^r}{1+e^r}dr = \int \frac{1}{u}du = \ln|u| + C = \ln(1 + e^r) + C$

26. $\int \frac{1}{1+e^x}dx = \int \frac{e^{-x}}{e^{-x}+1}dx$; let $u = e^{-x} + 1 \Rightarrow du = -e^{-x}dx \Rightarrow -du = e^{-x}dx$;
$\int \frac{e^{-x}}{e^{-x}+1}dx = -\int \frac{1}{u}du = -\ln|u| + C = -\ln(e^{-x} + 1) + C$

27. $\int_0^1 2^{-\theta}d\theta = \int_0^1 \left(\frac{1}{2}\right)^{\theta}d\theta = \left[\frac{\left(\frac{1}{2}\right)^{\theta}}{\ln\left(\frac{1}{2}\right)}\right]_0^1 = \frac{\frac{1}{2}}{\ln\left(\frac{1}{2}\right)} - \frac{1}{\ln\left(\frac{1}{2}\right)} = -\frac{\frac{1}{2}}{\ln\left(\frac{1}{2}\right)} = \frac{-1}{2(\ln 1 - \ln 2)} = \frac{1}{2\ln 2}$

[redacted]

28. $\int_{-2}^{0} 5^{-\theta}\, d\theta = \int_{-2}^{0} \left(\frac{1}{5}\right)^{\theta} d\theta = \left[\frac{\left(\frac{1}{5}\right)^{\theta}}{\ln\left(\frac{1}{5}\right)}\right]_{-2}^{0} = \frac{1}{\ln\left(\frac{1}{5}\right)} - \frac{\left(\frac{1}{5}\right)^{-2}}{\ln\left(\frac{1}{5}\right)} = \frac{1}{\ln\left(\frac{1}{5}\right)}(1-25) = \frac{-24}{\ln 1 - \ln 5} = \frac{24}{\ln 5}$

29. Let $u = x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$; $x = 1 \Rightarrow u = 1$, $x = \sqrt{2} \Rightarrow u = 2$;

$\int_{1}^{\sqrt{2}} x2^{(x^2)}\,dx = \int_{1}^{2} \left(\frac{1}{2}\right) 2^{u}\,du = \frac{1}{2}\left[\frac{2^{u}}{\ln 2}\right]_{1}^{2} = \left(\frac{1}{2\ln 2}\right)(2^2 - 2^1) = \frac{1}{\ln 2}$

30. Let $u = x^{1/2} \Rightarrow du = \frac{1}{2}x^{-1/2}dx \Rightarrow 2\,du = \frac{dx}{\sqrt{x}}$; $x = 1 \Rightarrow u = 1, x = 4 \Rightarrow u = 2$;

$\int_{1}^{4} \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_{1}^{4} 2^{x^{1/2}} \cdot x^{-1/2}\,dx = 2\int_{1}^{2} 2^{u}\,du = \left[\frac{2^{(u+1)}}{\ln 2}\right]_{1}^{2} = \left(\frac{1}{\ln 2}\right)(2^3 - 2^2) = \frac{4}{\ln 2}$

31. Let $u = \cos t \Rightarrow du = -\sin t\,dt \Rightarrow -du = \sin t\,dt$; $t = 0 \Rightarrow u = 1$, $t = \frac{\pi}{2} \Rightarrow u = 0$;

$\int_{0}^{\pi/2} 7^{\cos t} \sin t\,dt = -\int_{1}^{0} 7^{u}\,du = \left[-\frac{7^{u}}{\ln 7}\right]_{1}^{0} = \left(\frac{-1}{\ln 7}\right)(7^0 - 7) = \frac{6}{\ln 7}$

32. Let $u = \tan t \Rightarrow du = \sec^2 t\,dt$; $t = 0 \Rightarrow u = 0$, $t = \frac{\pi}{4} \Rightarrow u = 1$;

$\int_{0}^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t\,dt = \int_{0}^{1} \left(\frac{1}{3}\right)^{u} du = \left[\frac{\left(\frac{1}{3}\right)^{u}}{\ln\left(\frac{1}{3}\right)}\right]_{0}^{1} = \left(-\frac{1}{\ln 3}\right)\left[\left(\frac{1}{3}\right)^{1} - \left(\frac{1}{3}\right)^{0}\right] = \frac{2}{3\ln 3}$

33. Let $u = x^{2x} \Rightarrow \ln u = 2x \ln x \Rightarrow \frac{1}{u}\frac{du}{dx} = 2\ln x + (2x)\left(\frac{1}{x}\right) \Rightarrow \frac{du}{dx} = 2u(\ln x + 1) \Rightarrow \frac{1}{2}du = x^{2x}(1 + \ln x)dx$;

$x = 2 \Rightarrow u = 2^4 = 16$, $x = 4 \Rightarrow u = 4^8 = 65,536$;

$\int_{2}^{4} x^{2x}(1 + \ln x)dx = \frac{1}{2}\int_{16}^{65,536} du = \frac{1}{2}[u]_{16}^{65,536} = \frac{1}{2}(65,536 - 16) = \frac{65,520}{2} = 32,760$

34. Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$; $x = 1 \Rightarrow u = 0, x = 2 \Rightarrow u = \ln 2$;

$\int_{1}^{2} \frac{2^{\ln x}}{x}\,dx = \int_{0}^{\ln 2} 2^{u}\,du = \left[\frac{2^{u}}{\ln 2}\right]_{0}^{\ln 2} = \left(\frac{1}{\ln 2}\right)(2^{\ln 2} - 2^0) = \frac{2^{\ln 2} - 1}{\ln 2}$

35. $\int_{0}^{3} \left(\sqrt{2} + 1\right) x^{\sqrt{2}}\,dx = \left[x^{\left(\sqrt{2}+1\right)}\right]_{0}^{3} = 3^{\left(\sqrt{2}+1\right)}$

36. $\int_{1}^{e} x^{(\ln 2)-1}\,dx = \left[\frac{x^{\ln 2}}{\ln 2}\right]_{1}^{e} = \frac{e^{\ln 2} - 1^{\ln 2}}{\ln 2} = \frac{2-1}{\ln 2} = \frac{1}{\ln 2}$

37. $\int \frac{\log_{10} x}{x}\,dx = \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right)dx$; $\left[u = \ln x \Rightarrow du = \frac{1}{x}dx\right]$

$\rightarrow \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right)dx = \frac{1}{\ln 10}\int u\,du = \left(\frac{1}{\ln 10}\right)\left(\frac{1}{2}u^2\right) + C = \frac{(\ln x)^2}{2\ln 10} + C$

38. $\int_{1}^{4} \frac{\log_2 x}{x}\,dx = \int_{1}^{4} \left(\frac{\ln x}{\ln 2}\right)\left(\frac{1}{x}\right)dx$; $\left[u = \ln x \Rightarrow du = \frac{1}{x}dx;\ x = 1 \Rightarrow u = 0,\ x = 4 \Rightarrow u = \ln 4\right]$

$\rightarrow \int_{1}^{4} \left(\frac{\ln x}{\ln 2}\right)\left(\frac{1}{x}\right)dx = \int_{0}^{\ln 4} \left(\frac{1}{\ln 2}\right)u\,du = \left(\frac{1}{\ln 2}\right)\left[\frac{1}{2}u^2\right]_{0}^{\ln 4} = \left(\frac{1}{\ln 2}\right)\left[\frac{1}{2}(\ln 4)^2\right] = \frac{(\ln 4)^2}{2\ln 2} = \frac{(\ln 4)^2}{\ln 4} = \ln 4$

[redacted]

39. $\int_1^4 \frac{\ln 2 \log_2 x}{x}\,dx = \int_1^4 \left(\frac{\ln 2}{x}\right)\left(\frac{\ln x}{\ln 2}\right)dx = \int_1^4 \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^4 = \frac{1}{2}[(\ln 4)^2 - (\ln 1)^2] = \frac{1}{2}(\ln 4)^2 = \frac{1}{2}(2\ln 2)^2$
$= 2(\ln 2)^2$

40. $\int_1^e \frac{2\ln 10(\log_{10} x)}{x}\,dx = \int_1^e \frac{(\ln 10)(2\ln x)}{(\ln 10)}\left(\frac{1}{x}\right)dx = [(\ln x)^2]_1^e = (\ln e)^2 - (\ln 1)^2 = 1$

41. $\int_0^2 \frac{\log_2(x+2)}{x+2}\,dx = \frac{1}{\ln 2}\int_0^2 [\ln(x+2)]\left(\frac{1}{x+2}\right)dx = \left(\frac{1}{\ln 2}\right)\left[\frac{(\ln(x+2))^2}{2}\right]_0^2 = \left(\frac{1}{\ln 2}\right)\left[\frac{(\ln 4)^2}{2} - \frac{(\ln 2)^2}{2}\right]$
$= \left(\frac{1}{\ln 2}\right)\left[\frac{4(\ln 2)^2}{2} - \frac{(\ln 2)^2}{2}\right] = \frac{3}{2}\ln 2$

42. $\int_{1/10}^{10} \frac{\log_{10}(10x)}{x}\,dx = \frac{10}{\ln 10}\int_{1/10}^{10} [\ln(10x)]\left(\frac{1}{10x}\right)dx = \left(\frac{10}{\ln 10}\right)\left[\frac{(\ln(10x))^2}{20}\right]_{1/10}^{10} = \left(\frac{10}{\ln 10}\right)\left[\frac{(\ln 100)^2}{20} - \frac{(\ln 1)^2}{2}\right]$
$= \left(\frac{10}{\ln 10}\right)\left[\frac{4(\ln 10)^2}{20}\right] = 2\ln 10$

43. $\int_0^9 \frac{2\log_{10}(x+1)}{x+1}\,dx = \frac{2}{\ln 10}\int_0^9 \ln(x+1)\left(\frac{1}{x+1}\right)dx = \left(\frac{2}{\ln 10}\right)\left[\frac{(\ln(x+1))^2}{2}\right]_0^9 = \left(\frac{2}{\ln 10}\right)\left[\frac{(\ln 10)^2}{2} - \frac{(\ln 1)^2}{2}\right] = \ln 10$

44. $\int_2^3 \frac{2\log_2(x-1)}{x-1}\,dx = \frac{2}{\ln 2}\int_2^3 \ln(x-1)\left(\frac{1}{x-1}\right)dx = \left(\frac{2}{\ln 2}\right)\left[\frac{(\ln(x-1))^2}{2}\right]_2^3 = \left(\frac{2}{\ln 2}\right)\left[\frac{(\ln 2)^2}{2} - \frac{(\ln 1)^2}{2}\right] = \ln 2$

45. $\int \frac{dx}{x\log_{10} x} = \int \left(\frac{\ln 10}{\ln x}\right)\left(\frac{1}{x}\right)dx = (\ln 10)\int \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)dx;\ \left[u = \ln x \Rightarrow du = \frac{1}{x}dx\right]$
$\to (\ln 10)\int \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)dx = (\ln 10)\int \frac{1}{u}\,du = (\ln 10)\ln|u| + C = (\ln 10)\ln|\ln x| + C$

46. $\int \frac{dx}{x(\log_8 x)^2} = \int \frac{dx}{x\left(\frac{\ln x}{\ln 8}\right)^2} = (\ln 8)^2 \int \frac{(\ln x)^{-2}}{x}\,dx = (\ln 8)^2\,\frac{(\ln x)^{-1}}{-1} + C = -\frac{(\ln 8)^2}{\ln x} + C$

47. $\frac{dy}{dt} = e^t \sin(e^t - 2) \Rightarrow y = \int e^t \sin(e^t - 2)\,dt;$
let $u = e^t - 2 \Rightarrow du = e^t\,dt \Rightarrow y = \int \sin u\,du = -\cos u + C = -\cos(e^t - 2) + C;$
$y(\ln 2) = 0 \Rightarrow -\cos(e^{\ln 2} - 2) + C = 0 \Rightarrow -\cos(2-2) + C = 0 \Rightarrow C = \cos 0 = 1$; thus, $y = 1 - \cos(e^t - 2)$

48. $\frac{dy}{dt} = e^{-t}\sec^2(\pi e^{-t}) \Rightarrow y = \int e^{-t}\sec^2(\pi e^{-t})\,dt;$
let $u = \pi e^{-t} \Rightarrow du = -\pi e^{-t}\,dt \Rightarrow -\frac{1}{\pi}du = e^{-t}\,dt \Rightarrow y = -\frac{1}{\pi}\int \sec^2 u\,du = -\frac{1}{\pi}\tan u + C = -\frac{1}{\pi}\tan(\pi e^{-t}) + C;$
$y(\ln 4) = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi}\tan(\pi e^{-\ln 4}) + C = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi}\tan\left(\pi \cdot \frac{1}{4}\right) + C = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi}(1) + C = \frac{2}{\pi} \Rightarrow C = \frac{3}{\pi};$
thus, $y = \frac{3}{\pi} - \frac{1}{\pi}\tan(\pi e^{-t})$

[redacted].