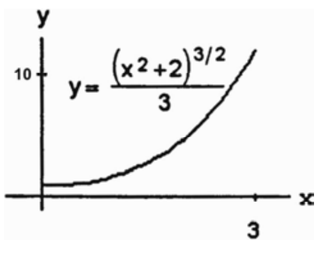
## 6.3 ARC LENGTH

1. $\frac{dy}{dx} = \frac{1}{3} \cdot \frac{3}{2}\left(x^2 + 2\right)^{1/2} \cdot 2x = \sqrt{\left(x^2 + 2\right)} \cdot x$

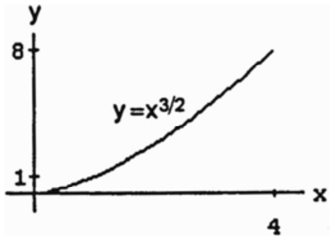
$\Rightarrow L = \int_0^3 \sqrt{1 + \left(x^2 + 2\right)x^2}\, dx = \int_0^3 \sqrt{1 + 2x^2 + x^4}\, dx$

$= \int_0^3 \sqrt{\left(1 + x^2\right)^2}\, dx = \int_0^3 \left(1 + x^2\right) dx = \left[x + \frac{x^3}{3}\right]_0^3$

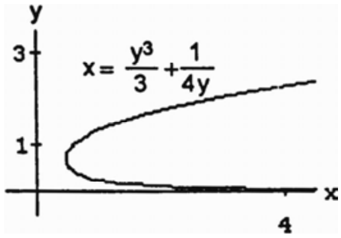
$= 3 + \frac{27}{3} = 12$



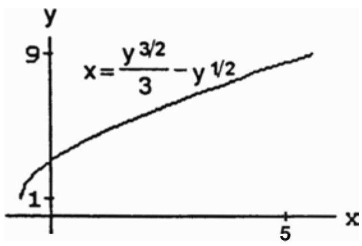
2. $\frac{dy}{dx} = \frac{3}{2}\sqrt{x} \Rightarrow L = \int_0^4 \sqrt{1 + \frac{9}{4}x}\ dx;$
$\left[u = 1 + \frac{9}{4}x \Rightarrow du = \frac{9}{4}dx \Rightarrow \frac{4}{9}du = dx;\right.$
$x = 0 \Rightarrow u = 1;\ x = 4 \Rightarrow u = 10]$
$\rightarrow L = \int_1^{10} u^{1/2}\left(\frac{4}{9}du\right) = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{8}{27}\left(10\sqrt{10} - 1\right)$



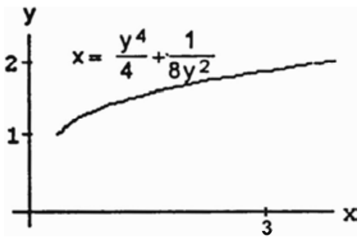

3. $\frac{dx}{dy} = y^2 - \frac{1}{4y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^4 - \frac{1}{2} + \frac{1}{16y^4}$
$\Rightarrow L = \int_1^3 \sqrt{1 + y^4 - \frac{1}{2} + \frac{1}{16y^4}}\ dy = \int_1^3 \sqrt{y^4 + \frac{1}{2} + \frac{1}{16y^4}}\ dy$
$= \int_1^3 \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2}\ dy = \int_1^3 \left(y^2 + \frac{1}{4y^2}\right) dy$
$= \left[\frac{y^3}{3} - \frac{y^{-1}}{4}\right]_1^3 = \left(\frac{27}{3} - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = 9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4}$
$= 9 + \frac{(-1-4+3)}{12} = 9 + \frac{(-2)}{12} = \frac{53}{6}$




4. $\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)$
$\Rightarrow L = \int_1^9 \sqrt{1 + \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)}\ dy = \int_1^9 \sqrt{\frac{1}{4}\left(y + 2 + \frac{1}{y}\right)}\ dy$
$= \int_1^9 \frac{1}{2}\sqrt{\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2}\ dy = \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right) dy$
$= \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \left[\frac{y^{3/2}}{3} + y^{1/2}\right]_1^9$
$= \left(\frac{3^3}{3} + 3\right) - \left(\frac{1}{3} + 1\right) = 11 - \frac{1}{3} = \frac{32}{3}$




5. $\frac{dx}{dy} = y^3 - \frac{1}{4y^3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^6 - \frac{1}{2} + \frac{1}{16y^6}$
$\Rightarrow L = \int_1^2 \sqrt{1 + y^6 - \frac{1}{2} + \frac{1}{16y^6}}\ dy = \int_1^2 \sqrt{y^6 + \frac{1}{2} + \frac{1}{16y^6}}\ dy$
$= \int_1^2 \sqrt{\left(y^3 + \frac{y^{-3}}{4}\right)^2}\ dy = \int_1^2 \left(y^3 + \frac{y^{-3}}{4}\right) dy = \left[\frac{y^4}{4} - \frac{y^{-2}}{8}\right]_1^2$
$= \left(\frac{16}{4} - \frac{1}{(16)(2)}\right) - \left(\frac{1}{4} - \frac{1}{8}\right) = 4 - \frac{1}{32} - \frac{1}{4} + \frac{1}{8}$
$= \frac{128-1-8+4}{32} = \frac{123}{32}$



6. $\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y^4 - 2 + y^{-4}\right)$

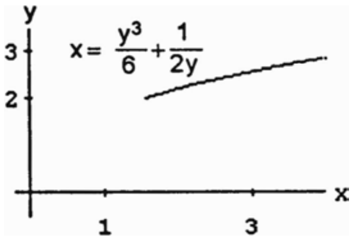
$\Rightarrow L = \int_2^3 \sqrt{1 + \frac{1}{4}\left(y^4 - 2 + y^{-4}\right)}\, dy$

$= \int_2^3 \sqrt{\frac{1}{4}\left(y^4 + 2 + y^{-4}\right)}\, dy$

$= \frac{1}{2}\int_2^3 \sqrt{\left(y^2 + y^{-2}\right)^2}\, dy = \frac{1}{2}\int_2^3 \left(y^2 + y^{-2}\right) dy$

$= \frac{1}{2}\left[\frac{y^3}{3} - y^{-1}\right]_2^3 = \frac{1}{2}\left[\left(\frac{27}{3} - \frac{1}{3}\right) - \left(\frac{8}{3} - \frac{1}{2}\right)\right]$

$= \frac{1}{2}\left(\frac{26}{3} - \frac{8}{3} + \frac{1}{2}\right) = \frac{1}{2}\left(6 + \frac{1}{2}\right) = \frac{13}{4}$




7. $\frac{dy}{dx} = x^{1/3} - \frac{1}{4}x^{-1/3} \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^{2/3} - \frac{1}{2} + \frac{x^{-2/3}}{16}$

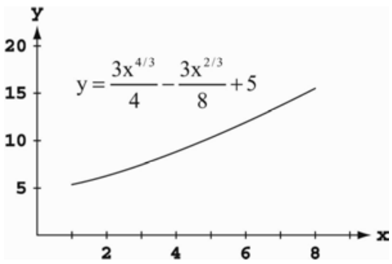
$\Rightarrow L = \int_1^8 \sqrt{1 + x^{2/3} - \frac{1}{2} + \frac{x^{-2/3}}{16}}\, dx$

$= \int_1^8 \sqrt{x^{2/3} + \frac{1}{2} + \frac{x^{-2/3}}{16}}\, dx = \int_1^8 \sqrt{\left(x^{1/3} + \frac{1}{4}x^{-1/3}\right)^2}\, dx$

$= \int_1^8 \left(x^{1/3} + \frac{1}{4}x^{-1/3}\right) dx = \left[\frac{3}{4}x^{4/3} + \frac{3}{8}x^{2/3}\right]_1^8$

$= \frac{3}{8}\left[2x^{4/3} + x^{2/3}\right]_1^8 = \frac{3}{8}\left[\left(2\cdot 2^4 + 2^2\right) - (2+1)\right]$

$= \frac{3}{8}(32 + 4 - 3) = \frac{99}{8}$




8. $\frac{dy}{dx} = x^2 + 2x + 1 - \frac{4}{(4x+4)^2} = x^2 + 2x + 1 - \frac{1}{4}\frac{1}{(1+x)^2}$

$= (1+x)^2 - \frac{1}{4}\frac{1}{(1+x)^2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = (1+x)^4 - \frac{1}{2} + \frac{1}{16(1+x)^4}$

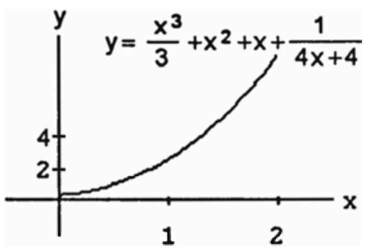
$\Rightarrow L = \int_0^2 \sqrt{1 + (1+x)^4 - \frac{1}{2} + \frac{(1+x)^{-4}}{16}}\, dx$

$= \int_0^2 \sqrt{(1+x)^4 + \frac{1}{2} + \frac{(1+x)^{-4}}{16}}\, dx$

$= \int_0^2 \sqrt{\left[(1+x)^2 + \frac{(1+x)^{-2}}{4}\right]^2}\, dx$

$= \int_0^2 \left[(1+x)^2 + \frac{(1+x)^{-2}}{4}\right] dx;\ [u = 1 + x \Rightarrow du = dx;\ x = 0 \Rightarrow u = 1,\ x = 2 \Rightarrow u = 3] \rightarrow L = \int_1^3 \left(u^2 + \frac{1}{4}u^{-2}\right) du$

$= \left[\frac{u^3}{3} - \frac{1}{4}u^{-1}\right]_1^3 = \left(9 - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{108-1-4+3}{12} = \frac{106}{12} = \frac{53}{6}$

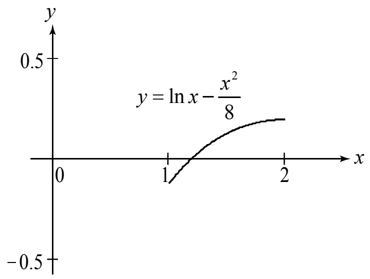


9. $\frac{dx}{dy} = \frac{1}{x} - \frac{x}{4} = \left(\frac{dx}{dy}\right)^2 = \left(\frac{1}{x} - \frac{x}{4}\right)^2 = \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16}$

$\Rightarrow L = \int_1^2 \sqrt{1 + \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16}}\, dx = \int_1^2 \sqrt{\frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16}}\, dx =$

$\int_1^2 \sqrt{\left(\frac{1}{x} + \frac{x}{4}\right)^2}\, dx = \int_1^2 \left(\frac{1}{x} + \frac{x}{4}\right) dx = \left[\ln|x| + \frac{x^2}{8}\right]_1^2 =$

$\left(\ln 2 + \frac{4}{8}\right) - \left(\ln 1 - \frac{1}{8}\right) = \ln 2 + \frac{3}{8}$



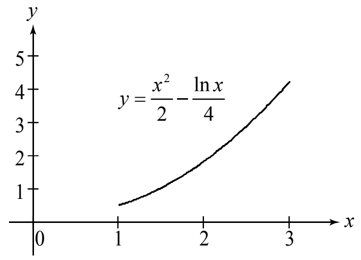

10. $\frac{dy}{dx} = x - \frac{1}{4x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(x - \frac{1}{4x}\right)^2 = x^2 - \frac{1}{2} + \frac{1}{16x^2}$

$\Rightarrow L = \int_1^3 \sqrt{1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}}\, dx =$

$= \int_1^3 \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}}\, dx = \int_1^3 \sqrt{\left(x + \frac{1}{4x}\right)^2}\, dx =$

$\int_1^3 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{1}{4}\ln|x|\right]_1^3 =$

$\left(\frac{9}{2} + \frac{1}{4}\ln 3\right) - \left(\frac{1}{2} + \frac{1}{4}\ln 1\right) = 4 + \frac{1}{4}\ln 3$

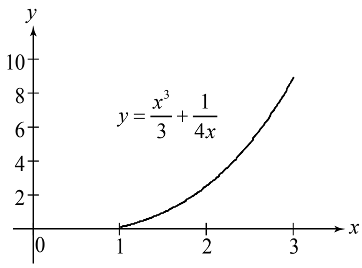



11. $\frac{dy}{dx} = x^2 - \frac{1}{4x^2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(x^2 - \frac{1}{4x^2}\right)^2 = x^4 - \frac{1}{2} + \frac{1}{16x^4}$

$\Rightarrow L = \int_1^3 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}}\, dx =$

$\int_1^3 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}}\, dx = \int_1^3 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx =$

$\int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^3 = \left(9 + \frac{1}{12}\right) - \left(\frac{1}{3} + \frac{1}{3}\right) = \frac{53}{6}$



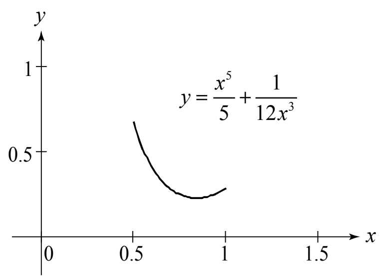

12. $\frac{dy}{dx} = x^4 - \frac{1}{4x^4} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(x^4 - \frac{1}{4x^4}\right)^2 = x^8 - \frac{1}{2} + \frac{1}{16x^8}$

$\Rightarrow L = \int_{1/2}^1 \sqrt{1 + x^8 - \frac{1}{2} + \frac{1}{16x^8}}\, dx =$

$\int_{1/2}^1 \sqrt{x^8 + \frac{1}{2} + \frac{1}{16x^8}}\, dx = \int_{1/2}^1 \sqrt{\left(x^4 + \frac{1}{4x^4}\right)^2}\, dx =$

$\int_{1/2}^1 \left(x^4 + \frac{1}{4x^4}\right) dx = \left[\frac{x^5}{5} - \frac{1}{12x^3}\right]_{1/2}^1 =$

$\left(\frac{1}{5} - \frac{1}{12}\right) - \left(\frac{1}{160} - \frac{2}{3}\right) = \frac{373}{480}$




13. $\frac{dx}{dy} = \sqrt{\sec^4 y - 1} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \sec^4 y - 1$

$\Rightarrow L = \int_{-\pi/4}^{\pi/4} \sqrt{1 + \left(\sec^4 y - 1\right)}\, dy = \int_{-\pi/4}^{\pi/4} \sec^2 y\, dy$

$= [\tan y]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$