5.5 Solutions to selected problems.

(18) Let $u = 5s+4$ so, $\frac{du}{ds} = 5 \Rightarrow \frac{1}{5}du = ds$

So, $\int \frac{1}{\sqrt{5s+4}}\,ds = \int \frac{1}{\sqrt{u}}\frac{1}{5}du = \frac{1}{5}\int u^{-\frac{1}{2}}du$

$= \frac{1}{5}\cdot\frac{1}{-\frac{1}{2}+1}u^{-\frac{1}{2}+1} + C = \frac{2}{5}u^{\frac{1}{2}} + C$

$= \frac{2}{5}(5s+4)^{\frac{1}{2}} + C = \frac{2}{5}\sqrt{5s+4} + C$

(20) Let $u = 7-3y^2$ so, $\frac{du}{dy} = -6y \Rightarrow \frac{-1}{6y}du = dy$

So, $\int 3y\sqrt{7-3y^2}\,dy = \int 3y\sqrt{u}\cdot\left(-\frac{1}{6y}\right)du = -\frac{1}{2}\int\sqrt{u}\,du$

$= -\frac{1}{2}\int u^{\frac{1}{2}}du = -\frac{1}{2}\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(7-3y^2)^{3/2} + C$

(21) Let $u = 1+\sqrt{x}$ so, $\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow 2\sqrt{x}\,du = dx$

So, $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx = \int \frac{1}{\cancel{\sqrt{x}}\,u^2}2\cancel{\sqrt{x}}\,du = 2\int\frac{1}{u^2}du = 2\int u^{-2}du$

$= 2\cdot\frac{u^{-2+1}}{-2+1} + C = -2u^{-1} + C = -2(1+\sqrt{x})^{-1} + C.$

(22) Let $u = 3z+4$ so, $\frac{du}{dz} = 3 \Rightarrow \frac{1}{3}du = dz$

So, $\int \cos(3z+4)\,dz = \int \cos(u)\cdot\frac{1}{3}du = \frac{1}{3}\int\cos(u)\,du$

$= \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3z+4) + C.$

(24) Let $u = \tan x$, so $\frac{du}{dx} = \sec^2 x \Rightarrow du = \sec^2 x\, dx$

so, $\int \tan^2 x \cdot \sec^2 x\, dx = \int u^2\, du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$

(26) Let $u = \tan\frac{x}{2}$, so $\frac{du}{dx} = \sec^2\frac{x}{2}\cdot\frac{1}{2} \Rightarrow 2du = \sec^2 x\, dx$

So, $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\, dx = 2\int u^7 du = \frac{2}{8}u^8 + C = \frac{1}{4}u^8 + C$

$= \frac{1}{4}\tan^8\frac{x}{2} + C$

(28) Let $u = 7 - \frac{r^5}{10}$, so $\frac{du}{dr} = -\frac{5}{10}r^4 \Rightarrow \frac{du}{dr} = -\frac{1}{2}r^4$

$\Rightarrow du = -\frac{1}{2}r^4\, dr \Rightarrow -\frac{2}{r^4}\, du = dr$

So, $\int r^4\left(7 - \frac{r^5}{10}\right)^3 dr = \int r^4 u^3\left(-\frac{2}{r^4}\, du\right) = -2\int u^3 du$

$= -2\cdot\frac{1}{4}u^4 + C = -\frac{1}{2}u^4 + C = -\frac{1}{2}\left(7 - \frac{r^5}{10}\right)^4 + C$

(30) Let $u = \frac{v - \pi}{2}$, so $\frac{du}{dv} = \frac{1}{2} \Rightarrow 2du = dv$

So, $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right) dv = 2\int \csc(u)\cot(u)\, dv$

$= 2\left[-\csc(u)\right] + C = -2\csc(u) + C$

$= -2\csc\left(\frac{v-\pi}{2}\right) + C$

(32) Let $u = \sec z$, so $\frac{du}{dz} = \sec z\tan z \Rightarrow du = \sec z\tan z\, dz$

So, $\int \frac{\sec z\tan z}{\sqrt{\sec z}}\, dz = \int \frac{1}{\sqrt{u}}\, du = 2\sqrt{u} + C = 2\sqrt{\sec z} + C$

(34) Let $u = \sqrt{t} + 3$, so $\frac{du}{dt} = \frac{1}{2\sqrt{t}} \Rightarrow 2\sqrt{t}\, du = dt$

so, $\int \frac{1}{\sqrt{t}} \cos(\sqrt{t}+3)\, dt = \int \frac{1}{\sqrt{t}} \cos(u) \cdot 2\sqrt{t}\, du = 2\int \cos u\, du$

$= 2\sin(u) + C = 2\sin(\sqrt{t}+3) + C$

(36) Let $u = \sin\sqrt{\theta}$, so $\frac{du}{d\theta} = \cos\sqrt{\theta} \cdot \frac{1}{2\sqrt{\theta}} \Rightarrow \frac{2\sqrt{\theta}}{\cos\sqrt{\theta}}\, du = d\theta$

So, $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\,\sin^2\sqrt{\theta}}\, d\theta = \int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\,u^2} \cdot \frac{2\sqrt{\theta}}{\cos\sqrt{\theta}}\, du = 2\int \frac{1}{u^2}\, du$

$= 2\int u^{-2}\, du = 2 \cdot \frac{u^{-1}}{-1} + C = -\frac{2}{u} + C = -\frac{2}{\sin\sqrt{\theta}} + C$

(38) $\int \sqrt{\frac{x-1}{x^5}}\, dx = \int \sqrt{\frac{1}{x^4}\left(\frac{x-1}{x}\right)}\, dx = \int \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\, dx$

Let $u = 1 - \frac{1}{x}$

$\frac{du}{dx} = -\frac{1}{x^2}$

$-x^2 du = dx$

$= \int \frac{1}{x^2}\sqrt{u} \cdot (-x^2 du) = -\int \sqrt{u}\, du$

$= -\frac{u^{3/2}}{3/2} + C = -\frac{2}{3} u^{3/2} + C = -\frac{2}{3}\left(1-\frac{1}{x}\right)^{3/2} + C$

(40) $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\, dx = \int \frac{1}{x^3}\sqrt{1-\frac{1}{x^2}}\, dx = \int \frac{1}{x^3}\sqrt{u} \cdot \frac{x^3}{2}\, du$

$u = 1 - \frac{1}{x^2}$

$\frac{du}{dx} = \frac{2}{x^3}$

$\frac{x^3}{2} du = dx$

$= \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C$

$= \frac{1}{3} u^{3/2} + C = \frac{1}{3}\left(1-\frac{1}{x^2}\right)^{3/2} + C$

(42) $\int \sqrt{\frac{x^4}{x^3-1}}\,dx = \int \frac{x^2}{\sqrt{x^3-1}}\,dx = \int \frac{x^2}{\sqrt{u}} \cdot \frac{1}{3x^2}\,du$

$u = x^3 - 1$

$\frac{du}{dx} = 3x^2$

$\frac{1}{3x^2}du = dx$

$= \frac{1}{3}\int \frac{1}{\sqrt{u}}\,du = \frac{2}{3}\sqrt{u} + C = \frac{2}{3}\sqrt{x^3-1} + C$

(44) $\int x\sqrt{4-x}\,dx = \int (4-u)\sqrt{u}\,(-du) = -\int (4\sqrt{u} - u\sqrt{u})\,du$

$u = 4 - x \Rightarrow x = 4 - u$

$\frac{du}{dx} = -1$

$-du = dx$

$= -4\int u^{\frac{1}{2}}\,du + \int u^{\frac{3}{2}}\,du$

$= -4 \cdot \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C$

$= -\frac{8}{3}(4-x)^{3/2} + \frac{2}{5}(4-x)^{5/2} + C$

(46) $\int (x+5)(x-5)^{1/3}\,dx = \int (u+10) \cdot u^{1/3}\,du$

$u = x - 5 \Rightarrow u + 5 = x$

$u + 10 = x + 5$

$\frac{du}{dx} = 1$

$= \int u^{4/3}\,du + 10\int u^{1/3}\,du$

$= \frac{3}{7}u^{7/3} + 10 \cdot \frac{3}{4}u^{4/3} + C$

$= \frac{3}{7}(x-5)^{7/3} + \frac{15}{2}(x-5)^{4/3} + C$

(48) $\int 3x^5\sqrt{x^3+1}\,dx = \int 3x^2 \cdot x^3\sqrt{x^3+1}\,dx$

$u = x^3 - 1 \Rightarrow u + 1 = x^3$

$\frac{du}{dx} = 3x^2$

$\frac{1}{3x^2}du = dx$

$= \int 3x^2 \cdot (u+1)\sqrt{u} \cdot \frac{1}{3x^2}\,du$

$= \int (u+1)\sqrt{u}\,du$

$= \int u^{3/2}\,du + \int u^{\frac{1}{2}}\,du$

$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(x^3-1)^{5/2} + \frac{2}{3}(x^3-1)^{3/2} + C$

(50) $\int \frac{x}{(x-4)^3}\,dx = \int \frac{u+4}{u^3}\,du = \int u^{-2} + 4u^{-3}\,du$

$u = x-4 \Rightarrow u+4 = x$

$\frac{du}{dx} = 1$

$du = dx$

$= \frac{u^{-1}}{-1} + \frac{4u^{-2}}{-2} + C$

$= -u^{-1} - 2u^{-2} + C$

$= -(x-4)^{-1} - 2(x-4)^{-2} + C$

(52) $\int (\sin 2\theta) e^{\sin^2\theta}\,d\theta = \int e^{\sin^2\theta} \cdot \sin 2\theta\,d\theta = \int e^u\,du$

$u = \sin^2\theta$

$\frac{du}{d\theta} = 2\sin\theta \cdot \cos\theta$

$\frac{du}{d\theta} = \sin 2\theta$

$du = \sin 2\theta\,d\theta$

$= e^u + C$

$= e^{\sin^2\theta} + C$

(Trig. Identity : $\sin 2\theta = 2\sin\theta\cos\theta$.)

(54) $\int \frac{1}{x^2} e^{\frac{1}{x}} \sec(1+e^{\frac{1}{x}}) \tan(1+e^{\frac{1}{x}})\,dx$

$u = 1 + e^{\frac{1}{x}}$

$\frac{du}{dx} = e^{\frac{1}{x}} \cdot \left(\frac{1}{x}\right)'$

$\frac{du}{dx} = e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)$

$-du = e^{\frac{1}{x}}\left(\frac{1}{x^2}\right)dx$

$= \int \sec(1+e^{\frac{1}{x}}) \tan(1+e^{\frac{1}{x}})\, e^{\frac{1}{x}}\left(\frac{1}{x^2}\right) dx$

$= \int \sec(u)\tan(u) \cdot (-du)$

$= -\int \sec(u)\tan(u)\,du$

$= -\sec(u) + C = -\sec(1+e^{\frac{1}{x}}) + C$

(56) $\int \frac{\ln\sqrt{t}}{t}\,dt = \int \frac{\ln t^{\frac{1}{2}}}{t}\,dt = \int \frac{\frac{1}{2}\ln t}{t}\,dt$

$u = \ln t$

$\frac{du}{dt} = \frac{1}{t}$

$du = \frac{1}{t}\,dt$

$= \frac{1}{2}\int \ln t \, \frac{dt}{t}$

$= \frac{1}{2}\int u\,du = \frac{1}{2} \cdot \frac{1}{2}u^2 + C$

$= \frac{1}{4}u^2 + C = \frac{1}{4}(\ln t)^2 + C$

56 Alternate solution

Let $u = \ln\sqrt{t}$

So, $\frac{du}{dx} = \frac{1}{\sqrt{t}} \cdot \frac{1}{2\sqrt{t}} = \frac{1}{2t}$

$\frac{du}{dt} = \frac{1}{2t}$

$2t\,du = dt$

So, $\int \frac{\ln\sqrt{t}}{t}\,dt = \int \frac{u}{t}(2t\,du)$

$= 2\int u\,du$

$= 2 \cdot \frac{u^{1+1}}{1+1} + C$

$= u^2 + C$

$= (\ln\sqrt{t})^2 + C$

Note that this gives the same answer as previous method because $(\ln\sqrt{t})^2 = (\frac{1}{2}\ln t)^2 = \frac{1}{4}(\ln t)^2$

(58) $\int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{1}{x\sqrt{(x^2)^2-1}}\,dx = \int \frac{x}{x^2\sqrt{(x^2)^2-1}}\,dx$

Let $u = x^2$

$\frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

$= \int \frac{1}{x^2\sqrt{(x^2)^2-1}}\,x\,dx = \int \frac{1}{u\sqrt{u^2-1}}\,\frac{1}{2}\,du$

$= \frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}}\,du = \frac{1}{2}\sec^{-1}(u) + C$

$= \frac{1}{2}\sec^{-1}(x^2) + C$

(60) $\int \frac{1}{\sqrt{e^{2\theta}-1}}\,d\theta = \int \frac{e^\theta}{e^\theta\sqrt{(e^\theta)^2-1}}\,d\theta = \int \frac{1}{e^\theta\sqrt{(e^\theta)^2-1}}\,e^\theta d\theta$

$u = e^\theta$

$\frac{du}{d\theta} = e^\theta$

$du = e^\theta\,d\theta$

$= \int \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1}(u) + C$

$= \sec^{-1}(e^\theta) + C$

(62) $\int \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx = -\int e^u\,du = -e^u + C$

$= -e^{\cos^{-1}x} + C$

$u = \cos^{-1}x$

$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$-du = \frac{1}{\sqrt{1-x^2}}\,dx$

(64) $\int \frac{\sqrt{\tan^{-1} x}}{1+x^2}\, dx = \int \sqrt{\tan^{-1} x} \cdot \frac{1}{1+x^2}\, dx$

$u = \tan^{-1} x$

$\frac{du}{dx} = \frac{1}{1+x^2}$

$du = \frac{1}{1+x^2}\, dx$

$= \int \sqrt{u}\, du$

$= \frac{2}{3} u^{3/2} + C$

$= \frac{2}{3} \left(\tan^{-1} x\right)^{3/2} + C$

(66) $\int \frac{dy}{(\sin^{-1} y)\sqrt{1-y^2}} = \int \frac{1}{(\sin^{-1} y)} \cdot \frac{1}{\sqrt{1-y^2}}\, dy$

$u = \sin^{-1} y$

$\frac{du}{dy} = \frac{1}{\sqrt{1-y^2}}$

$du = \frac{1}{\sqrt{1-y^2}}\, dy$

$= \int \frac{1}{u} \cdot du$

$= \ln(u) + C$

$= \ln(\sin^{-1} y) + C$