## 6.2 VOLUMES USING CYLINDRICAL SHELLS

1. For the sketch given, $a = 0, b = 2$;

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^2 2\pi x\left(1+\frac{x^2}{4}\right) dx = 2\pi\int_0^2 \left(x+\frac{x^3}{4}\right) dx = 2\pi\left[\frac{x^2}{2}+\frac{x^4}{16}\right]_0^2 = 2\pi\left(\frac{4}{2}+\frac{16}{16}\right)$$
$$= 2\pi \cdot 3 = 6\pi$$

2. For the sketch given, $a = 0, b = 2$;

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^2 2\pi x\left(2-\frac{x^2}{4}\right) dx = 2\pi\int_0^2 \left(2x-\frac{x^3}{4}\right) dx = 2\pi\left[x^2-\frac{x^4}{16}\right]_0^2 = 2\pi(4-1) = 6\pi$$

3. For the sketch given, $c = 0, d = \sqrt{2}$;

$$V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^{\sqrt{2}} 2\pi y\cdot\left(y^2\right) dy = 2\pi\int_0^{\sqrt{2}} y^3 dy = 2\pi\left[\frac{y^4}{4}\right]_0^{\sqrt{2}} = 2\pi$$

4. For the sketch given, $c = 0, d = \sqrt{3}$;

$$V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^{\sqrt{3}} 2\pi y\cdot\left[3-\left(3-y^2\right)\right] dy = 2\pi\int_0^{\sqrt{3}} y^3 dy = 2\pi\left[\frac{y^4}{4}\right]_0^{\sqrt{3}} = \frac{9\pi}{2}$$

5. For the sketch given, $a = 0, b = \sqrt{3}$;

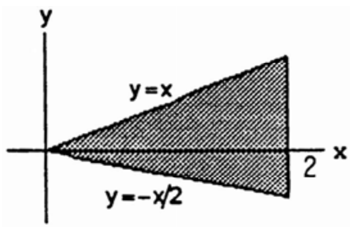
$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^{\sqrt{3}} 2\pi x\cdot\left(\sqrt{x^2+1}\right)dx;$$
$$\left[u = x^2+1 \Rightarrow du = 2x\,dx;\ x = 0 \Rightarrow u = 1,\ x = \sqrt{3} \Rightarrow u = 4\right]$$
$$\to V = \pi\int_1^4 u^{1/2} du = \pi\left[\frac{2}{3}u^{3/2}\right]_1^4 = \frac{2\pi}{3}\left(4^{3/2}-1\right) = \left(\frac{2\pi}{3}\right)(8-1) = \frac{14\pi}{3}$$

6. For the sketch given, $a = 0, b = 3$;

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^3 2\pi x\left(\frac{9x}{\sqrt{x^3+9}}\right) dx;$$
$$[u = x^3+9 \Rightarrow du = 3x^2\,dx \Rightarrow 3\,du = 9x^2\,dx;\ x = 0 \Rightarrow u = 9,\ x = 3 \Rightarrow u = 36]$$
$$\to V = 2\pi\int_9^{36} 3u^{-1/2} du = 6\pi\left[2u^{1/2}\right]_9^{36} = 12\pi\left(\sqrt{36}-\sqrt{9}\right) = 36\pi$$

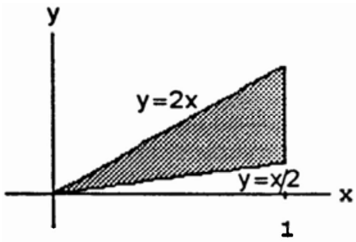
7. $a = 0, b = 2$;

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^2 2\pi x\left[x-\left(-\frac{x}{2}\right)\right] dx$$
$$= \int_0^2 2\pi x^2\cdot\frac{3}{2}\,dx = \pi\int_0^2 3x^2 dx = \pi\left[x^3\right]_0^2 = 8\pi$$



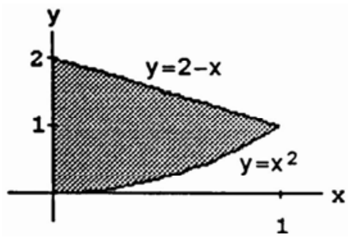
8. $a = 0, b = 1;$

$$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x\left(2x - \tfrac{x}{2}\right) dx$$
$$= \pi\int_0^1 2\left(\tfrac{3x^2}{2}\right) dx = \pi\int_0^1 3x^2 dx = \pi\left[x^3\right]_0^1 = \pi$$



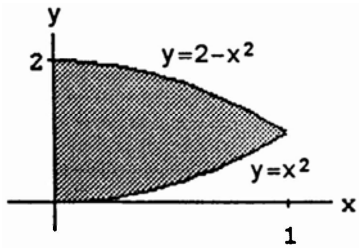

9. $a = 0, b = 1;$

$$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x\left[(2-x) - x^2\right] dx$$
$$= 2\pi\int_0^1\left(2x - x^2 - x^3\right) dx = 2\pi\left[x^2 - \tfrac{x^3}{3} - \tfrac{x^4}{4}\right]_0^1$$
$$= 2\pi\left(1 - \tfrac{1}{3} - \tfrac{1}{4}\right) = 2\pi\left(\tfrac{12-4-3}{12}\right) = \tfrac{10\pi}{12} = \tfrac{5\pi}{6}$$



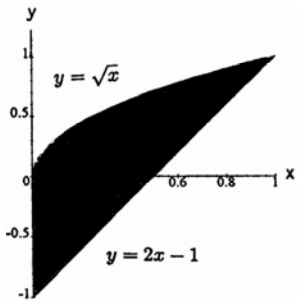

10. $a = 0, b = 1;$

$$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x\left[\left(2 - x^2\right) - x^2\right] dx$$
$$= 2\pi\int_0^1 x\left(2 - 2x^2\right) dx = 4\pi\int_0^1\left(x - x^3\right) dx$$
$$= 4\pi\left[\tfrac{x^2}{2} - \tfrac{x^4}{4}\right]_0^1 = 4\pi\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \pi$$



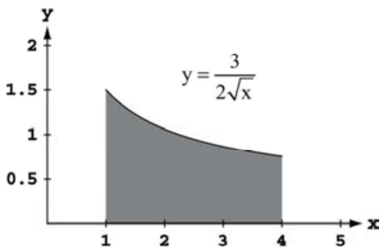

11. $a = 0, b = 1;$

$$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x\left[\sqrt{x} - (2x-1)\right] dx$$
$$= 2\pi\int_0^1\left(x^{3/2} - 2x^2 + x\right) dx = 2\pi\left[\tfrac{2}{5}x^{5/2} - \tfrac{2}{3}x^3 + \tfrac{1}{2}x^2\right]_0^1$$
$$= 2\pi\left(\tfrac{2}{5} - \tfrac{2}{3} + \tfrac{1}{2}\right) = 2\pi\left(\tfrac{12-20+15}{30}\right) = \tfrac{7\pi}{15}$$




12. $a = 1, b = 4;$

$$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_1^4 2\pi x\left(\tfrac{3}{2}x^{-1/2}\right) dx$$
$$= 3\pi\int_1^4 x^{1/2}\, dx = 3\pi\left[\tfrac{2}{3}x^{3/2}\right]_1^4 = 2\pi\left(4^{3/2} - 1\right)$$
$$= 2\pi(8-1) = 14\pi$$




13. (a) $x f(x) = \begin{cases} x \cdot \frac{\sin x}{x}, & 0 < x \le \pi \\ x, & x = 0 \end{cases} \Rightarrow x f(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ 0, & x = 0 \end{cases}$; since $\sin 0 = 0$ we have

$$x f(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ \sin x, & x = 0 \end{cases} \Rightarrow x f(x) = \sin x, 0 \le x \le \pi$$

(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^{\pi} 2\pi x \cdot f(x)\, dx$ and $x \cdot f(x) = \sin x,\ 0 \le x \le \pi$ by part (a)

$\Rightarrow V = 2\pi \int_0^{\pi} \sin x\, dx = 2\pi\left[-\cos x\right]_0^{\pi} = 2\pi(-\cos \pi + \cos 0) = 4\pi$

14. (a) $xg(x) = \begin{cases} x \cdot \frac{\tan^2 x}{x}, & 0 < x \le \frac{\pi}{4} \\ x \cdot 0, & x = 0 \end{cases} \Rightarrow xg(x) = \begin{cases} \tan^2 x, & 0 < x \le \pi/4 \\ 0, & x = 0 \end{cases}$; since $\tan 0 = 0$ we have

$xg(x) = \begin{cases} \tan^2 x, & 0 < x \le \pi/4 \\ \tan^2 x, & x = 0 \end{cases} \Rightarrow xg(x) = \tan^2 x,\ 0 \le x \le \pi/4$

(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^{\pi/4} 2\pi x \cdot g(x)\, dx$ and $x \cdot g(x) = \tan^2 x,\ 0 \le x \le \pi/4$ by part (a)

$\Rightarrow V = 2\pi \int_0^{\pi/4} \tan^2 x\, dx = 2\pi \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx = 2\pi\left[\tan x - x\right]_0^{\pi/4} = 2\pi\left(1 - \frac{\pi}{4}\right) = \frac{4\pi - \pi^2}{2}$

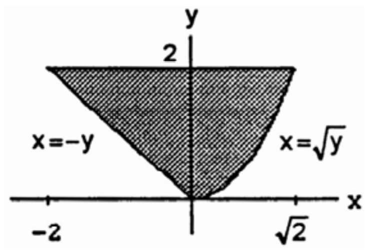
15. $c = 0,\ d = 2;$

$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y\left[\sqrt{y} - (-y)\right] dy$

$= 2\pi \int_0^2 \left(y^{3/2} + y^2\right) dy = 2\pi\left[\frac{2y^{5/2}}{5} + \frac{y^3}{3}\right]_0^2$

$= 2\pi\left[\frac{2}{5}\left(\sqrt{2}\right)^5 + \frac{2^3}{3}\right] = 2\pi\left(\frac{8\sqrt{2}}{5} + \frac{8}{3}\right) = 16\pi\left(\frac{\sqrt{2}}{5} + \frac{1}{3}\right)$

$= \frac{16\pi}{15}\left(3\sqrt{2} + 5\right)$

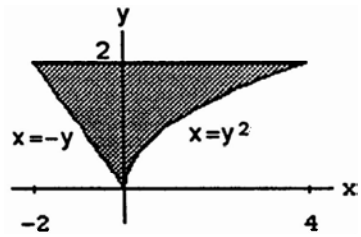



16. $c = 0,\ d = 2;$

$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y\left[y^2 - (-y)\right] dy$

$= 2\pi \int_0^2 \left(y^3 + y^2\right) dy = 2\pi\left[\frac{y^4}{4} + \frac{y^3}{3}\right]_0^2 = 16\pi\left(\frac{2}{4} + \frac{1}{3}\right)$

$= 16\pi\left(\frac{5}{6}\right) = \frac{40\pi}{3}$

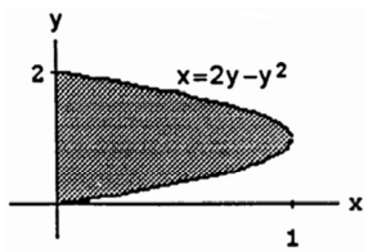



17. $c = 0,\ d = 2;$

$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y\left(2y - y^2\right) dy$

$= 2\pi \int_0^2 \left(2y^2 - y^3\right) dy = 2\pi\left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = 2\pi\left(\frac{16}{3} - \frac{16}{4}\right)$

$= 32\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{32\pi}{12} = \frac{8\pi}{3}$

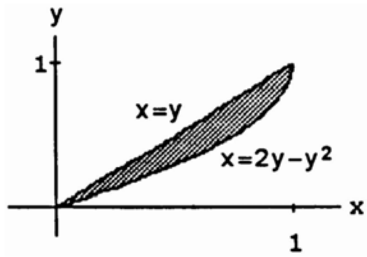


18. $c=0, d=1;$

$V=\int_c^d 2\pi\left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right)dy=\int_0^1 2\pi y\left(2y-y^2-y\right)dy$

$=2\pi\int_0^1 y\left(y-y^2\right)dy=2\pi\int_0^1\left(y^2-y^3\right)dy$

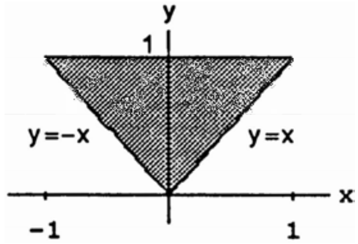
$=2\pi\left[\frac{y^3}{3}-\frac{y^4}{4}\right]_0^1=2\pi\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{\pi}{6}$




19. $c=0, d=1;$

$V=\int_c^d 2\pi\left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right)dy=2\pi\int_0^1 y\,[y-(-y)]\,dy$

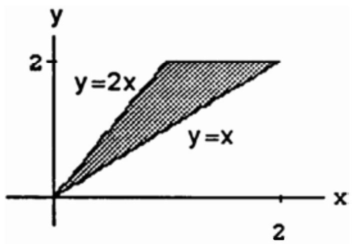
$=2\pi\int_0^1 2y^2dy=\frac{4\pi}{3}\left[y^3\right]_0^1=\frac{4\pi}{3}$




20. $c=0, d=2;$

$V=\int_c^d 2\pi\left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right)dy=\int_0^2 2\pi\, y\left(y-\frac{y}{2}\right)dy$

$=2\pi\int_0^2\frac{y^2}{2}\,dy=\frac{\pi}{3}\left[y^3\right]_0^2=\frac{8\pi}{3}$

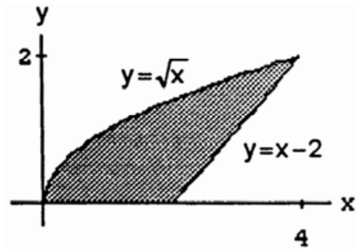



21. $c=0, d=2;$

$V=\int_c^d 2\pi\left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right)dy=\int_0^2 2\pi y\left[(2+y)-y^2\right]dy$

$=2\pi\int_0^2\left(2y+y^2-y^3\right)dy=2\pi\left[y^2+\frac{y^3}{3}-\frac{y^4}{4}\right]_0^2$

$=2\pi\left(4+\frac{8}{3}-\frac{16}{4}\right)=\frac{\pi}{6}(48+32-48)=\frac{16\pi}{3}$

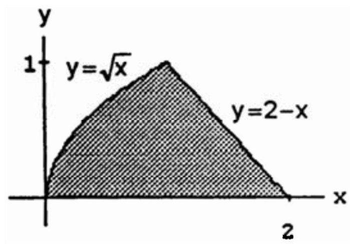



22. $c=0, d=1;$

$V=\int_c^d 2\pi\left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right)dy=\int_0^1 2\pi y\left[(2-y)-y^2\right]dy$

$=2\pi\int_0^1\left(2y-y^2-y^3\right)dy=2\pi\left[y^2-\frac{y^3}{3}-\frac{y^4}{4}\right]_0^1$

$=2\pi\left(1-\frac{1}{3}-\frac{1}{4}\right)=\frac{\pi}{6}(12-4-3)=\frac{5\pi}{6}$




23. (a) $V=\int_a^b 2\pi\left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right)dx=\int_0^2 2\pi\, x\,(3x)dx=6\pi\int_0^2 x^2dx=2\pi\left[x^3\right]_0^2=16\pi$

(b) $V=\int_a^b 2\pi\left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right)dx=\int_0^2 2\pi\,(4-x)\,(3x)dx=6\pi\int_0^2\left(4x-x^2\right)dx=6\pi\left[2x^2-\frac{1}{3}x^3\right]_0^2$

$=6\pi\left(8-\frac{8}{3}\right)=32\pi$

(c) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^2 2\pi(x+1)\,(3x)dx = 6\pi\int_0^2 \left(x^2+x\right)dx = 6\pi\left[\frac{1}{3}x^3+\frac{1}{2}x^2\right]_0^2$
$= 6\pi\left(\frac{8}{3}+2\right) = 28\pi$

(d) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^6 2\pi\ y\left(2-\frac{1}{3}y\right)dy = 2\pi\int_0^6 \left(2y-\frac{1}{3}y^2\right)dy = 2\pi\left[y^2-\frac{1}{9}y^3\right]_0^6$
$= 2\pi(36-24) = 24\pi$

(e) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^6 2\pi\ (7-y)\left(2-\frac{1}{3}y\right)dy = 2\pi\int_0^6 \left(14-\frac{13}{3}y+\frac{1}{3}y^2\right)dy$
$= 2\pi\left[14y-\frac{13}{6}y^2+\frac{1}{9}y^3\right]_0^6 = 2\pi(84-78+24) = 60\pi$

(f) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^6 2\pi\ (y+2)\left(2-\frac{1}{3}y\right)dy = 2\pi\int_0^6 \left(4+\frac{4}{3}y-\frac{1}{3}y^2\right)dy$
$= 2\pi\left[4y+\frac{2}{3}y^2-\frac{1}{9}y^3\right]_0^6 = 2\pi(24+24-24) = 48\pi$

24. (a) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^2 2\pi\ x\left(8-x^3\right)dx = 2\pi\int_0^2 \left(8x-x^4\right)dx = 2\pi\left[4x^2-\frac{1}{5}x^5\right]_0^2$
$= 2\pi\left(16-\frac{32}{5}\right) = \frac{96\pi}{5}$

(b) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^2 2\pi\ (3-x)\left(8-x^3\right)dx = 2\pi\int_0^2 \left(24-8x-3x^3+x^4\right)dx$
$= 2\pi\left[24x-4x^2-\frac{3}{4}x^4+\frac{1}{5}x^5\right]_0^2 = 2\pi\left(48-16-12+\frac{32}{5}\right) = \frac{264\pi}{5}$

(c) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^2 2\pi\ (x+2)\left(8-x^3\right)dx = 2\pi\int_0^2 \left(16+8x-2x^3-x^4\right)dx$
$= 2\pi\left[16x+4x^2-\frac{1}{2}x^4-\frac{1}{5}x^5\right]_0^2 = 2\pi\left(32+16-8-\frac{32}{5}\right) = \frac{336\pi}{5}$

(d) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^8 2\pi\ y\cdot y^{1/3}dy = 2\pi\int_0^8 y^{4/3}dy = \frac{6\pi}{7}\left[y^{7/3}\right]_0^8 = \frac{6\pi}{7}(128) = \frac{768\pi}{7}$

(e) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^8 2\pi\ (8-y)\ y^{1/3}dy = 2\pi\int_0^8 \left(8y^{1/3}-y^{4/3}\right)dy = 2\pi\left[6y^{4/3}-\frac{3}{7}y^{7/3}\right]_0^8$
$= 2\pi\left(96-\frac{384}{7}\right) = \frac{576\pi}{7}$

(f) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^8 2\pi\ (y+1)\ y^{1/3}dx = 2\pi\int_0^8 \left(y^{4/3}+y^{1/3}\right)dy = 2\pi\left[\frac{3}{7}y^{7/3}+\frac{3}{4}y^{4/3}\right]_0^8$
$= 2\pi\left(\frac{384}{7}+12\right) = \frac{936\pi}{7}$

25. (a) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_{-1}^2 2\pi\ (2-x)\left(x+2-x^2\right)dx = 2\pi\int_{-1}^2 \left(4-3x^2+x^3\right)dx$
$= 2\pi\left[4x-x^3+\frac{1}{4}x^4\right]_{-1}^2 = 2\pi(8-8+4)-2\pi\left(-4+1+\frac{1}{4}\right) = \frac{27\pi}{2}$

(b) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_{-1}^2 2\pi\ (x+1)\left(x+2-x^2\right)dx = 2\pi\int_{-1}^2 \left(2+3x-x^3\right)dx$
$= 2\pi\left[2x+\frac{3}{2}x^2-\frac{1}{4}x^4\right]_{-1}^2 = 2\pi\ (4+6-4)-2\pi\left(-2+\frac{3}{2}-\frac{1}{4}\right) = \frac{27\pi}{2}$

(c) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi\ y\left(\sqrt{y}-\left(-\sqrt{y}\right)\right)dy+\int_1^4 2\pi\ y\left(\sqrt{y}-(y-2)\right)dy$

$=4\pi\int_0^1 y^{3/2}dy+2\pi\int_1^4\left(y^{3/2}-y^2+2y\right)dy=\frac{8\pi}{5}\left[y^{5/2}\right]_0^1+2\pi\left[\frac{2}{5}y^{5/2}-\frac{1}{3}y^3+y^2\right]_1^4$

$=\frac{8\pi}{5}(1)+2\pi\left(\frac{64}{5}-\frac{64}{3}+16\right)-2\pi\left(\frac{2}{5}-\frac{1}{3}+1\right)=\frac{72\pi}{5}$

(d) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi\ (4-y)\left(\sqrt{y}-\left(-\sqrt{y}\right)\right)dy+\int_1^4 2\pi\ (4-y)\left(\sqrt{y}-(y-2)\right)dy$

$=4\pi\int_0^1\left(4\sqrt{y}-y^{3/2}\right)dy+2\pi\int_1^4\left(y^2-y^{3/2}-6y+4\sqrt{y}+8\right)dy$

$=4\pi\left[\frac{8}{3}y^{3/2}-\frac{2}{5}y^{5/2}\right]_0^1+2\pi\left[\frac{1}{3}y^3-\frac{2}{5}y^{5/2}-3y^2+\frac{8}{3}y^{3/2}+8y\right]_1^4$

$=4\pi\left(\frac{8}{3}-\frac{2}{5}\right)+2\pi\left(\frac{64}{3}-\frac{64}{5}-48+\frac{64}{3}+32\right)-2\pi\left(\frac{1}{3}-\frac{2}{5}-3+\frac{8}{3}+8\right)=\frac{108\pi}{5}.$

26. (a) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_{-1}^1 2\pi\ (1-x)\left(4-3x^2-x^4\right)dx=2\pi\int_{-1}^1\left(x^5-x^4+3x^3-3x^2-4x+4\right)dx$

$=2\pi\left[\frac{1}{6}x^6-\frac{1}{5}x^5+\frac{3}{4}x^4-x^3-2x^2+4x\right]_{-1}^1=2\pi\left(\frac{1}{6}-\frac{1}{5}+\frac{3}{4}-1-2+4\right)-2\pi\left(\frac{1}{6}+\frac{1}{5}+\frac{3}{4}+1-2-4\right)=\frac{56\pi}{5}$

(b) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi y\left(\sqrt[4]{y}-\left(-\sqrt[4]{y}\right)\right)dy+\int_1^4 2\pi\ y\left[\sqrt{\frac{4-y}{3}}-\left(-\sqrt{\frac{4-y}{3}}\right)\right]dy$

$=4\pi\int_0^1 y^{5/4}dy+\frac{4\pi}{\sqrt{3}}\int_1^4 y\sqrt{4-y}dy\quad[u=4-y\Rightarrow y=4-u\Rightarrow du=-du;\ y=1\Rightarrow u=3,\ y=4\Rightarrow u=0]$

$=\frac{16\pi}{9}\left[y^{9/4}\right]_0^1-\frac{4\pi}{\sqrt{3}}\int_3^0(4-u)\sqrt{u}\,du=\frac{16\pi}{9}(1)+\frac{4\pi}{\sqrt{3}}\int_0^3\left(4\sqrt{u}-u^{3/2}\right)du=\frac{16\pi}{9}+\frac{4\pi}{\sqrt{3}}\left[\frac{8}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right]_0^3$

$=\frac{16\pi}{9}+\frac{4\pi}{\sqrt{3}}\left(8\sqrt{3}-\frac{18}{5}\sqrt{3}\right)=\frac{16\pi}{9}+\frac{88\pi}{5}=\frac{872\pi}{45}$

27. (a) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi y\cdot 12\left(y^2-y^3\right)dy=24\pi\int_0^1\left(y^3-y^4\right)dy=24\pi\left[\frac{y^4}{4}-\frac{y^5}{5}\right]_0^1$

$=24\pi\left(\frac{1}{4}-\frac{1}{5}\right)=\frac{24\pi}{20}=\frac{6\pi}{5}$

(b) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi(1-y)\left[12\left(y^2-y^3\right)\right]dy=24\pi\int_0^1(1-y)\left(y^2-y^3\right)dy$

$=24\pi\int_0^1\left(y^2-2y^3+y^4\right)dy=24\pi\left[\frac{y^3}{3}-\frac{y^4}{2}+\frac{y^5}{5}\right]_0^1=24\pi\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)=24\pi\left(\frac{1}{30}\right)=\frac{4\pi}{5}$

(c) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi\left(\frac{8}{5}-y\right)\left[12\left(y^2-y^3\right)\right]dy=24\pi\int_0^1\left(\frac{8}{5}-y\right)\left(y^2-y^3\right)dy$

$=24\pi\int_0^1\left(\frac{8}{5}y^2-\frac{13}{5}y^3+y^4\right)dy=24\pi\left[\frac{8}{15}y^3-\frac{13}{20}y^4+\frac{y^5}{5}\right]_0^1=24\pi\left(\frac{8}{15}-\frac{13}{20}+\frac{1}{5}\right)$

$=\frac{24\pi}{60}(32-39+12)=\frac{24\pi}{12}=2\pi$

(d) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^1 2\pi\left(y+\frac{2}{5}\right)\left[12\left(y^2-y^3\right)\right]dy=24\pi\int_0^1\left(y+\frac{2}{5}\right)\left(y^2-y^3\right)dy$

$=24\pi\int_0^1\left(y^3-y^4+\frac{2}{5}y^2-\frac{2}{5}y^3\right)dy=24\pi\int_0^1\left(\frac{2}{5}y^2+\frac{3}{5}y^3-y^4\right)dy=24\pi\left[\frac{2}{15}y^3+\frac{3}{20}y^4-\frac{y^5}{5}\right]_0^1$

$=24\pi\left(\frac{2}{15}+\frac{3}{20}-\frac{1}{5}\right)=\frac{24\pi}{60}(8+9-12)=\frac{24\pi}{12}=2\pi$

28. (a) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^2 2\pi y\left[\frac{y^2}{2}-\left(\frac{y^4}{4}-\frac{y^2}{2}\right)\right]dy=\int_0^2 2\pi y\left(y^2-\frac{y^4}{4}\right)dy=2\pi\int_0^2\left(y^3-\frac{y^5}{4}\right)dy$

$=2\pi\left[\frac{y^4}{4}-\frac{y^6}{24}\right]_0^2=2\pi\left(\frac{2^4}{4}-\frac{2^6}{24}\right)=32\pi\left(\frac{1}{4}-\frac{4}{24}\right)=32\pi\left(\frac{1}{4}-\frac{1}{6}\right)=32\pi\left(\frac{2}{24}\right)=\frac{8\pi}{3}$

(b) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^2 2\pi(2-y)\left[\frac{y^2}{2}-\left(\frac{y^4}{4}-\frac{y^2}{2}\right)\right]dy=\int_0^2 2\pi(2-y)\left(y^2-\frac{y^4}{4}\right)dy$

$=2\pi\int_0^2\left(2y^2-\frac{y^4}{2}-y^3+\frac{y^5}{4}\right)dy=2\pi\left[\frac{2y^3}{3}-\frac{y^5}{10}-\frac{y^4}{4}+\frac{y^6}{24}\right]_0^2=2\pi\left(\frac{16}{3}-\frac{32}{10}-\frac{16}{4}+\frac{64}{24}\right)=\frac{8\pi}{5}$

(c) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^2 2\pi(5-y)\left[\frac{y^2}{2}-\left(\frac{y^4}{4}-\frac{y^2}{2}\right)\right]dy=\int_0^2 2\pi(5-y)\left(y^2-\frac{y^4}{4}\right)dy$

$=2\pi\int_0^2\left(5y^2-\frac{5}{4}y^4-y^3+\frac{y^5}{4}\right)dy=2\pi\left[\frac{5y^3}{3}-\frac{5y^5}{20}-\frac{y^4}{4}+\frac{y^6}{24}\right]_0^2=2\pi\left(\frac{40}{3}-\frac{160}{20}-\frac{16}{4}+\frac{64}{24}\right)=8\pi$

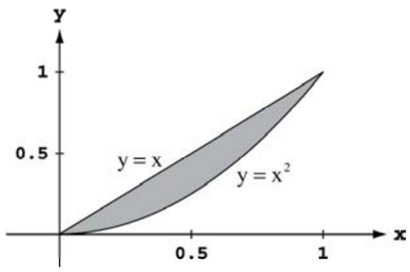
(d) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^2 2\pi\left(y+\frac{5}{8}\right)\left[\frac{y^2}{2}-\left(\frac{y^4}{4}-\frac{y^2}{2}\right)\right]dy=\int_0^2 2\pi\left(y+\frac{5}{8}\right)\left(y^2-\frac{y^4}{4}\right)dy$

$=2\pi\int_0^2\left(y^3-\frac{y^5}{4}+\frac{5}{8}y^2-\frac{5}{32}y^4\right)dy=2\pi\left[\frac{y^4}{4}-\frac{y^6}{24}+\frac{5y^3}{24}-\frac{5y^5}{160}\right]_0^2=2\pi\left(\frac{16}{4}-\frac{64}{24}+\frac{40}{24}-\frac{160}{160}\right)=4\pi$

29. (a) About $x$-axis: $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy$

$=\int_0^1 2\pi y\left(\sqrt{y}-y\right)dy=2\pi\int_0^1\left(y^{3/2}-y^2\right)dy$

$=2\pi\left[\frac{2}{5}y^{5/2}-\frac{1}{3}y^3\right]_0^1=2\pi\left(\frac{2}{5}-\frac{1}{3}\right)=\frac{2\pi}{15}$




About $y$-axis: $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx$

$=\int_0^1 2\pi x\left(x-x^2\right)dx=2\pi\int_0^1\left(x^2-x^3\right)dx$

$=2\pi\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=2\pi\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{\pi}{6}$

(b) About $x$-axis: $R(x)=x$ and $r(x)=x^2\Rightarrow V=\int_a^b\pi\left([R(x)]^2-[r(x)]^2\right)dx=\int_0^1\pi\left(x^2-x^4\right)dx$

$=\pi\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1=\pi\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{2\pi}{15}$

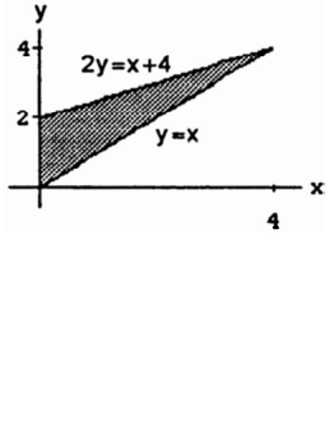
About $y$-axis: $R(y)=\sqrt{y}$ and $r(y)=y\Rightarrow V=\int_c^d\pi\left([R(y)]^2-[r(y)]^2\right)dy=\int_0^1\pi\left(y-y^2\right)dy$

$=\pi\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{\pi}{6}$

30. (a) $V=\int_a^b\pi\left([R(x)]^2-[r(x)]^2\right)dx=\pi\int_0^4\left[\left(\frac{x}{2}+2\right)^2-x^2\right]dx$

$=\pi\int_0^4\left(-\frac{3}{4}x^2+2x+4\right)dx=\pi\left[-\frac{x^3}{4}+x^2+4x\right]_0^4$

$=\pi(-16+16+16)=16\pi$



(b) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^4 2\pi x\left(\frac{x}{2}+2-x\right)dx=\int_0^4 2\pi x\left(2-\frac{x}{2}\right)dx=2\pi\int_0^4\left(2x-\frac{x^2}{2}\right)dx$

$=2\pi\left[x^2-\frac{x^3}{6}\right]_0^4=2\pi\left(16-\frac{64}{6}\right)=\frac{32\pi}{3}$

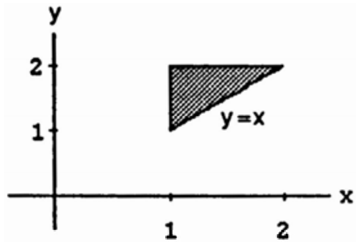
(c) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^4 2\pi(4-x)\left(\frac{x}{2}+2-x\right)dx=\int_0^4 2\pi(4-x)\left(2-\frac{x}{2}\right)dx=2\pi\int_0^4\left(8-4x-\frac{x^2}{2}\right)dx$

$=2\pi\left[8x-2x^2+\frac{x^3}{6}\right]_0^4=2\pi\left(32-32+\frac{64}{6}\right)=\frac{64\pi}{3}$

(d) $V=\int_a^b \pi\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx=\pi\int_0^4\left[(8-x)^2-\left(6-\frac{x}{2}\right)^2\right]dx=\pi\int_0^4\left[\left(64-16x+x^2\right)-\left(36-6x+\frac{x^2}{4}\right)\right]dx$

$\pi\int_0^4\left(\frac{3}{4}x^2-10x+28\right)dx=\pi\left[\frac{x^3}{4}-5x^2+28x\right]_0^4=\pi[16-(5)(16)+(7)(16)]=\pi(3)(16)=48\pi$

31. (a) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_1^2 2\pi y(y-1)\,dy$




$=2\pi\int_1^2\left(y^2-y\right)dy=2\pi\left[\frac{y^3}{3}-\frac{y^2}{2}\right]_1^2$

$=2\pi\left[\left(\frac{8}{3}-\frac{4}{2}\right)-\left(\frac{1}{3}-\frac{1}{2}\right)\right]$

$=2\pi\left(\frac{7}{3}-2+\frac{1}{2}\right)=\frac{\pi}{3}(14-12+3)=\frac{5\pi}{3}$

(b) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_1^2 2\pi x(2-x)\,dx=2\pi\int_1^2\left(2x-x^2\right)dx=2\pi\left[x^2-\frac{x^3}{3}\right]_1^2$

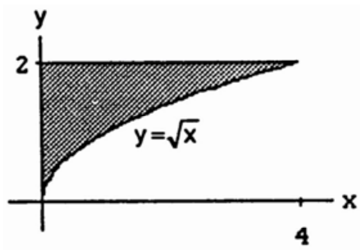
$=2\pi\left[\left(4-\frac{8}{3}\right)-\left(1-\frac{1}{3}\right)\right]=2\pi\left[\left(\frac{12-8}{3}\right)-\left(\frac{3-1}{3}\right)\right]=2\pi\left(\frac{4}{3}-\frac{2}{3}\right)=\frac{4\pi}{3}$

(c) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_1^2 2\pi\left(\frac{10}{3}-x\right)(2-x)\,dx=2\pi\int_1^2\left(\frac{20}{3}-\frac{16}{3}x+x^2\right)dx$

$=2\pi\left[\frac{20}{3}x-\frac{8}{3}x^2+\frac{1}{3}x^3\right]_1^2=2\pi\left[\left(\frac{40}{3}-\frac{32}{3}+\frac{8}{3}\right)-\left(\frac{20}{3}-\frac{8}{3}+\frac{1}{3}\right)\right]=2\pi\left(\frac{3}{3}\right)=2\pi$

(d) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_1^2 2\pi(y-1)(y-1)\,dy=2\pi\int_1^2(y-1)^2=2\pi\left[\frac{(y-1)^3}{3}\right]_1^2=\frac{2\pi}{3}$

32. (a) $V=\int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dy=\int_0^2 2\pi y\left(y^2-0\right)dy$




$=2\pi\int_0^2 y^3dy=2\pi\left[\frac{y^4}{4}\right]_0^2=2\pi\left(\frac{2^4}{4}\right)=8\pi$

(b) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx$

$=\int_0^4 2\pi x\left(2-\sqrt{x}\right)dx=2\pi\int_0^4\left(2x-x^{3/2}\right)dx$

$=2\pi\left[x^2-\frac{2}{5}x^{5/2}\right]_0^4=2\pi\left(16-\frac{2\cdot 2^5}{5}\right)$

$=2\pi\left(16-\frac{64}{5}\right)=\frac{2\pi}{5}(80-64)=\frac{32\pi}{5}$

(c) $V=\int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right)dx=\int_0^4 2\pi(4-x)\left(2-\sqrt{x}\right)dx=2\pi\int_0^4\left(8-4x^{1/2}-2x+x^{3/2}\right)dx$

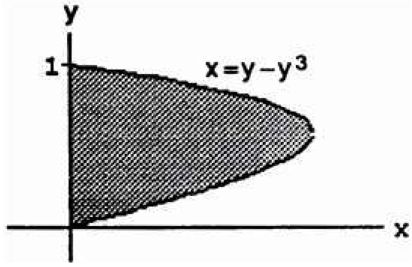
$=2\pi\left[8x-\frac{8}{3}x^{3/2}-x^2+\frac{2}{5}x^{5/2}\right]_0^4=2\pi\left(32-\frac{64}{3}-16+\frac{64}{5}\right)=\frac{2\pi}{15}(240-320+192)=\frac{2\pi}{15}(112)=\frac{224\pi}{15}$

(d) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi(2-y)\left(y^2\right) dy = 2\pi \int_0^2 \left(2y^2 - y^3\right) dy = 2\pi \left[\frac{2}{3}y^3 - \frac{y^4}{4}\right]_0^2$

$= 2\pi\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{32\pi}{12}(4-3) = \frac{8\pi}{3}$

33. (a) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi y\left(y - y^3\right) dy$

$= \int_0^1 2\pi\left(y^2 - y^4\right) dy = 2\pi\left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{4\pi}{15}$




(b) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi(1-y)\left(y - y^3\right) dy$

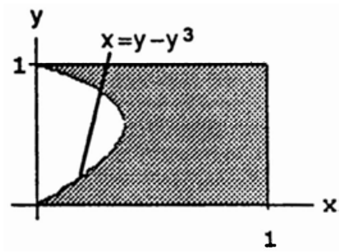
$= 2\pi \int_0^1 \left(y - y^2 - y^3 + y^4\right) dy = 2\pi\left[\frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5}\right]_0^1$

$= 2\pi\left(\frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\right) = \frac{2\pi}{60}(30 - 20 - 15 + 12) = \frac{7\pi}{30}$

34. (a) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi y\left[1 - \left(y - y^3\right)\right] dy$

$= 2\pi \int_0^1 \left(y - y^2 + y^4\right) dy = 2\pi\left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{y^5}{5}\right]_0^1$

$= 2\pi\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{5}\right) = \frac{2\pi}{30}(15 - 10 + 6) = \frac{11\pi}{15}$




(b) Use the washer method:

$V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \int_0^1 \pi\left[1^2 - \left(y - y^3\right)^2\right] dy = \pi \int_0^1 \left(1 - y^2 - y^6 + 2y^4\right) dy$

$= \pi\left[y - \frac{y^3}{3} - \frac{y^7}{7} + \frac{2y^5}{5}\right]_0^1 = \pi\left(1 - \frac{1}{3} - \frac{1}{7} + \frac{2}{5}\right) = \frac{\pi}{105}(105 - 35 - 15 + 42) = \frac{97\pi}{105}$

(c) Use the washer method:

$V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \int_0^1 \pi\left[\left[1 - \left(y - y^3\right)\right]^2 - 0\right] dy = \pi \int_0^1 \left[1 - 2\left(y - y^3\right) + \left(y - y^3\right)^2\right] dy$

$= \pi \int_0^1 \left(1 + y^2 + y^6 - 2y + 2y^3 - 2y^4\right) dy = \pi\left[y + \frac{y^3}{3} + \frac{y^7}{7} - y^2 + \frac{y^4}{2} - \frac{2y^5}{5}\right]_0^1$

$= \pi\left(1 + \frac{1}{3} + \frac{1}{7} - 1 + \frac{1}{2} - \frac{2}{5}\right) = \frac{\pi}{210}(70 + 30 + 105 - 2 \cdot 42) = \frac{121\pi}{210}$

(d) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi(1-y)\left[1 - \left(y - y^3\right)\right] dy = 2\pi \int_0^1 (1-y)\left(1 - y + y^3\right) dy$

$= 2\pi \int_0^1 \left(1 - y + y^3 - y + y^2 - y^4\right) dy = 2\pi \int_0^1 \left(1 - 2y + y^2 + y^3 - y^4\right) dy = 2\pi\left[y - y^2 + \frac{y^3}{3} + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$

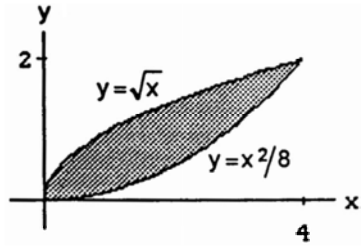
$= 2\pi\left(1 - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{5}\right) = \frac{2\pi}{60}(20 + 15 - 12) = \frac{23\pi}{30}$

35. (a) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^1 2\pi y\left(\sqrt{8y}-y^2\right)dy$

$=2\pi\int_0^2\left(2\sqrt{2}y^{3/2}-y^3\right)dy=2\pi\left[\frac{4\sqrt{2}}{5}y^{5/2}-\frac{y^4}{4}\right]_0^2$

$=2\pi\left(\frac{4\sqrt{2}\cdot(\sqrt{2})^5}{5}-\frac{2^4}{4}\right)=2\pi\left(\frac{4\cdot 2^3}{5}-\frac{4\cdot 4}{4}\right)$

$=2\pi\cdot 4\left(\frac{8}{5}-1\right)=\frac{8\pi}{5}(8-5)=\frac{24\pi}{5}$



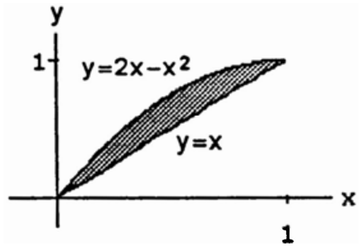

(b) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^4 2\pi x\left(\sqrt{x}-\frac{x^2}{8}\right)dx=2\pi\int_0^4\left(x^{3/2}-\frac{x^3}{8}\right)dx=2\pi\left[\frac{2}{5}x^{5/2}-\frac{x^4}{32}\right]_0^4$

$=2\pi\left(\frac{2\cdot 2^5}{5}-\frac{4^4}{32}\right)=2\pi\left(\frac{2^6}{5}-\frac{2^8}{32}\right)=\frac{\pi\cdot 2^7}{160}(32-20)=\frac{\pi\cdot 2^9\cdot 3}{160}=\frac{\pi\cdot 2^4\cdot 3}{5}=\frac{48\pi}{5}$

36. (a) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^1 2\pi x\left[\left(2x-x^2\right)-x\right]dx$

$=2\pi\int_0^1 x\left(x-x^2\right)dx=2\pi\int_0^1\left(x^2-x^3\right)dx$

$=2\pi\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=2\pi\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{\pi}{6}$




(b) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^1 2\pi(1-x)\left[\left(2x-x^2\right)-x\right]dx=2\pi\int_0^1(1-x)\left(x-x^2\right)dx$

$=2\pi\int_0^1\left(x-2x^2+x^3\right)dx=2\pi\left[\frac{x^2}{2}-\frac{2}{3}x^3+\frac{x^4}{4}\right]_0^1=2\pi\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)=\frac{2\pi}{12}(6-8+3)=\frac{\pi}{6}$

37. (a) $V=\int_a^b \pi\left([R(x)]^2-[r(x)]^2\right)dx=\pi\int_{1/16}^1\left(x^{-1/2}-1\right)dx$

$=\pi\left[2x^{1/2}-x\right]_{1/16}^1=\pi\left[(2-1)-\left(2\cdot\frac{1}{4}-\frac{1}{16}\right)\right]$

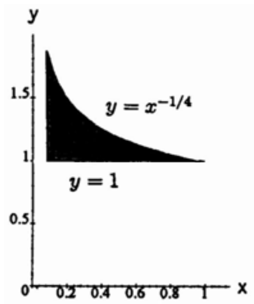
$=\pi\left(1-\frac{7}{16}\right)=\frac{9\pi}{16}$

(b) $V=\int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_0^1 2\pi y\left(\frac{1}{y^4}-\frac{1}{16}\right)dy$

$=2\pi\int_1^2\left(y^{-3}-\frac{y}{16}\right)dy=2\pi\left[-\frac{1}{2}y^{-2}-\frac{y^2}{32}\right]_1^2$

$=2\pi\left[\left(-\frac{1}{8}-\frac{1}{8}\right)-\left(-\frac{1}{2}-\frac{1}{32}\right)\right]=2\pi\left(\frac{1}{4}+\frac{1}{32}\right)$

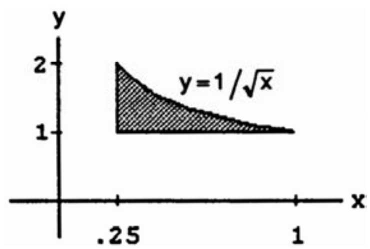
$=\frac{2\pi}{32}(8+1)=\frac{9\pi}{16}$




38. (a) $V=\int_c^d \pi\left([R(y)]^2-[r(y)]^2\right)dy=\int_1^2\pi\left(\frac{1}{y^4}-\frac{1}{16}\right)dy$

$=\pi\left[-\frac{1}{3}y^{-3}-\frac{y}{16}\right]_1^2=\pi\left[\left(-\frac{1}{24}-\frac{1}{8}\right)-\left(-\frac{1}{3}-\frac{1}{16}\right)\right]$

$=\frac{\pi}{48}(-2-6+16+3)=\frac{11\pi}{48}$



(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_{1/4}^{1} 2\pi\left(\frac{1}{\sqrt{x}} - 1\right) dx = 2\pi \int_{1/4}^{1} \left(x^{1/2} - x\right) dx = 2\pi\left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_{1/4}^{1}$
$= 2\pi\left[\left(\frac{2}{3} - \frac{1}{2}\right) - \left(\frac{2}{3}\cdot\frac{1}{8} - \frac{1}{32}\right)\right] = \pi\left(\frac{4}{3} - 1 - \frac{1}{6} + \frac{1}{16}\right) = \frac{\pi}{48}(4\cdot 16 - 48 - 8 + 3) = \frac{11\pi}{48}$

39. (a) *Disk*: $V = V_1 - V_2$
$V_1 = \int_{a_1}^{b_1} \pi\left[R_1(x)\right]^2 dx$ and $V_2 = \int_{a_2}^{b_2} \pi\left[R_2(x)\right]^2 dx$ with $R_1(x) = \sqrt{\frac{x+2}{3}}$ and $R_2(x) = \sqrt{x}$,
$a_1 = -2, b_1 = 1; a_2 = 0, b_2 = 1 \Rightarrow$ two integrals are required

(b) *Washer*: $V = V_1 + V_2$
$V_1 = \int_{a_1}^{b_1} \pi\left(\left[R_1(x)\right]^2 - \left[r_1(x)\right]^2\right) dx$ with $R_1(x) = \sqrt{\frac{x+2}{3}}$ and $r_1(x) = 0$; $a_1 = -2$ and $b_1 = 0$;
$V_2 = \int_{a_2}^{b_2} \pi\left(\left[R_2(x)\right]^2 - \left[r_2(x)\right]^2\right) dx$ with $R_2(x) = \sqrt{\frac{x+2}{3}}$ and $r_2(x) = \sqrt{x}$; $a_2 = 0$ and $b_2 = 1$
$\Rightarrow$ two integrals are required

(c) *Shell*: $V = \int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_c^d 2\pi y \left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy$ where shell height $= y^2 - \left(3y^2 - 2\right) = 2 - 2y^2$;
$c = 0$ and $d = 1$. Only *one* integral is required. It is, therefore preferable to use the *shell* method.
However, whichever method you use, you will get $V = \pi$.

40. (a) *Disk*: $V = V_1 - V_2 - V_3$
$V_i = \int_{c_i}^{d_i} \pi\left[R_i(y)\right]^2 dy$, $i = 1, 2, 3$ with $R_1(y) = 1$ and $c_1 = -1, d_1 = 1$; $R_2(y) = \sqrt{y}$ and $c_2 = 0$ and $d_2 = 1$;
$R_3(y) = (-y)^{1/4}$ and $c_3 = -1, d_3 = 0 \Rightarrow$ three integrals are required

(b) *Washer*: $V = V_1 + V_2$
$V_i = \int_{c_i}^{d_i} \pi\left(\left[R_i(y)\right]^2 - \left[r_i(y)\right]^2\right) dy$, $i = 1, 2$ with $R_1(y) = 1, r_1(y) = \sqrt{y}, c_1 = 0$ and $d_1 = 1$;
$R_2(y) = 1$, $r_2(y) = (-y)^{1/4}$, $c_2 = -1$ and $d_2 = 0 \Rightarrow$ two integrals are required

(c) *Shell*: $V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_a^b 2\pi x \left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx$, where shell height $= x^2 - \left(-x^4\right) = x^2 + x^4$, $a = 0$
and $b = 1 \Rightarrow$ only one integral is required. It is, therefore preferable to use the *shell* method.
However, whichever method you use, you will get $V = \frac{5\pi}{6}$.

41. (a) $V = \int_a^b \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right) dx = \int_{-4}^{4} \pi\left[\left(\sqrt{25 - x^2}\right)^2 - (3)^2\right] dx = \pi\int_{-4}^{4}\left(25 - x^2 - 9\right) dx = \pi\int_{-4}^{4}\left(16 - x^2\right) dx$
$= \pi\left[16x - \frac{1}{3}x^3\right]_{-4}^{4} = \pi\left(64 - \frac{64}{3}\right) - \pi\left(-64 + \frac{64}{3}\right) = \frac{256\pi}{3}$

(b) Volume of sphere $= \frac{4}{3}\pi(5)^3 = \frac{500\pi}{3} \Rightarrow$ Volume of portion removed $= \frac{500\pi}{3} - \frac{256\pi}{3} = \frac{244\pi}{3}$

42. $V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_1^{\sqrt{1+\pi}} 2\pi\, x \sin\left(x^2 - 1\right) dx$; $\left[u = x^2 - 1 \Rightarrow du = 2x\, dx;\right.$
$x = 1 \Rightarrow u = 0, x = \sqrt{1+\pi} \Rightarrow u = \pi\left.\right] \to \pi\int_0^{\pi} \sin u\, du = -\pi\left[\cos u\right]_0^{\pi} = -\pi(-1-1) = 2\pi$