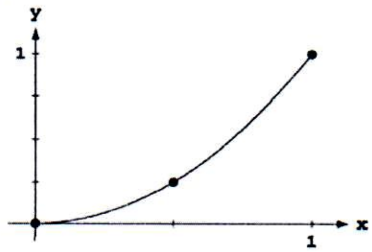
CHAPTER 5 INTEGRATION

5.1 AREA AND ESTIMATING WITH FINITE SUMS

1. $f(x) = x^2$



Since f is increasing on [0, 1], we use left endpoints to obtain lower sums and right endpoints to obtain upper sums.

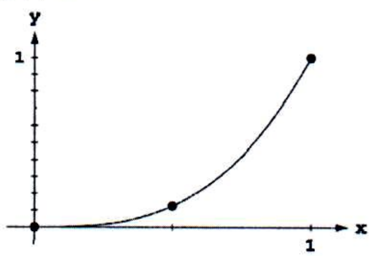
(a) $\Delta x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\Delta x = \frac{i}{2} \Rightarrow$ a lower sum is $\sum_{i=0}^{1} \left(\frac{i}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{2}\left(0^2 + \left(\frac{1}{2}\right)^2\right) = \frac{1}{8}$

(b) $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\Delta x = \frac{i}{4} \Rightarrow$ a lower sum is $\sum_{i=0}^{3} \left(\frac{i}{4}\right)^2 \cdot \frac{1}{4} = \frac{1}{4}\left(0^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2\right) = \frac{1}{4} \cdot \frac{7}{8} = \frac{7}{32}$

(c) $\Delta x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\Delta x = \frac{i}{2} \Rightarrow$ an upper sum is $\sum_{i=1}^{2} \left(\frac{i}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 + 1^2\right) = \frac{5}{8}$

(d) $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\Delta x = \frac{i}{4} \Rightarrow$ an upper sum is $\sum_{i=1}^{4} \left(\frac{i}{4}\right)^2 \cdot \frac{1}{4} = \frac{1}{4}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2\right) = \frac{1}{4} \cdot \left(\frac{30}{16}\right) = \frac{15}{32}$

2. $f(x) = x^3$



Since f is increasing on [0, 1], we use left endpoints to obtain lower sums and right endpoints to obtain upper sums.

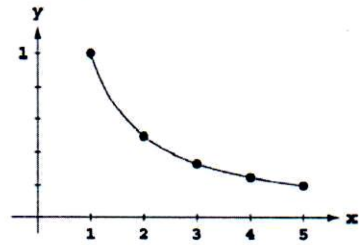
(a) $\Delta x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\Delta x = \frac{i}{2} \Rightarrow$ a lower sum is $\sum_{i=0}^{1} \left(\frac{i}{2}\right)^3 \cdot \frac{1}{2} = \frac{1}{2}\left(0^3 + \left(\frac{1}{2}\right)^3\right) = \frac{1}{16}$

(b) $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\Delta x = \frac{i}{4} \Rightarrow$ a lower sum is $\sum_{i=0}^{3} \left(\frac{i}{4}\right)^3 \cdot \frac{1}{4} = \frac{1}{4}\left(0^3 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{3}{4}\right)^3\right) = \frac{36}{256} = \frac{9}{64}$

(c) $\Delta x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\Delta x = \frac{i}{2} \Rightarrow$ an upper sum is $\sum_{i=1}^{2} \left(\frac{i}{2}\right)^3 \cdot \frac{1}{2} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^3 + 1^3\right) = \frac{1}{2} \cdot \frac{9}{8} = \frac{9}{16}$

(d) $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\Delta x = \frac{i}{4} \Rightarrow$ an upper sum is $\sum_{i=1}^{4} \left(\frac{i}{4}\right)^3 \cdot \frac{1}{4} = \frac{1}{4}\left(\left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{3}{4}\right)^3 + 1^3\right) = \frac{100}{256} = \frac{25}{64}$

3. $f(x) = \frac{1}{x}$



Since f is decreasing on [1, 5], we use left endpoints to obtain upper sums and right endpoints to obtain lower sums.

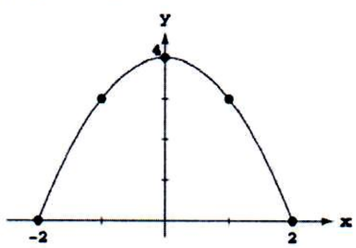
(a) $\Delta x = \frac{5-1}{2} = 2$ and $x_i = 1 + i\Delta x = 1 + 2i \Rightarrow$ a lower sum is $\sum_{i=1}^{2} \frac{1}{x_i} \cdot 2 = 2\left(\frac{1}{3} + \frac{1}{5}\right) = \frac{16}{15}$

(b) $\Delta x = \frac{5-1}{4} = 1$ and $x_i = 1 + i\Delta x = 1 + i \Rightarrow$ a lower sum is $\sum_{i=1}^{4} \frac{1}{x_i} \cdot 1 = 1\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \frac{77}{60}$

(c) $\Delta x = \frac{5-1}{2} = 2$ and $x_i = 1 + i\Delta x = 1 + 2i \Rightarrow$ an upper sum is $\sum_{i=0}^{1} \frac{1}{x_i} \cdot 2 = 2\left(1 + \frac{1}{3}\right) = \frac{8}{3}$

(d) $\Delta x = \frac{5-1}{4} = 1$ and $x_i = 1 + i\Delta x = 1 + i \Rightarrow$ an upper sum is $\sum_{i=0}^{3} \frac{1}{x_i} \cdot 1 = 1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{25}{12}$

4. $f(x) = 4 - x^2$



Since f is increasing on [−2, 0] and decreasing on [0, 2], we use left endpoints on [−2, 0] and right endpoints on [0, 2] to obtain lower sums and use right endpoints on [−2, 0] and left endpoints on [0, 2] to obtain upper sums.

(a) $\Delta x = \frac{2-(-2)}{2} = 2$ and $x_i = -2 + i\Delta x = -2 + 2i \Rightarrow$ a lower sum is $2 \cdot (4 - (-2)^2) + 2 \cdot (4 - 2^2) = 0$

(b) $\Delta x = \frac{2-(-2)}{4} = 1$ and $x_i = -2 + i\Delta x = -2 + i \Rightarrow$ a lower sum is $\sum_{i=0}^{1} (4 - (x_i)^2) \cdot 1 + \sum_{i=3}^{4} (4 - (x_i)^2) \cdot 1$

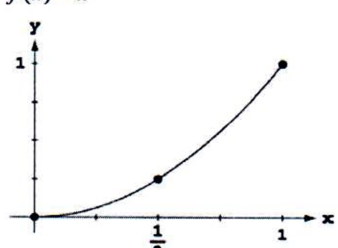
$= 1((4 - (-2)^2) + (4 - (-1)^2) + (4 - 1^2) + (4 - 2^2)) = 6$

(c) $\Delta x = \frac{2-(-2)}{2} = 2$ and $x_i = -2 + i\Delta x = -2 + 2i \Rightarrow$ an upper sum is $2 \cdot (4 - (0)^2) + 2 \cdot (4 - 0^2) = 16$

(d) $\Delta x = \frac{2-(-2)}{4} = 1$ and $x_i = -2 + i\Delta x = -2 + i \Rightarrow$ an upper sum is $\sum_{i=1}^{2} (4 - (x_i)^2) \cdot 1 + \sum_{i=2}^{3} (4 - (x_i)^2) \cdot 1$

$= 1((4 - (-1)^2) + (4 - 0^2) + (4 - 0^2) + (4 - 1^2)) = 14$

5. $f(x) = x^2$



Using 2 rectangles $\Rightarrow \Delta x = \frac{1-0}{2} = \frac{1}{2}$

$\Rightarrow \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right) = \frac{1}{2}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right) = \frac{10}{32} = \frac{5}{16}$

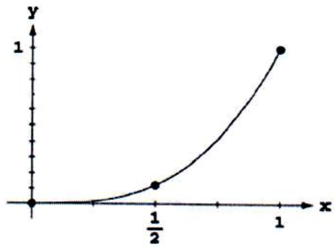
Using 4 rectangles $\Rightarrow \Delta x = \frac{1-0}{4} = \frac{1}{4}$

$\Rightarrow \frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right)$

$= \frac{1}{4}\left(\left(\frac{1}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{5}{8}\right)^2 + \left(\frac{7}{8}\right)^2\right) = \frac{21}{64}$

[redacted]

6. $f(x) = x^3$



Using 2 rectangles $\Rightarrow \Delta x = \frac{1-0}{2} = \frac{1}{2}$

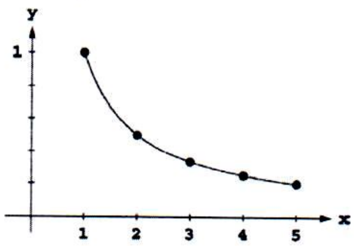
$\Rightarrow \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right) = \frac{1}{2}\left(\left(\frac{1}{4}\right)^3 + \left(\frac{3}{4}\right)^3\right) = \frac{28}{2\cdot 64} = \frac{7}{32}$

Using 4 rectangles $\Rightarrow \Delta x = \frac{1-0}{4} = \frac{1}{4}$

$\Rightarrow \frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right)$

$= \frac{1}{4}\left(\frac{1^3+3^3+5^3+7^3}{8^3}\right) = \frac{496}{4\cdot 8^3} = \frac{124}{8^3} = \frac{31}{128}$

7. $f(x) = \frac{1}{x}$



Using 2 rectangles $\Rightarrow \Delta x = \frac{5-1}{2} = 2 \Rightarrow 2(f(2) + f(4))$

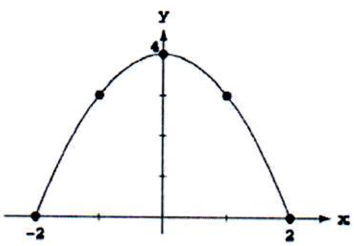
$= 2\left(\frac{1}{2} + \frac{1}{4}\right) = \frac{3}{2}$

Using 4 rectangles $\Rightarrow \Delta x = \frac{5-1}{4} = 1$

$\Rightarrow 1\left(f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right) + f\left(\frac{9}{2}\right)\right) = 1\left(\frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9}\right)$

$= \frac{1488}{3\cdot 5\cdot 7\cdot 9} = \frac{496}{5\cdot 7\cdot 9} = \frac{496}{315}$

8. $f(x) = 4 - x^2$



Using 2 rectangles $\Rightarrow \Delta x = \frac{2-(-2)}{2} = 2$

$\Rightarrow 2(f(-1) + f(1)) = 2(3+3) = 12$

Using 4 rectangles $\Rightarrow \Delta x = \frac{2-(-2)}{4} = 1$

$\Rightarrow 1\left(f\left(-\frac{3}{2}\right) + f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right)\right)$

$= 1\left(\left(4 - \left(-\frac{3}{2}\right)^2\right) + \left(4 - \left(-\frac{1}{2}\right)^2\right) + \left(4 - \left(\frac{1}{2}\right)^2\right) + \left(4 - \left(\frac{3}{2}\right)^2\right)\right)$

$= 16 - \left(\frac{9}{4}\cdot 2 + \frac{1}{4}\cdot 2\right) = 16 - \frac{10}{2} = 11$

9. (a) $D \approx (0)(1) + (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) = 87$ inches
   (b) $D \approx (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) + (0)(1) = 87$ inches

10. (a) $D \approx (1)(300) + (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300) + (1.0)(300) + (1.8)(300) + (1.5)(300) + (1.2)(300) = 5220$ meters (NOTE: 5 minutes = 300 seconds)
   (b) $D \approx (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300) + (1.0)(300) + (1.8)(300) + (1.5)(300) + (1.2)(300) + (0)(300) = 4920$ meters (NOTE: 5 minutes = 300 seconds)

11. (a) $D \approx (0)(10) + (44)(10) + (15)(10) + (35)(10) + (30)(10) + (44)(10) + (35)(10) + (15)(10) + (22)(10) + (35)(10) + (44)(10) + (30)(10) = 3490$ feet ≈ 0.66 miles
   (b) $D \approx (44)(10) + (15)(10) + (35)(10) + (30)(10) + (44)(10) + (35)(10) + (15)(10) + (22)(10) + (35)(10) + (44)(10) + (30)(10) + (35)(10) = 3840$ feet ≈ 0.73 miles

12. (a) The distance traveled will be the area under the curve. We will use the approximate velocities at the midpoints of each time interval to approximate this area using rectangles. Thus, $D \approx (20)(0.001) + (50)(0.001) + (72)(0.001) + (90)(0.001) + (102)(0.001) + (112)(0.001) + (120)(0.001) + (128)(0.001) + (134)(0.001) + (139)(0.001) \approx 0.967$ miles
   (b) Roughly, after 0.0063 hours, the car would have gone 0.484 miles, where 0.0060 hours = 22.7 sec. At 22.7 sec, the velocity was approximately 120 mi/hr.

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13. (a) Because the acceleration is decreasing, an upper estimate is obtained using left endpoints in summing acceleration $\cdot \Delta t$. Thus, $\Delta t = 1$ and speed $\approx [32.00 + 19.41 + 11.77 + 7.14 + 4.33](1) = 74.65$ ft/sec
(b) Using right endpoints we obtain a lower estimate: speed $\approx [19.41 + 11.77 + 7.14 + 4.33 + 2.63](1) = 45.28$ ft/sec
(c) Upper estimates for the speed at each second are:

t	0	1	2	3	4	5
v	0	32.00	51.41	63.18	70.32	74.65

Thus, the distance fallen when $t = 3$ seconds is $s \approx [32.00 + 51.41 + 63.18](1) = 146.59$ ft.

14. (a) The speed is a decreasing function of time $\Rightarrow$ right endpoints give a lower estimate for the height (distance) attained. Also

t	0	1	2	3	4	5
v	400	368	336	304	272	240

gives the time-velocity table by subtracting the constant $g = 32$ from the speed at each time increment $\Delta t = 1\,\text{sec}$. Thus, the speed ≈ 240 ft/sec after 5 seconds.
(b) A lower estimate for height attained is $h \approx [368 + 336 + 304 + 272 + 240](1) = 1520$ ft.

15. Partition [0, 2] into the four subintervals [0, 0.5], [0.5, 1], [1, 1.5], and [1.5, 2]. The midpoints of these subintervals are $m_1 = 0.25$, $m_2 = 0.75$, $m_3 = 1.25$, and $m_4 = 1.75$. The heights of the four approximating rectangles are $f(m_1) = (0.25)^3 = \frac{1}{64}$, $f(m_2) = (0.75)^3 = \frac{27}{64}$, $f(m_3) = (1.25)^3 = \frac{125}{64}$, and $f(m_4) = (1.75)^3 = \frac{343}{64}$.
Notice that the average value is approximated by $\frac{1}{2}\left[\left(\frac{1}{4}\right)^3\left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)^3\left(\frac{1}{2}\right) + \left(\frac{5}{4}\right)^3\left(\frac{1}{2}\right) + \left(\frac{7}{4}\right)^3\left(\frac{1}{2}\right)\right] = \frac{31}{16}$
$= \frac{1}{\text{length of } [0,2]} \cdot \left[\begin{matrix}\text{approximate area under} \\ \text{curve } f(x) = x^3\end{matrix}\right]$. We use this observation in solving the next several exercises.

16. Partition [1, 9] into the four subintervals [1, 3], [3, 5], [5, 7], and [7, 9]. The midpoints of these subintervals are $m_1 = 2$, $m_2 = 4$, $m_3 = 6$, and $m_4 = 8$. The heights of the four approximating rectangles are $f(m_1) = \frac{1}{2}$, $f(m_2) = \frac{1}{4}$, $f(m_3) = \frac{1}{6}$, and $f(m_4) = \frac{1}{8}$. The width of each rectangle is $\Delta x = 2$. Thus,
Area $\approx 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 2\left(\frac{1}{6}\right) + 2\left(\frac{1}{8}\right) = \frac{25}{12} \Rightarrow$ average value $\approx \frac{\text{area}}{\text{length of } [1,9]} = \frac{\left(\frac{25}{12}\right)}{8} = \frac{25}{96}$.

17. Partition [0, 2] into the four subintervals [0, 0.5], [0.5, 1], [1, 1.5], and [1.5, 2]. The midpoints of the subintervals are $m_1 = 0.25$, $m_2 = 0.75$, $m_3 = 1.25$, and $m_4 = 1.75$. The heights of the four approximating rectangles are $f(m_1) = \frac{1}{2} + \sin^2 \frac{\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$, $f(m_2) = \frac{1}{2} + \sin^2 \frac{3\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$, $f(m_3) = \frac{1}{2} + \sin^2 \frac{5\pi}{4}$ $= \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$, and $f(m_4) = \frac{1}{2} + \sin^2 \frac{7\pi}{4} = \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$. The width of each rectangle is $\Delta x = \frac{1}{2}$.
Thus, Area $\approx (1 + 1 + 1 + 1)\left(\frac{1}{2}\right) = 2 \Rightarrow$ average value $\approx \frac{\text{area}}{\text{length of } [0, 2]} = \frac{2}{2} = 1$.

18. Partition [0, 4] into the four subintervals [0, 1], [1, 2], [2, 3], and [3, 4]. The midpoints of the subintervals are $m_1 = \frac{1}{2}$, $m_2 = \frac{3}{2}$, $m_3 = \frac{5}{2}$, and $m_4 = \frac{7}{2}$. The heights of the four approximating rectangles are
$f(m_1) = 1 - \left(\cos\left(\frac{\pi\left(\frac{1}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{\pi}{8}\right)\right)^4 = 0.27145$ (to 5 decimal places), $f(m_2) = 1 - \left(\cos\left(\frac{\pi\left(\frac{3}{2}\right)}{4}\right)\right)^4$
$= 1 - \left(\cos\left(\frac{3\pi}{8}\right)\right)^4 = 0.97855$, $f(m_3) = 1 - \left(\cos\left(\frac{\pi\left(\frac{5}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{5\pi}{8}\right)\right)^4 = 0.97855$, and

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