```
for n in N do
  Xlist := [ a+1.*(b-a)/n*i $ i=0..n ];
  Ylist := map( f, Xlist );
end do:
for n in N do                                              # (c)
  Avg[n] := evalf(add(y,y=Ylist)/nops(Ylist));
end do;
avg := FunctionAverage( f(x), x=a..b, output=value );
evalf( avg );
FunctionAverage(f(x),x=a..b, output=plot);                # (d)
fsolve( f(x)=avg, x=0.5 );
fsolve( f(x)=avg, x=2.5 );
fsolve( f(x)=Avg[1000], x=0.5 );
fsolve( f(x)=Avg[1000], x=2.5 );
```

Mathematica: (assigned function and values for a and b may vary):
Symbols for π, →, powers, roots, fractions, etc. are available in Palettes.
Never insert a space between the name of a function and its argument.

```
Clear[x]
f[x_]:=x Sin[1/x]
{a, b}={π/4, π}
Plot[f[x],{x, a, b}]
```

The following code computes the value of the function for each interval midpoint and then finds the average. Each sequence of commands for a different value of n (number of subdivisions) should be placed in a separate cell.

```
n =100; dx = (b – a) /n;
values = Table[N[f[x]],{x, a + dx/2, b, dx}]
average=Sum[values[[i]],{i, 1, Length[values]}] / n
n =200; dx = (b – a) /n;
values = Table[N[f[x]],{x, a + dx/2, b, dx}]
average=Sum[values[[i]],{i, 1, Length[values]}] / n
n =1000; dx = (b – a) /n;
values=Table[N[f[x]],{x, a + dx/2, b, dx}]
average=Sum[values[[i]],{i, 1, Length[values]}] / n
FindRoot[f[x] == average,{x, a}]
```

## 5.2 SIGMA NOTATION AND LIMITS OF FINITE SUMS

1. $\sum_{k=1}^{2} \frac{6k}{k+1} = \frac{6(1)}{1+1} + \frac{6(2)}{2+1} = \frac{6}{2} + \frac{12}{3} = 7$

2. $\sum_{k=1}^{3} \frac{k-1}{k} = \frac{1-1}{1} + \frac{2-1}{2} + \frac{3-1}{3} = 0 + \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

3. $\sum_{k=1}^{4} \cos k\pi = \cos(1\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) = -1+1-1+1 = 0$

[redacted]

4. $\sum_{k=1}^{5} \sin k\pi = \sin(1\pi) + \sin(2\pi) + \sin(3\pi) + \sin(4\pi) + \sin(5\pi) = 0+0+0+0+0 = 0$

5. $\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k} = (-1)^{1+1} \sin \frac{\pi}{1} + (-1)^{2+1} \sin \frac{\pi}{2} + (-1)^{3+1} \sin \frac{\pi}{3} = 0 - 1 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-2}{2}$

6. $\sum_{k=1}^{4} (-1)^{k} \cos k\pi = (-1)^{1} \cos(1\pi) + (-1)^{2} \cos(2\pi) + (-1)^{3} \cos(3\pi) + (-1)^{4} \cos(4\pi) = -(-1) + 1 - (-1) + 1 = 4$

7. (a) $\sum_{k=1}^{6} 2^{k-1} = 2^{1-1} + 2^{2-1} + 2^{3-1} + 2^{4-1} + 2^{5-1} + 2^{6-1} = 1+2+4+8+16+32$

   (b) $\sum_{k=0}^{5} 2^{k} = 2^{0} + 2^{1} + 2^{2} + 2^{3} + 2^{4} + 2^{5} = 1+2+4+8+16+32$

   (c) $\sum_{k=1}^{4} 2^{k+1} = 2^{-1+1} + 2^{0+1} + 2^{1+1} + 2^{2+1} + 2^{3+1} + 2^{4+1} = 1+2+4+8+16+32$

   All of them represent $1+2+4+8+16+32$

8. (a) $\sum_{k=1}^{6} (-2)^{k-1} = (-2)^{1-1} + (-2)^{2-1} + (-2)^{3-1} + (-2)^{4-1} + (-2)^{5-1} + (-2)^{6-1} = 1-2+4-8+16-32$

   (b) $\sum_{k=0}^{5} (-1)^{k} 2^{k} = (-1)^{0}2^{0} + (-1)^{1}2^{1} + (-1)^{2}2^{2} + (-1)^{3}2^{3} + (-1)^{4}2^{4} + (-1)^{5}2^{5} = 1-2+4-8+16-32$

   (c) $\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2} = (-1)^{-2+1}2^{-2+2} + (-1)^{-1+1}2^{-1+2} + (-1)^{0+1}2^{0+2} + (-1)^{1+1}2^{1+2} + (-1)^{2+1}2^{2+2} + (-1)^{3+1}2^{3+2}$

   $= -1+2-4+8-16+32$;

   (a) and (b) represent $1-2+4-8+16-32$; (c) is not equivalent to the other two

9. (a) $\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1} = \frac{(-1)^{2-1}}{2-1} + \frac{(-1)^{3-1}}{3-1} + \frac{(-1)^{4-1}}{4-1} = -1 + \frac{1}{2} - \frac{1}{3}$

   (b) $\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1} = \frac{(-1)^{0}}{0+1} + \frac{(-1)^{1}}{1+1} + \frac{(-1)^{2}}{2+1} = 1 - \frac{1}{2} + \frac{1}{3}$

   (c) $\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2} = \frac{(-1)^{-1}}{-1+2} + \frac{(-1)^{0}}{0+2} + \frac{(-1)^{1}}{1+2} = -1 + \frac{1}{2} - \frac{1}{3}$

   (a) and (c) are equivalent; (b) is not equivalent to the other two.

10. (a) $\sum_{k=1}^{4} (k-1)^{2} = (1-1)^{2} + (2-1)^{2} + (3-1)^{2} + (4-1)^{2} = 0+1+4+9$

    (b) $\sum_{k=-1}^{3} (k+1)^{2} = (-1+1)^{2} + (0+1)^{2} + (1+1)^{2} + (2+1)^{2} + (3+1)^{2} = 0+1+4+9+16$

    (c) $\sum_{k=-3}^{-1} k^{2} = (-3)^{2} + (-2)^{2} + (-1)^{2} = 9+4+1$

    (a) and (c) are equivalent to each other; (b) is not equivalent to the other two.

11. $\sum_{k=1}^{6} k$

12. $\sum_{k=1}^{4} k^{2}$

13. $\sum_{k=1}^{4} \frac{1}{2^{k}}$

14. $\sum_{k=1}^{5} 2k$

15. $\sum_{k=1}^{5} (-1)^{k+1} \frac{1}{k}$

16. $\sum_{k=1}^{5} (-1)^{k} \frac{k}{5}$

[redacted]

17. (a) $\sum_{k=1}^{n} 3a_k = 3\sum_{k=1}^{n} a_k = 3(-5) = -15$

(b) $\sum_{k=1}^{n} \frac{b_k}{6} = \frac{1}{6}\sum_{k=1}^{n} b_k = \frac{1}{6}(6) = 1$

(c) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = -5 + 6 = 1$

(d) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k = -5 - 6 = -11$

(e) $\sum_{k=1}^{n} (b_k - 2a_k) = \sum_{k=1}^{n} b_k - 2\sum_{k=1}^{n} a_k = 6 - 2(-5) = 16$

18. (a) $\sum_{k=1}^{n} 8a_k = 8\sum_{k=1}^{n} a_k = 8(0) = 0$

(b) $\sum_{k=1}^{n} 250b_k = 250\sum_{k=1}^{n} b_k = 250(1) = 250$

(c) $\sum_{k=1}^{n} (a_k + 1) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} 1 = 0 + n = n$

(d) $\sum_{k=1}^{n} (b_k - 1) = \sum_{k=1}^{n} b_k - \sum_{k=1}^{n} 1 = 1 - n$

19. (a) $\sum_{k=1}^{10} k = \frac{10(10+1)}{2} = 55$

(b) $\sum_{k=1}^{10} k^2 = \frac{10(10+1)(2(10)+1)}{6} = 385$

(c) $\sum_{k=1}^{10} k^3 = \left[\frac{10(10+1)}{2}\right]^2 = 55^2 = 3025$

20. (a) $\sum_{k=1}^{13} k = \frac{13(13+1)}{2} = 91$

(b) $\sum_{k=1}^{13} k^2 = \frac{13(13+1)(2(13)+1)}{6} = 819$

(c) $\sum_{k=1}^{13} k^3 = \left[\frac{13(13+1)}{2}\right]^2 = 91^2 = 8281$

21. $\sum_{k=1}^{7} -2k = -2\sum_{k=1}^{7} k = -2\left(\frac{7(7+1)}{2}\right) = -56$

22. $\sum_{k=1}^{5} \frac{\pi k}{15} = \frac{\pi}{15}\sum_{k=1}^{5} k = \frac{\pi}{15}\left(\frac{5(5+1)}{2}\right) = \pi$

23. $\sum_{k=1}^{6} (3 - k^2) = \sum_{k=1}^{6} 3 - \sum_{k=1}^{6} k^2 = 3(6) - \frac{6(6+1)(2(6)+1)}{6} = -73$

24. $\sum_{k=1}^{6} (k^2 - 5) = \sum_{k=1}^{6} k^2 - \sum_{k=1}^{6} 5 = \frac{6(6+1)(2(6)+1)}{6} - 5(6) = 61$

25. $\sum_{k=1}^{5} k(3k+5) = \sum_{k=1}^{5} (3k^2 + 5k) = 3\sum_{k=1}^{5} k^2 + 5\sum_{k=1}^{5} k = 3\left(\frac{5(5+1)(2(5)+1)}{6}\right) + 5\left(\frac{5(5+1)}{2}\right) = 240$

26. $\sum_{k=1}^{7} k(2k+1) = \sum_{k=1}^{7} (2k^2 + k) = 2\sum_{k=1}^{7} k^2 + \sum_{k=1}^{7} k = 2\left(\frac{7(7+1)(2(7)+1)}{6}\right) + \frac{7(7+1)}{2} = 308$

27. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3 = \frac{1}{225}\sum_{k=1}^{5} k^3 + \left(\sum_{k=1}^{5} k\right)^3 = \frac{1}{225}\left(\frac{5(5+1)}{2}\right)^2 + \left(\frac{5(5+1)}{2}\right)^3 = 3376$

28. $\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4} = \left(\sum_{k=1}^{7} k\right)^2 - \frac{1}{4}\sum_{k=1}^{7} k^3 = \left(\frac{7(7+1)}{2}\right)^2 - \frac{1}{4}\left(\frac{7(7+1)}{2}\right)^2 = 588$

[redacted]

29. (a) $\sum\limits_{k=1}^{7} 3 = 3(7) = 21$  (b) $\sum\limits_{k=1}^{500} 7 = 7(500) = 3500$

(c) Let $j = k-2 \Rightarrow k = j+2$; if $k = 3 \Rightarrow j = 1$ and if $k = 264 \Rightarrow j = 262 \Rightarrow \sum\limits_{k=3}^{264} 10 = \sum\limits_{j=1}^{262} 10 = 10(262) = 2620$

30. (a) Let $j = k-8 \Rightarrow k = j+8$; if $k = 9 \Rightarrow j = 1$ and if $k = 36 \Rightarrow j = 28 \Rightarrow \sum\limits_{k=9}^{36} k = \sum\limits_{j=1}^{28} (j+8) = \sum\limits_{j=1}^{28} j + \sum\limits_{j=1}^{28} 8$
$= \frac{28(28+1)}{2} + 8(28) = 630$

(b) Let $j = k-2 \Rightarrow k = j+2$; if $k = 3 \Rightarrow j = 1$ and if $k = 17 \Rightarrow j = 15 \Rightarrow \sum\limits_{k=3}^{17} k^2 = \sum\limits_{j=1}^{15} (j+2)^2$
$= \sum\limits_{j=1}^{15} (j^2+4j+4) = \sum\limits_{j=1}^{15} j^2 + \sum\limits_{j=1}^{15} 4j + \sum\limits_{j=1}^{15} 4 = \frac{15(15+1)(2(15)+1)}{6} + 4\cdot\frac{15(15+1)}{2} + 4(15) = 1240 + 480 + 60 = 1780$

(c) Let $j = k-17 \Rightarrow k = j+17$; if $k = 18 \Rightarrow j = 1$ and if $k = 71 \Rightarrow j = 54 \Rightarrow \sum\limits_{k=3}^{71} k(k-1)$
$= \sum\limits_{j=1}^{54} (j+17)((j+17)-1) = \sum\limits_{j=1}^{54} (j^2+33j+272) = \sum\limits_{j=1}^{54} j^2 + \sum\limits_{j=1}^{54} 33j + \sum\limits_{j=1}^{54} 272$
$= \frac{54(54+1)(2(54)+1)}{6} + 33\cdot\frac{54(54+1)}{2} + 272(54) = 53955 + 49005 + 14688 = 117648$

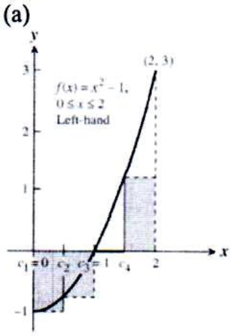
31. (a) $\sum\limits_{k=1}^{n} 4 = 4n$  (b) $\sum\limits_{k=1}^{n} c = cn$

(c) $\sum\limits_{k=1}^{n} (k-1) = \sum\limits_{k=1}^{n} k - \sum\limits_{k=1}^{n} 1 = \frac{n(n+1)}{2} - n = \frac{n^2-n}{2}$

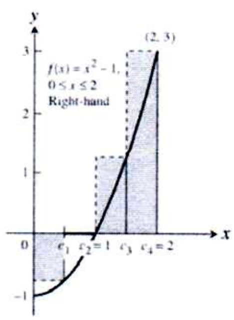
32. (a) $\sum\limits_{k=1}^{n} \left(\frac{1}{n}+2n\right) = \left(\frac{1}{n}+2n\right)n = 1+2n^2$  (b) $\sum\limits_{k=1}^{n} \frac{c}{n} = \frac{c}{n}\cdot n = c$

(c) $\sum\limits_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2n}$

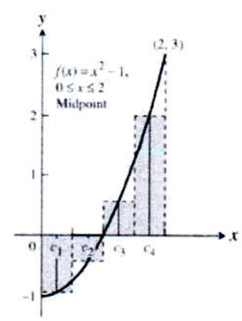
33. (a)




(b)

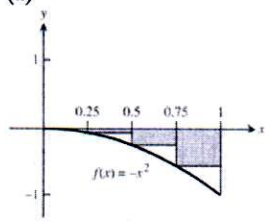



(c)

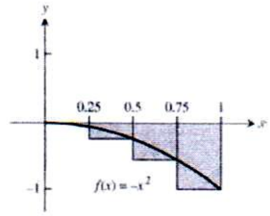



[redacted]

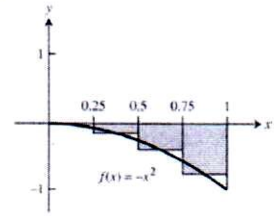
34. (a)



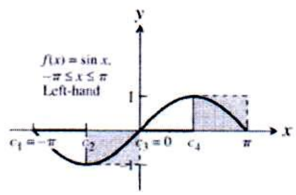

(b)



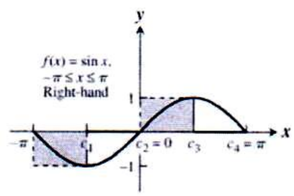

(c)



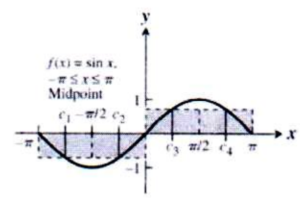

35. (a)



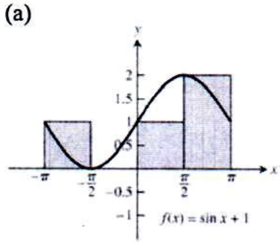

(b)



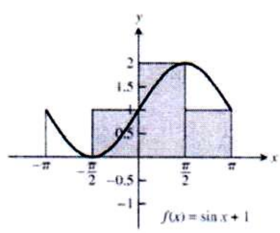

(c)



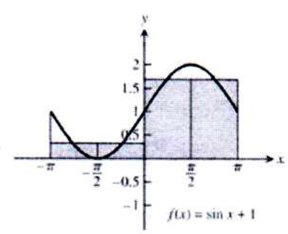

36. (a)




(b)



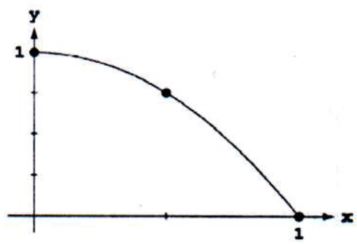

(c)




37. $|x_1 - x_0| = |1.2 - 0| = 1.2$, $|x_2 - x_1| = |1.5 - 1.2| = 0.3$, $|x_3 - x_2| = |2.3 - 1.5| = 0.8$, $|x_4 - x_3| = |2.6 - 2.3| = 0.3$, and $|x_5 - x_4| = |3 - 2.6| = 0.4$; the largest is $\|P\| = 1.2$.

38. $|x_1 - x_0| = |-1.6 - (-2)| = 0.4$, $|x_2 - x_1| = |-0.5 - (-1.6)| = 1.1$, $|x_3 - x_2| = |0 - (-0.5)| = 0.5$, $|x_4 - x_3| = |0.8 - 0| = 0.8$, and $|x_5 - x_4| = |1 - 0.8| = 0.2$; the largest is $\|P\| = 1.1$.

39. $f(x) = 1 - x^2$



Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\Delta x = \frac{i}{n}$. The right-hand sum is

$$\sum_{i=1}^{n}\left(1 - c_i^2\right)\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \left(\frac{i}{n}\right)^2\right) = \frac{1}{n^3}\sum_{i=1}^{n}\left(n^2 - i^2\right)$$

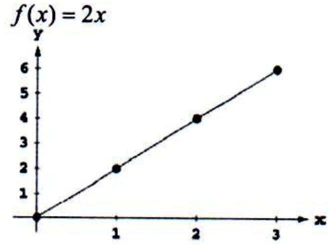
$$= \frac{n^3}{n^3} - \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 1 - \frac{n(n+1)(2n+1)}{6n^3} = 1 - \frac{2n^3+3n^2+n}{6n^3}$$

$$= 1 - \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}. \text{ Thus, } \lim_{n\to\infty}\sum_{i=1}^{n}\left(1 - c_i^2\right)\frac{1}{n}$$

$$= \lim_{n\to\infty}\left(1 - \frac{2-\frac{3}{n}+\frac{1}{n^2}}{6}\right) = 1 - \frac{1}{3} = \frac{2}{3}$$

[redacted]

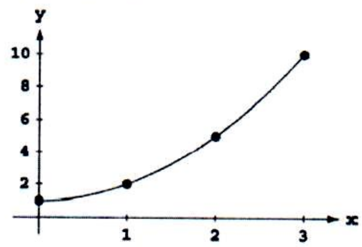
40. $f(x) = 2x$




Let $\Delta x = \frac{3-0}{n} = \frac{3}{n}$ and $c_i = i\Delta x = \frac{3i}{n}$. The right-hand sum is $\sum_{i=1}^{n} 2c_i\left(\frac{3}{n}\right) = \sum_{i=1}^{n} \frac{6i}{n}\cdot\frac{3}{n} = \frac{18}{n^2}\sum_{i=1}^{n} i = \frac{18}{n^2}\cdot\frac{n(n+1)}{2} = \frac{9n^2+9n}{n^2}$.

Thus, $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6i}{n}\cdot\frac{3}{n} = \lim_{n\to\infty} \frac{9n^2+9n}{n^2} = \lim_{n\to\infty}\left(9+\frac{9}{n}\right) = 9$.

41. $f(x) = x^2 + 1$




Let$\Delta x = \frac{3-0}{n} = \frac{3}{n}$ and $c_i = i\Delta x = \frac{3i}{n}$. The right-hand sum is

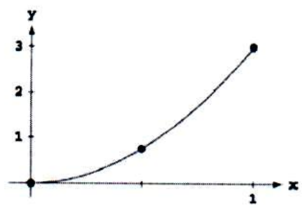
$$\sum_{i=1}^{n}\left(c_i^2+1\right)\frac{3}{n} = \sum_{i=1}^{n}\left(\left(\frac{3i}{n}\right)^2+1\right)\frac{3}{n} = \frac{3}{n}\sum_{i=1}^{n}\left(\frac{9i^2}{n^2}+1\right)$$

$$= \frac{27}{n}\sum_{i=1}^{n} i^2 + \frac{3}{n}\cdot n = \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + 3 = \frac{9(2n^3+3n^2+n)}{2n^3} + 3$$

$$= \frac{18+\frac{27}{n}+\frac{9}{n^2}}{2} + 3.$$ Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(c_i^2+1\right)\frac{3}{n}$

$$= \lim_{n\to\infty}\left(\frac{18+\frac{27}{n}+\frac{9}{n^2}}{2}+3\right) = 9+3 = 12.$$

42. $f(x) = 3x^2$



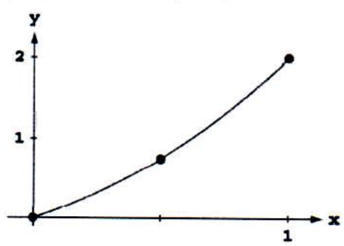

Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\Delta x = \frac{i}{n}$. The right-hand sum is

$$\sum_{i=1}^{n} 3c_i^2\left(\frac{1}{n}\right) = \sum_{i=1}^{n} 3\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) = \frac{3}{n^3}\sum_{i=1}^{n} i^2 = \frac{3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{2n^3+3n^2+n}{2n^3} = \frac{2+\frac{3}{n}+\frac{1}{n^2}}{2}.$$ Thus, $\lim_{n\to\infty}\sum_{i=1}^{n} 3c_i^2\left(\frac{1}{n}\right)$

$$= \lim_{n\to\infty}\left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{2}\right) = \frac{2}{2} = 1.$$

43. $f(x) = x + x^2 = x(1+x)$




Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\Delta x = \frac{i}{n}$. The right-hand sum is

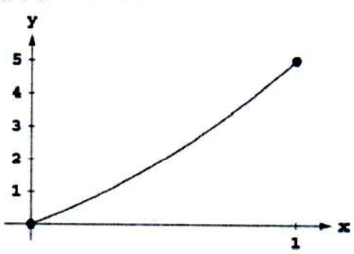
$$\sum_{i=1}^{n}\left(c_i + c_i^2\right)\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{i}{n}+\left(\frac{i}{n}\right)^2\right)\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{n^2+n}{2n^2} + \frac{2n^3+3n^2+n}{6n^3}$$

$$= \frac{1+\frac{1}{n}}{2} + \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}.$$ Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(c_i + c_i^2\right)\frac{1}{n}$

$$= \lim_{n\to\infty}\left[\left(\frac{1+\frac{1}{n}}{2}\right) + \left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}\right)\right] = \frac{1}{2}+\frac{2}{6} = \frac{5}{6}.$$

44. $f(x) = 3x + 2x^2$




Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\Delta x = \frac{i}{n}$. The right-hand sum is

$$\sum_{i=1}^{n}\left(3c_i + 2c_i^2\right)\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{3i}{n}+2\left(\frac{i}{n}\right)^2\right)\frac{1}{n} = \frac{3}{n^2}\sum_{i=1}^{n} i + \frac{2}{n^3}\sum_{i=1}^{n} i^2$$

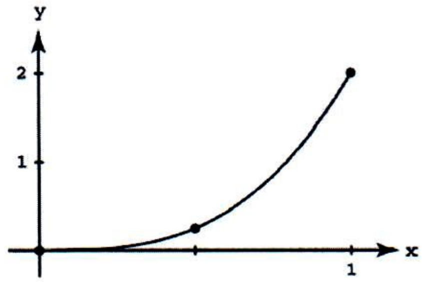
$$= \frac{3}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{2}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{3n^2+3n}{2n^2} + \frac{2n^2+3n+1}{3n^2}$$

$$= \frac{3+\frac{3}{n}}{2} + \frac{2+\frac{3}{n}+\frac{1}{n^2}}{3}.$$ Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(3c_i + 2c_i^2\right)\frac{1}{n}$

$$= \lim_{n\to\infty}\left[\left(\frac{3+\frac{3}{n}}{2}\right) + \left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{3}\right)\right] = \frac{3}{2}+\frac{2}{3} = \frac{13}{6}.$$

[redacted]

45. $f(x) = 2x^3$



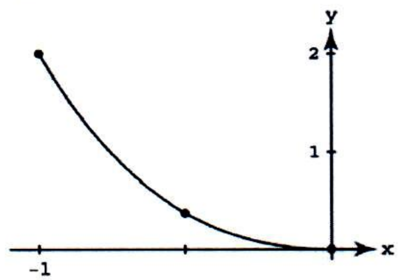

Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $c_i = i\Delta x = \frac{i}{n}$. The right-hand sum is

$$\sum_{i=1}^{n}\left(2c_i^3\right)\frac{1}{n} = \sum_{i=1}^{n}\left(2\left(\frac{i}{n}\right)^3\right)\frac{1}{n} = \frac{2}{n^4}\sum_{i=1}^{n} i^3 = \frac{2}{n^4}\left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{2n^2(n^2+2n+1)}{4n^4} = \frac{n^2+2n+1}{2n^2} = \frac{1+\frac{2}{n}+\frac{1}{n^2}}{2}.$$ Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(2c_i^3\right)\frac{1}{n}$

$$= \lim_{n\to\infty}\left[\frac{1+\frac{2}{n}+\frac{1}{n^2}}{2}\right] = \frac{1}{2}.$$

46. $f(x) = x^2 - x^3$




Let $\Delta x = \frac{0-(-1)}{n} = \frac{1}{n}$ and $c_i = -1 + i\Delta x = -1 + \frac{i}{n}$.

The right-hand sum is $\sum_{i=1}^{n}\left(c_i^2 - c_i^3\right)\frac{1}{n}$

$$= \sum_{i=1}^{n}\left(\left(-1+\frac{i}{n}\right)^2 - \left(-1+\frac{i}{n}\right)^3\right)\frac{1}{n} = \sum_{i=1}^{n}\left(2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right)\frac{1}{n}$$

$$= \sum_{i=1}^{n}\left(\frac{2}{n} - \frac{5i}{n^2} + \frac{4i^2}{n^3} - \frac{i^3}{n^4}\right) = \sum_{i=1}^{n}\frac{2}{n} - \frac{5}{n^2}\sum_{i=1}^{n} i + \frac{4}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= \frac{2}{n}(n) - \frac{5}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{4}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2$$

$$= 2 - \frac{5n+5}{2n} + \frac{4n^2+6n+2}{3n^2} - \frac{n^2+2n+1}{4n^2} = 2 - \frac{5+\frac{5}{n}}{2} + \frac{4+\frac{6}{n}+\frac{2}{n^2}}{3} - \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}.$$ Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(c_i^2 - c_i^3\right)\frac{1}{n}$

$$= \lim_{n\to\infty}\left[2 - \frac{5+\frac{5}{n}}{2} + \frac{4+\frac{6}{n}+\frac{2}{n^2}}{3} - \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}\right] = 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}.$$

## 5.3 THE DEFINITE INTEGRAL

1. $\int_0^2 x^2\,dx$
2. $\int_{-1}^0 2x^3\,dx$
3. $\int_{-7}^5 (x^2 - 3x)\,dx$
4. $\int_1^4 \frac{1}{x}\,dx$
5. $\int_2^3 \frac{1}{1-x}\,dx$
6. $\int_0^1 \sqrt{4-x^2}\,dx$
7. $\int_{-\pi/4}^0 (\sec x)\,dx$
8. $\int_0^{\pi/4} (\tan x)\,dx$
9. (a) $\int_2^2 g(x)\,dx = 0$ (b) $\int_5^1 g(x)\,dx = -\int_1^5 g(x)\,dx = -8$

   (c) $\int_1^2 3f(x)\,dx = 3\int_1^2 f(x)\,dx = 3(-4) = -12$

   (d) $\int_2^5 f(x)\,dx = \int_1^5 f(x)\,dx - \int_1^2 f(x)\,dx = 6 - (-4) = 10$

   (e) $\int_1^5 [f(x) - g(x)]\,dx = \int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 6 - 8 = -2$

   (f) $\int_1^5 [4f(x) - g(x)]\,dx = 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 4(6) - 8 = 16$

[redacted]