CHAPTER 6 APPLICATIONS OF DEFINITE INTEGRALS

6.1 VOLUMES USING CROSS-SECTIONS

1. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left(\sqrt{x}-\left(-\sqrt{x}\right)\right)^2}{2} = 2x;\quad a = 0, b = 4;\ V = \int_a^b A(x)\,dx = \int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16$

2. $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi\left[\left(2-x^2\right)-x^2\right]^2}{4} = \frac{\pi\left[2\left(1-x^2\right)\right]^2}{4} = \pi\left(1-2x^2+x^4\right);\quad a = -1, b = 1;$

   $V = \int_a^b A(x)\,dx = \int_{-1}^1 \pi\left(1-2x^2+x^4\right)dx = \pi\left[x - \frac{2}{3}x^3 + \frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{16\pi}{15}$

3. $A(x) = (\text{edge})^2 = \left[\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right]^2 = \left(2\sqrt{1-x^2}\right)^2 = 4\left(1-x^2\right);\quad a = -1, b = 1;$

   $V = \int_a^b A(x)\,dx = \int_{-1}^1 4\left(1-x^2\right)dx = 4\left[x - \frac{x^3}{3}\right]_{-1}^1 = 8\left(1-\frac{1}{3}\right) = \frac{16}{3}$

4. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left[\sqrt{1-x^2}-\left(-\sqrt{1-x^2}\right)\right]^2}{2} = 2\frac{\left(2\sqrt{1-x^2}\right)^2}{2} = 2\left(1-x^2\right);\quad a = -1, b = 1;$

   $V = \int_a^b A(x)\,dx = 2\int_{-1}^1 \left(1-x^2\right)dx = 2\left[x - \frac{x^3}{3}\right]_{-1}^1 = 4\left(1-\frac{1}{3}\right) = \frac{8}{3}$

5. (a) STEP 1) $A(x) = \frac{1}{2}(\text{side})\cdot(\text{side})\cdot\left(\sin\frac{\pi}{3}\right) = \frac{1}{2}\cdot\left(2\sqrt{\sin x}\right)\cdot\left(2\sqrt{\sin x}\right)\left(\sin\frac{\pi}{3}\right) = \sqrt{3}\sin x$

   STEP 2) $a = 0, b = \pi$

   STEP 3) $V = \int_a^b A(x)\,dx = \sqrt{3}\int_0^\pi \sin x\,dx = \left[-\sqrt{3}\cos x\right]_0^\pi = \sqrt{3}(1+1) = 2\sqrt{3}$

   (b) STEP 1) $A(x) = (\text{side})^2 = \left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right) = 4\sin \text{x}$

   STEP 2) $a = 0, b = \pi$

   STEP 3) $V = \int_a^b A(x)\,dx = \int_0^\pi 4\sin x\,dx = \left[-4\cos x\right]_0^\pi = 8$

6. (a) STEP 1) $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi}{4}(\sec x - \tan x)^2 = \frac{\pi}{4}\left(\sec^2 x + \tan^2 x - 2\sec x\tan x\right)$

   $= \frac{\pi}{4}\left[\sec^2 x + \left(\sec^2 x - 1\right) - 2\frac{\sin x}{\cos^2 x}\right]$

   STEP 2) $a = -\frac{\pi}{3}, b = \frac{\pi}{3}$

   STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3} \frac{\pi}{4}\left(2\sec^2 x - 1 - \frac{2\sin x}{\cos^2 x}\right)dx = \frac{\pi}{4}\left[2\tan x - x + 2\left(-\frac{1}{\cos x}\right)\right]_{-\pi/3}^{\pi/3}$

   $= \frac{\pi}{4}\left[2\sqrt{3} - \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right) - \left(-2\sqrt{3} + \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right)\right)\right] = \frac{\pi}{4}\left(4\sqrt{3} - \frac{2\pi}{3}\right)$

(b) STEP 1) $A(x) = (\text{edge})^2 = (\sec x - \tan x)^2 = \left(2\sec^2 x - 1 - 2\frac{\sin x}{\cos^2 x}\right)$

STEP 2) $a = -\frac{\pi}{3}, b = \frac{\pi}{3}$

STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3}\left(2\sec^2 x - 1 - \frac{2\sin x}{\cos^2 x}\right)dx = 2\left(2\sqrt{3} - \frac{\pi}{3}\right) = 4\sqrt{3} - \frac{2\pi}{3}$

7. (a) STEP 1) $A(x) = (\text{length})\cdot(\text{height}) = (6-3x)\cdot(10) = 60 - 30x$

STEP 2) $a = 0, b = 2$

STEP 3) $V = \int_a^b A(x)\,dx = \int_0^2 (60-30x)\,dx = \left[60x - 15x^2\right]_0^2 = (120-60) - 0 = 60$

(b) STEP 1) $A(x) = (\text{length})\cdot(\text{height}) = (6-3x)\cdot\left(\frac{20-2(6-3x)}{2}\right) = (6-3x)(4+3x) = 24 + 6x - 9x^2$

STEP 2) $a = 0, b = 2$

STEP 3) $V = \int_a^b A(x)dx = \int_0^2\left(24 + 6x + 9x^2\right)dx = \left[24x + 3x^2 - 3x^3\right]_0^2 = (48 + 12 - 24) - 0 = 36$

8. (a) STEP 1) $A(x) = \frac{1}{2}(\text{base})\cdot(\text{height}) = \left(\sqrt{x} - \frac{x}{2}\right)\cdot(6) = 6\sqrt{x} - 3x$

STEP 2) $a = 0, b = 4$

STEP 3) $V = \int_a^b A(x)\,dx = \int_0^4\left(6x^{1/2} - 3x\right)dx = \left[4x^{3/2} - \frac{3}{2}x^2\right]_0^4 = (32-24) - 0 = 8$

(b) STEP 1) $A(x) = \frac{1}{2}\cdot\pi\left(\frac{\text{diameter}}{2}\right)^2 = \frac{1}{2}\cdot\pi\left(\frac{\sqrt{x}-\frac{x}{2}}{2}\right)^2 = \frac{\pi}{2}\cdot\frac{x - x^{3/2} + \frac{1}{4}x^2}{4} = \frac{\pi}{8}\left(x - x^{3/2} + \frac{1}{4}x^2\right)$

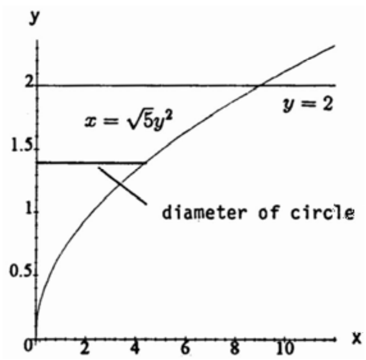
STEP 2) $a = 0, b = 4$

STEP 3) $V = \int_a^b A(x)\,dx = \frac{\pi}{8}\int_0^4\left(x - x^{3/2} + \frac{1}{4}x^2\right)dx = \left[\frac{1}{2}x^2 - \frac{2}{5}x^{5/2} + \frac{1}{12}x^3\right]_0^4 = \frac{\pi}{8}\left(8 - \frac{64}{5} + \frac{16}{3}\right) - \frac{\pi}{8}(0) = \frac{\pi}{15}$

9. $A(y) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{5}y^2 - 0\right)^2 = \frac{5\pi}{4}y^4;$

$c = 0,\ d = 2;\ V = \int_c^d A(y)\,dy$

$= \int_0^2 \frac{5\pi}{4}y^4\,dy = \left[\left(\frac{5\pi}{4}\right)\left(\frac{y^5}{5}\right)\right]_0^2 = \frac{\pi}{4}\left(2^5 - 0\right) = 8\pi$



10. $A(y) = \frac{1}{2}(\text{leg})(\text{leg}) = \frac{1}{2}\left[\sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right)\right]^2 = \frac{1}{2}\left(2\sqrt{1-y^2}\right)^2 = 2\left(1-y^2\right);\ c = -1, d = 1;$

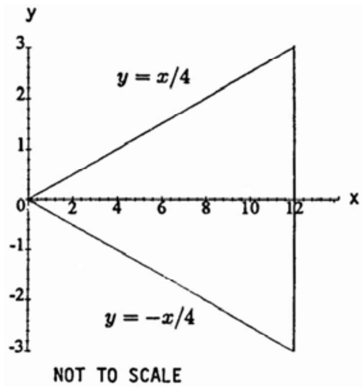
$V = \int_c^d A(y)\,dy = \int_{-1}^1 2\left(1-y^2\right)dy = 2\left[y - \frac{y^3}{3}\right]_{-1}^1 = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$

11. The slices perpendicular to the edge labeled 5 are triangles, and by similar triangles we have $\frac{b}{h}=\frac{4}{3} \Rightarrow h=\frac{3}{4}b$. The equation of the line through $(5, 0)$ and $(0, 4)$ is $y=-\frac{4}{5}x+4$, thus the length of the base $=-\frac{4}{5}x+4$ and the height $=\frac{3}{4}\left(-\frac{4}{5}x+4\right)=-\frac{3}{5}x+3$. Thus $A(x)=\frac{1}{2}(\text{base})\cdot(\text{height})=\frac{1}{2}\left(-\frac{4}{5}x+4\right)\cdot\left(-\frac{3}{5}x+3\right)$
$=\frac{6}{25}x^2-\frac{12}{5}x+6$ and $V=\int_a^b A(x)\,dx=\int_0^5\left(\frac{6}{25}x^2-\frac{12}{5}x+6\right)dx=\left[\frac{2}{25}x^3-\frac{6}{5}x^2+6x\right]_0^5=(10-30+30)-0=10$

12. The slices parallel to the base are squares. The cross section of the pyramid is a triangle, and by similar triangles we have $\frac{b}{h}=\frac{3}{5} \Rightarrow b=\frac{3}{5}h$. Thus $A(y)=(\text{base})^2=\left(\frac{3}{5}y\right)^2=\frac{9}{25}y^2 \Rightarrow V=\int_c^d A(y)\,dy=\int_0^5\frac{9}{25}y^2dy$
$=\left[\frac{3}{25}y^3\right]_0^5=15-0=15$

13. (a) It follows from Cavalieri's Principle that the volume of a column is the same as the volume of a right prism with a square base of side length s and altitude h. Thus,
STEP 1) $A(x)=(\text{sidelength})^2=s^2$;
STEP 2) $a=0, b=h$;
STEP 3) $V=\int_a^b A(x)\,dx=\int_0^h s^2\,dx=s^2h$
(b) From Cavalieri's Principle we conclude that the volume of the column is the same as the volume of the prism described above, regardless of the number of turns $\Rightarrow V=s^2h$

14. 1) The solid and the cone have the same altitude of 12.
2) The cross sections of the solid are disks of diameter $x-\left(\frac{x}{2}\right)=\frac{x}{2}$. If we place the vertex of the cone at the origin of the coordinate system and make its axis or symmetry coincide with the x-axis then the cone's cross sections will be circular disks of diameter $\frac{x}{4}-\left(-\frac{x}{4}\right)=\frac{x}{2}$ (see accompanying figure).
3) The solid and the cone have equal altitudes and identical parallel cross sections. From Cavalier's Principle we conclude that the solid and the cone have the same volume.



15. $R(x)=y=1-\frac{x}{2} \Rightarrow V=\int_0^2\pi[R(x)]^2dx=\pi\int_0^2\left(1-\frac{x}{2}\right)^2dx=\pi\int_0^2\left(1-x+\frac{x^2}{4}\right)dx=\pi\left[x-\frac{x^2}{2}+\frac{x^3}{12}\right]_0^2$
$=\pi\left(2-\frac{4}{2}+\frac{8}{12}\right)=\frac{2\pi}{3}$

16. $R(y)=x=\frac{3y}{2} \Rightarrow V=\int_0^2\pi[R(y)]^2dy=\pi\int_0^2\left(\frac{3y}{2}\right)^2dy=\pi\int_0^2\frac{9}{4}y^2dy=\pi\left[\frac{3}{4}y^3\right]_0^2=\pi\cdot\frac{3}{4}\cdot 8=6\pi$

17. $R(y)=\tan\left(\frac{\pi}{4}y\right)$; $u=\frac{\pi}{4}y \Rightarrow du=\frac{\pi}{4}dy \Rightarrow 4\,du=\pi\,dy$; $y=0 \Rightarrow u=0$, $y=1 \Rightarrow u=\frac{\pi}{4}$;
$V=\int_0^1\pi[R(y)]^2dy=\pi\int_0^1\left[\tan\left(\frac{\pi}{4}y\right)\right]^2dy=4\int_0^{\pi/4}\tan^2 u\,du=4\int_0^{\pi/4}\left(-1+\sec^2 u\right)du=4[-u+\tan u]_0^{\pi/4}$
$=4\left(-\frac{\pi}{4}+1-0\right)=4-\pi$

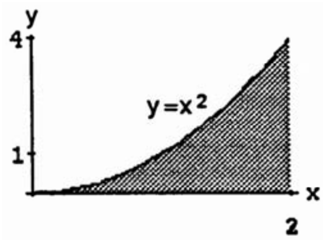
18. $R(x) = \sin x \cos x$; $R(x) = 0 \Rightarrow a = 0$ and $b = \frac{\pi}{2}$ are the limits of integration;

$V = \int_0^{\pi/2} \pi [R(x)]^2 dx = \pi \int_0^{\pi/2} (\sin x \cos x)^2 dx = \pi \int_0^{\pi/2} \frac{(\sin 2x)^2}{4} dx$; $\left[u = 2x \Rightarrow du = 2\,dx \Rightarrow \frac{du}{8} = \frac{dx}{4};\right.$

$\left. x = 0 \Rightarrow u = 0, x = \frac{\pi}{2} \Rightarrow u = \pi \right] \to V = \pi \int_0^{\pi} \frac{1}{8} \sin^2 u\, du = \frac{\pi}{8}\left[\frac{u}{2} - \frac{1}{4}\sin 2u\right]_0^{\pi} = \frac{\pi}{8}\left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{\pi^2}{16}$

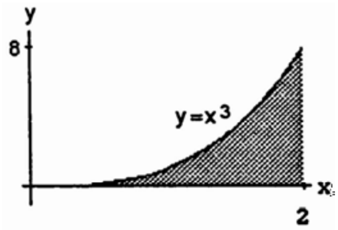
19. $R(x) = x^2 \Rightarrow V = \int_0^2 \pi [R(x)]^2 dx$

$= \pi \int_0^2 \left(x^2\right)^2 dx = \pi \int_0^2 x^4 dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$



20. $R(x) = x^3 \Rightarrow V = \int_0^2 \pi [R(x)]^2 dx$

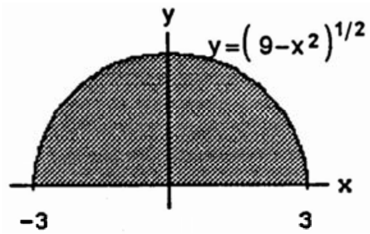
$= \pi \int_0^2 \left(x^3\right)^2 dx = \pi \int_0^2 x^6 dx = \pi \left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$



21. $R(x) = \sqrt{9 - x^2} \Rightarrow V = \int_{-3}^{3} \pi [R(x)]^2 dx$

$= \pi \int_{-3}^{3} \left(9 - x^2\right) dx = \pi \left[9x - \frac{x^3}{3}\right]_{-3}^{3}$

$= 2\pi \left[9(3) - \frac{27}{3}\right] = 2 \cdot \pi \cdot 18 = 36\pi$

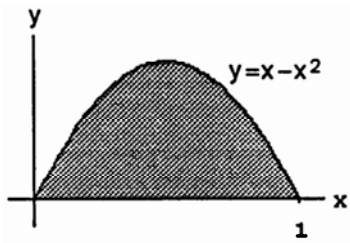


22. $R(x) = x - x^2 \Rightarrow V = \int_0^1 \pi [R(x)]^2 dx$

$= \pi \int_0^1 \left(x - x^2\right)^2 dx = \pi \int_0^1 \left(x^2 - 2x^3 + x^4\right) dx$

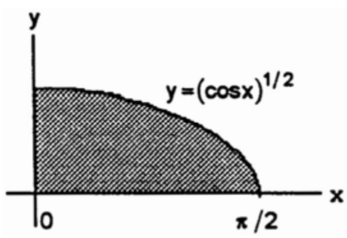
$= \pi \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$

$= \frac{\pi}{30}(10 - 15 + 6) = \frac{\pi}{30}$



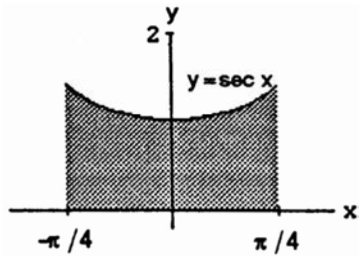
23. $R(x) = \sqrt{\cos x} \Rightarrow V = \int_0^{\pi/2} \pi [R(x)]^2 dx$

$= \pi \int_0^{\pi/2} \cos x\, dx = \pi [\sin x]_0^{\pi/2} = \pi(1 - 0) = \pi$



24. $R(x) = \sec x \Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi \left[R(x)\right]^2 dx$

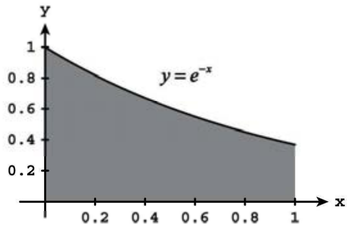
$= \pi \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx = \pi \left[\tan x\right]_{-\pi/4}^{\pi/4} = \pi[1-(-1)] = 2\pi$



25. $R(x) = e^{-x} \Rightarrow V = \int_0^1 \pi[R(x)]^2 dx = \pi \int_0^1 (e^{-x})^2 dx$

$= \pi \int_0^1 e^{-2x} dx = -\frac{\pi}{2} e^{-2x}\Big|_0^1 = -\frac{\pi}{2}(e^{-2} - 1)$

$= \frac{\pi}{2}\left(1 - \frac{1}{e^2}\right) = \frac{\pi(e^2-1)}{2e^2}$



26. $R(x) = \sqrt{\cot x} \Rightarrow V = \int_{\pi/6}^{\pi/2} \pi[R(x)]^2 dx = \pi \int_{\pi/6}^{\pi/2} \left(\sqrt{\cot x}\right)^2 dx = \pi \int_{\pi/6}^{\pi/2} \cot x \, dx = \pi \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x} dx = \pi[\ln(\sin x)]_{\pi/6}^{\pi/2}$

$= \pi\left(\ln 1 - \ln \frac{1}{2}\right) = \pi \ln 2$

27. $R(x) = \frac{1}{2\sqrt{x}} \Rightarrow V = \int_{1/4}^{4} \pi[R(x)]^2 dx = \pi \int_{1/4}^{4} \left(\frac{1}{2\sqrt{x}}\right)^2 dx = \frac{\pi}{4} \int_{1/4}^{4} \frac{1}{x} dx = \frac{\pi}{4}[\ln x]_{1/4}^{4} = \frac{\pi}{4}\left(\ln 4 - \ln \frac{1}{4}\right) = \frac{\pi}{2} \ln 4$

28. $R(x) = e^{x-1} \Rightarrow V = \int_1^3 \pi[R(x)]^2 dx = \pi \int_1^3 (e^{x-1})^2 dx = \pi \int_1^3 e^{2x-2} dx = \frac{\pi}{2}[e^{2x-2}]_1^3 = \frac{\pi}{2}(e^4 - 1) \approx 84.19$

29. $R(x) = \sqrt{2} - \sec x \tan x \Rightarrow V = \int_0^{\pi/4} \pi \left[R(x)\right]^2 dx$

$= \pi \int_0^{\pi/4} \left(\sqrt{2} - \sec x \tan x\right)^2 dx$

$= \pi \int_0^{\pi/4} \left(2 - 2\sqrt{2} \sec x \tan x + \sec^2 x \tan^2 x\right) dx$

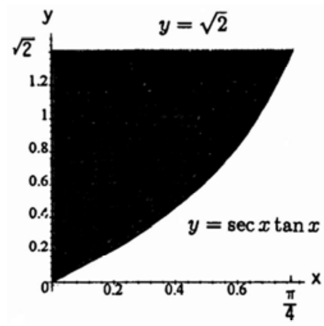
$= \pi \left(\int_0^{\pi/4} 2 \, dx - 2\sqrt{2} \int_0^{\pi/4} \sec x \tan x \, dx \right.$

$\left. + \int_0^{\pi/4} (\tan x)^2 \sec^2 x \, dx \right)$

$= \pi \left([2x]_0^{\pi/4} - 2\sqrt{2}\left[\sec x\right]_0^{\pi/4} + \left[\frac{\tan^3 x}{3}\right]_0^{\pi/4} \right)$

$= \pi\left[\left(\frac{\pi}{2} - 0\right) - 2\sqrt{2}\left(\sqrt{2} - 1\right) + \frac{1}{3}(1^3 - 0)\right]$

$= \pi\left(\frac{\pi}{2} + 2\sqrt{2} - \frac{11}{3}\right)$



30. $R(x) = 2 - 2\sin x = 2(1 - \sin x) \Rightarrow V = \int_0^{\pi/2} \pi [R(x)]^2 dx$

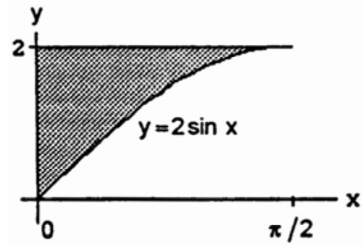
$= \pi \int_0^{\pi/2} 4(1 - \sin x)^2 dx = 4\pi \int_0^{\pi/2} \left(1 + \sin^2 x - 2\sin x\right) dx$

$= 4\pi \int_0^{\pi/2} \left[1 + \frac{1}{2}(1 - \cos 2x) - 2\sin x\right] dx$

$= 4\pi \int_0^{\pi/2} \left(\frac{3}{2} - \frac{\cos 2x}{2} - 2\sin x\right)$

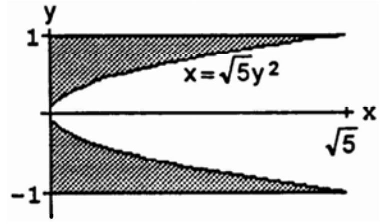
$= 4\pi \left[\frac{3}{2}x - \frac{\sin 2x}{4} + 2\cos x\right]_0^{\pi/2}$

$= 4\pi \left[\left(\frac{3\pi}{4} - 0 + 0\right) - (0 - 0 + 2)\right] = \pi(3\pi - 8)$



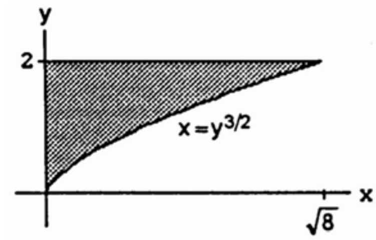
31. $R(y) = \sqrt{5}y^2 \Rightarrow V = \int_{-1}^{1} \pi [R(y)]^2 \, dy = \pi \int_{-1}^{1} 5y^4 \, dy$

$= \pi \left[y^5\right]_{-1}^{1} = \pi[1 - (-1)] = 2\pi$



32. $R(y) = y^{3/2} \Rightarrow V = \int_0^2 \pi [R(y)]^2 dy = \pi \int_0^2 y^3 dy$

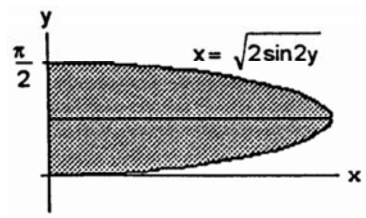
$= \pi \left[\frac{y^4}{4}\right]_0^2 = 4\pi$



33. $R(y) = \sqrt{2\sin 2y} \Rightarrow V = \int_0^{\pi/2} \pi [R(y)]^2 \, dy$

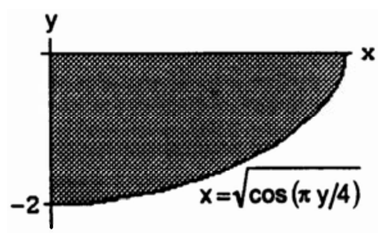
$= \pi \int_0^{\pi/2} 2\sin 2y \, dy = \pi [-\cos 2y]_0^{\pi/2}$

$= \pi[1 - (-1)] = 2\pi$



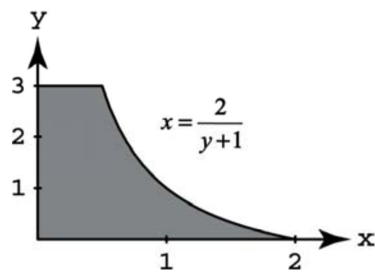
34. $R(y) = \sqrt{\cos \frac{\pi y}{4}} \Rightarrow V = \int_{-2}^{0} \pi [R(y)]^2 dy$

$= \pi \int_{-2}^{0} \cos\left(\frac{\pi y}{4}\right) dy = 4\left[\sin \frac{\pi y}{4}\right]_{-2}^{0} = 4[0 - (-1)] = 4$

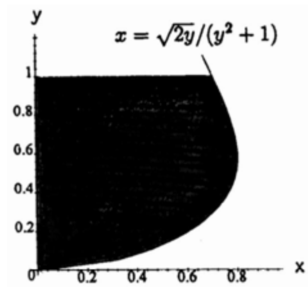


35. $R(y) = \frac{2}{y+1} \Rightarrow V = \int_0^3 \pi [R(y)]^2 \, dy = 4\pi \int_0^3 \frac{1}{(y+1)^2} \, dy$

$= 4\pi \left[-\frac{1}{y+1}\right]_0^3 = 4\pi \left[-\frac{1}{4} - (-1)\right] = 3\pi$



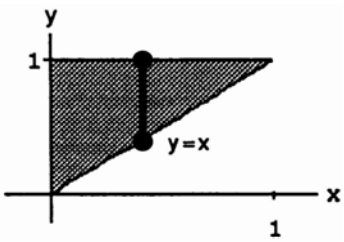
36. $R(y)=\sqrt{\frac{2y}{y^2+1}} \Rightarrow V=\int_0^1 \pi\left[R(y)\right]^2 dy$
$=\pi\int_0^1 2y\left(y^2+1\right)^{-2}dy$; $[u=y^2+1 \Rightarrow du=2y\,dy;$
$y=0\Rightarrow u=1,\ y=1\Rightarrow u=2]$
$\rightarrow V=\pi\int_1^2 u^{-2}du=\pi\left[-\frac{1}{u}\right]_1^2=\pi\left[-\frac{1}{2}-(-1)\right]=\frac{\pi}{2}$



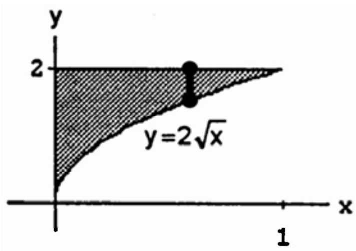
37. For the sketch given, $a=-\frac{\pi}{2}, b=\frac{\pi}{2}$; $R(x)=1$, $r(x)=\sqrt{\cos x}$; $V=\int_a^b \pi\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx$
$=\int_{-\pi/2}^{\pi/2}\pi(1-\cos x)\,dx=2\pi\int_0^{\pi/2}(1-\cos x)\,dx=2\pi\left[x-\sin x\right]_0^{\pi/2}=2\pi\left(\frac{\pi}{2}-1\right)=\pi^2-2\pi$

38. For the sketch given, $c=0, d=\frac{\pi}{4}$; $R(y)=1$, $r(y)=\tan y$; $V=\int_c^d \pi\left(\left[R(y)\right]^2-\left[r(y)\right]^2\right)dy$
$=\pi\int_0^{\pi/4}\left(1-\tan^2 y\right)dy=\pi\int_0^{\pi/4}\left(2-\sec^2 y\right)dy=\pi\left[2y-\tan y\right]_0^{\pi/4}=\pi\left(\frac{\pi}{2}-1\right)=\frac{\pi^2}{2}-\pi$

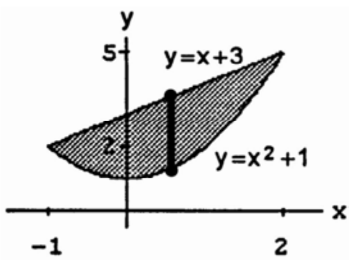
39. $r(x)=x$ and $R(x)=1\Rightarrow V=\int_0^1\pi\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx$
$=\int_0^1\pi\left(1-x^2\right)dx=\pi\left[x-\frac{x^3}{3}\right]_0^1=\pi\left[\left(1-\frac{1}{3}\right)-0\right]=\frac{2\pi}{3}$



40. $r(x)=2\sqrt{x}$ and $R(x)=2\Rightarrow V=\int_0^1\pi\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx$
$=\pi\int_0^1(4-4x)\,dx=4\pi\left[x-\frac{x^2}{2}\right]_0^1=4\pi\left(1-\frac{1}{2}\right)=2\pi$



41. $r(x)=x^2+1$ and $R(x)=x+3$
$\Rightarrow V=\int_{-1}^2\pi\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx$
$=\pi\int_{-1}^2\left[(x+3)^2-\left(x^2+1\right)^2\right]dx$
$=\pi\int_{-1}^2\left[\left(x^2+6x+9\right)-\left(x^4+2x^2+1\right)\right]dx$
$=\pi\int_{-1}^2\left(-x^4-x^2+6x+8\right)dx=\pi\left[-\frac{x^5}{5}-\frac{x^3}{3}+\frac{6x^2}{2}+8x\right]_{-1}^2$
$=\pi\left[\left(-\frac{32}{5}-\frac{8}{3}+\frac{24}{2}+16\right)-\left(\frac{1}{5}+\frac{1}{3}+\frac{6}{2}-8\right)\right]$
$=\pi\left(-\frac{33}{5}-3+28-3+8\right)=\pi\left(\frac{5\cdot 30-33}{5}\right)=\frac{117\pi}{5}$



42. $r(x) = 2 - x$ and $R(x) = 4 - x^2$

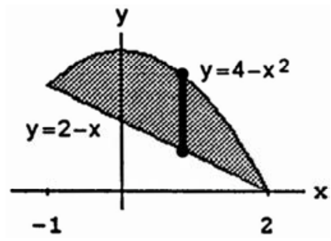
$\Rightarrow V = \int_{-1}^{2} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_{-1}^{2} \left[\left(4 - x^2\right)^2 - (2 - x)^2\right] dx$

$= \pi \int_{-1}^{2} \left[\left(16 - 8x^2 + x^4\right) - \left(4 - 4x + x^2\right)\right] dx$

$= \pi \int_{-1}^{2} \left(12 + 4x - 9x^2 + x^4\right) dx = \pi \left[12x + 2x^2 - 3x^3 + \frac{x^5}{5}\right]_{-1}^{2}$

$= \pi\left[\left(24 + 8 - 24 + \frac{32}{5}\right) - \left(-12 + 2 + 3 - \frac{1}{5}\right)\right] = \pi\left(15 + \frac{33}{5}\right) = \frac{108\pi}{5}$

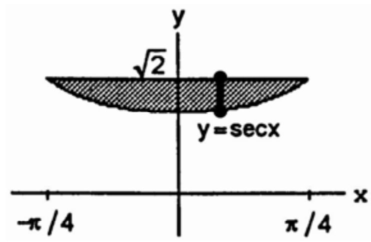


43. $r(x) = \sec x$ and $R(x) = \sqrt{2}$

$\Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

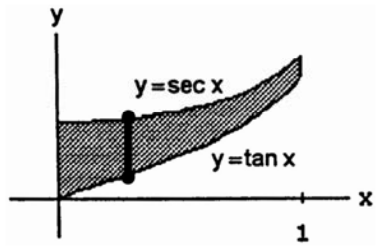
$= \pi \int_{-\pi/4}^{\pi/4} \left(2 - \sec^2 x\right) dx = \pi\left[2x - \tan x\right]_{-\pi/4}^{\pi/4}$

$= \pi\left[\left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right)\right] = \pi(\pi - 2)$



44. $R(x) = \sec x$ and $r(x) = \tan x \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

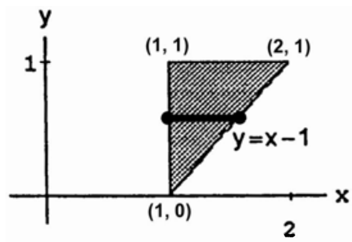
$= \pi \int_0^1 \left(\sec^2 x - \tan^2 x\right) dx = \pi \int_0^1 1\, dx = \pi\left[x\right]_0^1 = \pi$



45. $r(y) = 1$ and $R(y) = 1 + y \Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^1 \left[(1 + y)^2 - 1\right] dy = \pi \int_0^1 \left(1 + 2y + y^2 - 1\right) dy$

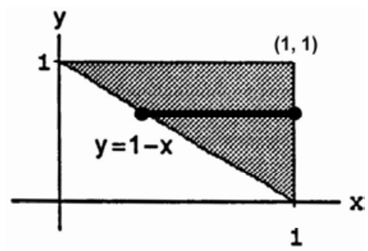
$= \pi \int_0^1 \left(2y + y^2\right) dy = \pi\left[y^2 + \frac{y^3}{3}\right]_0^1 = \pi\left(1 + \frac{1}{3}\right) = \frac{4\pi}{3}$



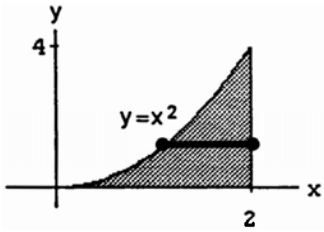
46. $R(y) = 1$ and $r(y) = 1 - y \Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^1 \left[1 - (1 - y)^2\right] dy = \pi \int_0^1 \left[1 - \left(1 - 2y + y^2\right)\right] dy$

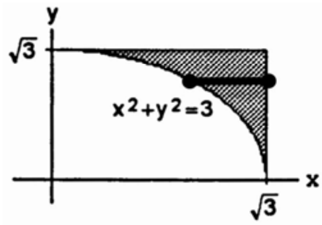
$= \pi \int_0^1 \left(2y - y^2\right) dy = \pi\left[y^2 - \frac{y^3}{3}\right]_0^1 = \pi\left(1 - \frac{1}{3}\right) = \frac{2\pi}{3}$



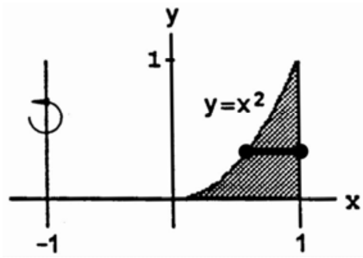
47. $R(y) = 2$ and $r(y) = \sqrt{y} \Rightarrow V = \int_0^4 \pi \left([R(y)]^2 - [r(y)]^2 \right) dy$
$= \pi \int_0^4 (4 - y)\, dy = \pi \left[4y - \frac{y^2}{2} \right]_0^4 = \pi(16 - 8) = 8\pi$



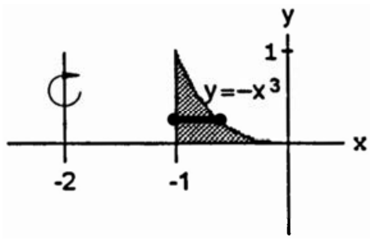
48. $R(y) = \sqrt{3}$ and $r(y) = \sqrt{3 - y^2}$
$\Rightarrow V = \int_0^{\sqrt{3}} \pi \left([R(y)]^2 - [r(y)]^2 \right) dy$
$= \pi \int_0^{\sqrt{3}} \left[3 - \left(3 - y^2\right) \right] dy = \pi \int_0^{\sqrt{3}} y^2\, dy = \pi \left[\frac{y^3}{3} \right]_0^{\sqrt{3}} = \pi\sqrt{3}$



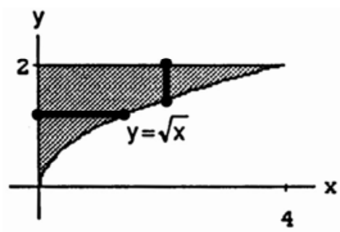
49. $R(y) = 2$ and $r(y) = 1 + \sqrt{y} \Rightarrow V = \int_0^1 \pi \left([R(y)]^2 - [r(y)]^2 \right) dy$
$= \pi \int_0^1 \left[4 - \left(1 + \sqrt{y}\right)^2 \right] dy = \pi \int_0^1 \left(4 - 1 - 2\sqrt{y} - y \right) dy$
$= \pi \int_0^1 \left(3 - 2\sqrt{y} - y \right) dy = \pi \left[3y - \frac{4}{3} y^{3/2} - \frac{y^2}{2} \right]_0^1$
$= \pi \left(3 - \frac{4}{3} - \frac{1}{2} \right) = \pi \left(\frac{18-8-3}{6} \right) = \frac{7\pi}{6}$



50. $R(y) = 2 - y^{1/3}$ and $r(y) = 1 \Rightarrow V = \int_0^1 \pi \left([R(y)]^2 - [r(y)]^2 \right) dy$
$= \pi \int_0^1 \left[\left(2 - y^{1/3}\right)^2 - 1 \right] dy = \pi \int_0^1 \left(4 - 4y^{1/3} + y^{2/3} - 1 \right) dy$
$= \pi \int_0^1 \left(3 - 4y^{1/3} + y^{2/3} \right) dy = \pi \left[3y - 3y^{4/3} + \frac{3y^{5/3}}{5} \right]_0^1$
$= \pi \left(3 - 3 + \frac{3}{5} \right) = \frac{3\pi}{5}$



51. (a) $r(x) = \sqrt{x}$ and $R(x) = 2$
$\Rightarrow V = \int_0^4 \pi \left([R(x)]^2 - [r(x)]^2 \right) dx$
$= \pi \int_0^4 (4 - x)\, dx = \pi \left[4x - \frac{x^2}{2} \right]_0^4 = \pi(16 - 8) = 8\pi$



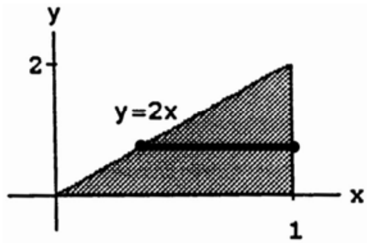
(b) $r(y) = 0$ and $R(y) = y^2 \Rightarrow V = \int_0^2 \pi \left([R(y)]^2 - [r(y)]^2 \right) dy = \pi \int_0^2 y^4 dy = \pi \left[\frac{y^5}{5} \right]_0^2 = \frac{32\pi}{5}$

(c) $r(x) = 0$ and $R(x) = 2 - \sqrt{x} \Rightarrow V = \int_0^4 \pi \left([R(x)]^2 - [r(x)]^2 \right) dx = \pi \int_0^4 \left(2 - \sqrt{x}\right)^2 dx$
$= \pi \int_0^4 \left(4 - 4\sqrt{x} + x \right) dx = \pi \left[4x - \frac{8x^{3/2}}{3} + \frac{x^2}{2} \right]_0^4 = \pi \left(16 - \frac{64}{3} + \frac{16}{2} \right) = \frac{8\pi}{3}$

(d) $r(y) = 4 - y^2$ and $R(y) = 4 \Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi \int_0^2 \left[16 - \left(4 - y^2\right)^2\right] dy$

$= \pi \int_0^2 \left(16 - 16 + 8y^2 - y^4\right) dy = \pi \int_0^2 \left(8y^2 - y^4\right) dy = \pi\left[\frac{8}{3}y^3 - \frac{y^5}{5}\right]_0^2 = \pi\left(\frac{64}{3} - \frac{32}{5}\right) = \frac{224\pi}{15}$

52. (a) $r(y) = 0$ and $R(y) = 1 - \frac{y}{2}$



$\Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^2 \left(1 - \frac{y}{2}\right)^2 dy = \pi \int_0^2 \left(1 - y + \frac{y^2}{4}\right) dy$

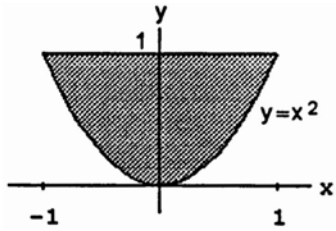
$= \pi\left[y - \frac{y^2}{2} + \frac{y^3}{12}\right]_0^2 = \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$

(b) $r(y) = 1$ and $R(y) = 2 - \frac{y}{2}$

$\Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi \int_0^2 \left[\left(2 - \frac{y}{2}\right)^2 - 1\right] dy = \pi \int_0^2 \left(4 - 2y + \frac{y^2}{4} - 1\right) dy$

$= \pi \int_0^2 \left(3 - 2y + \frac{y^2}{4}\right) dy = \pi\left[3y - y^2 + \frac{y^3}{12}\right]_0^2 = \pi\left(6 - 4 + \frac{8}{12}\right) = \pi\left(2 + \frac{2}{3}\right) = \frac{8\pi}{3}$

53. (a) $r(x) = 0$ and $R(x) = 1 - x^2 \Rightarrow V = \int_{-1}^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$



$= \pi \int_{-1}^1 \left(1 - x^2\right)^2 dx = \pi \int_{-1}^1 \left(1 - 2x^2 + x^4\right) dx$

$= \pi\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 2\pi\left(\frac{15-10+3}{15}\right) = \frac{16\pi}{15}$

(b) $r(x) = 1$ and $R(x) = 2 - x^2 \Rightarrow V = \int_{-1}^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^1 \left[\left(2 - x^2\right)^2 - 1\right] dx$

$= \pi \int_{-1}^1 \left(4 - 4x^2 + x^4 - 1\right) dx = \pi \int_{-1}^1 \left(3 - 4x^2 + x^4\right) dx = \pi\left[3x - \frac{4}{3}x^3 + \frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(3 - \frac{4}{3} + \frac{1}{5}\right)$

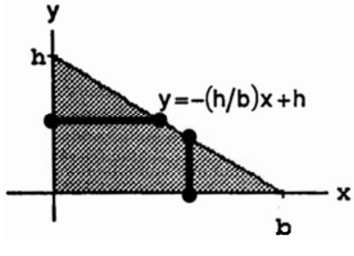
$= \frac{2\pi}{15}(45 - 20 + 3) = \frac{56\pi}{15}$

(c) $r(x) = 1 + x^2$ and $R(x) = 2 \Rightarrow V = \int_{-1}^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^1 \left[4 - \left(1 + x^2\right)^2\right] dx$

$= \pi \int_{-1}^1 \left(4 - 1 - 2x^2 - x^4\right) dx = \pi \int_{-1}^1 \left(3 - 2x^2 - x^4\right) dx = \pi\left[3x - \frac{2}{3}x^3 - \frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(3 - \frac{2}{3} - \frac{1}{5}\right)$

$= \frac{2\pi}{15}(45 - 10 - 3) = \frac{64\pi}{15}$

54. (a) $r(x) = 0$ and $R(x) = -\frac{h}{b}x + h$



$\Rightarrow V = \int_0^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_0^b \left(-\frac{h}{b}x + h\right)^2 dx = \pi \int_0^b \left(\frac{h^2}{b^2}x^2 - \frac{2h^2}{b}x + h^2\right) dx$

$= \pi h^2\left[\frac{x^3}{3b^2} - \frac{x^2}{b} + x\right]_0^b = \pi h^2\left(\frac{b}{3} - b + b\right) = \frac{\pi h^2 b}{3}$