Exercises 4.5 Solutions to selected problems

(2) $\lim_{x \to 0} \frac{\sin 5x}{x} \overset{L.R.}{=} \lim_{x \to 0} \frac{5\cos 5x}{1} = \frac{5\cos 0}{1} = 5$

Ch 2 $\lim_{x \to 0} \frac{\sin 5x}{x} = \lim_{5x \to 0} 5 \cdot \frac{\sin 5x}{5x} = 5 \cdot 1 = 5$

(4) $\lim_{x \to 1} \frac{x^3-1}{4x^3-x-3} \overset{L.R.}{=} \lim_{x \to 1} \frac{3x^2}{12x^2-1} = \frac{3 \cdot 1^2}{12 \cdot 1^2 - 1} = \frac{3}{11}$

Ch 2 $\lim_{x \to 1} \frac{x^3-1}{4x^3-x-3} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(4x^2+4x+3)} = \lim_{x \to 1} \frac{x^2+x+1}{4x^2+4x+3}$

$= \frac{1^2+1+1}{4 \cdot 1^2 + 4 \cdot 1 + 3} = \frac{3}{11}$

(6) $\lim_{x \to \infty} \frac{2x^2+3x}{x^3+x+1} \overset{L.R.}{=} \lim_{x \to \infty} \frac{4x+3}{3x^2+1} \overset{L.R.}{=} \lim_{x \to \infty} \frac{4}{6x} = 0$

Ch 2 $\lim_{x \to \infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x \to \infty} \frac{(2x^2+3x) \cdot \frac{1}{x^3}}{(x^3+x+1) \cdot \frac{1}{x^3}}$

$= \lim_{x \to \infty} \frac{\frac{2}{x} + \frac{3}{x^2}}{1 + \frac{1}{x^2} + \frac{1}{x^3}} = \frac{0+0}{1+0+0} = \frac{0}{1} = 0$

(8) $\lim_{x \to 2} \frac{x-2}{x^2-4}$ ($\frac{0}{0}$ form) $\overset{L.R.}{=} \lim_{x \to 2} \frac{1}{2x} = \frac{1}{2 \cdot 2} = \frac{1}{4}$

(10) $\lim_{t \to 1} \frac{3t^3-3}{4t^3-t-3}$ ($\frac{0}{0}$ form) $\overset{L.R.}{=} \lim_{t \to 1} \frac{9t^2}{12t^2-1} = \frac{9 \cdot 1^2}{12 \cdot 1^2 - 1}$

$= \frac{9}{11}$.

(12) $\lim_{x\to\infty} \dfrac{x - 8x^2}{12x^2 + 5x}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$ $\overset{L.R.}{=} \lim_{x\to\infty} \dfrac{1 - 16x}{24x + 5}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$\overset{L.R.}{=} \lim_{x\to\infty} \dfrac{-16}{24} = \dfrac{-16}{24} = -\dfrac{2}{3}$.

(14) $\lim_{t\to 0} \dfrac{\sin 5t}{2t}$ $\left(\frac{0}{0} \text{ form}\right)$ $\overset{L.R.}{=} \lim_{t\to 0} \dfrac{5\cos(5t)}{2} = \dfrac{5\cos(0)}{2}$

$= \dfrac{5}{2}$

(16) $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$ $\left(\frac{0}{0} \text{ form}\right)$ $\overset{L.R.}{=} \lim_{x\to 0} \dfrac{\cos x - 1}{3x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$\overset{L.R.}{=} \lim_{x\to 0} \dfrac{-\sin x}{6x}$ $\left(\frac{0}{0} \text{ form}\right)$ $\overset{L.R.}{=} \lim_{x\to 0} \dfrac{-\cos x}{6} = \dfrac{-\cos 0}{6}$

$= -\dfrac{1}{6}$

(18) $\lim_{\theta\to -\frac{\pi}{3}} \dfrac{3\theta + \pi}{\sin\left(\theta + \frac{\pi}{3}\right)}$ $\left(\frac{0}{0} \text{ form}\right)$

$\overset{L.R.}{=} \lim_{\theta\to -\frac{\pi}{3}} \dfrac{3}{\cos\left(\theta + \frac{\pi}{3}\right)} = \dfrac{3}{\cos\left(-\frac{\pi}{3} + \frac{\pi}{3}\right)} = \dfrac{3}{\cos 0} = \dfrac{3}{1} = 3$

(20) $\lim_{x\to 1} \dfrac{x - 1}{\ln x - \sin \pi x}$ $\left(\frac{0}{0} \text{ form}\right)$

$\overset{L.R.}{=} \lim_{x\to 1} \dfrac{1}{\frac{1}{x} - \pi\cos \pi x} = \dfrac{1}{1 - \pi\cos\pi} = \dfrac{1}{1 - \pi(-1)} = \dfrac{1}{1+\pi}$

(22) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\ln(\csc x)}{(x - \frac{\pi}{2})^2}$ $\left(\frac{0}{0} \text{ form because } \ln(\csc \frac{\pi}{2}) = 0 \right)$

$$\overset{\text{L.R.}}{=} \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\csc x} \cdot (-\csc x \cot x)}{2(x - \frac{\pi}{2}) \cdot 1} = \lim_{x \to \frac{\pi}{2}} \frac{-\cot x}{2(x - \frac{\pi}{2})} \quad \left(\frac{0}{0} \text{ form} \right)$$

$$\overset{\text{L.R.}}{=} \lim_{x \to \frac{\pi}{2}} \frac{\csc^2 x}{2} = \frac{1}{2}$$

(24) $\lim\limits_{t \to 0} \dfrac{t \sin t}{1 - \cos t}$ $\left(\frac{0}{0} \text{ form} \right)$ $\overset{\text{L.R.}}{=} \lim\limits_{t \to 0} \dfrac{t \cos t + \sin t}{\sin t}$ $\left(\frac{0}{0} \text{ form} \right)$

$$\overset{\text{L.R.}}{=} \lim_{t \to 0} \frac{t(-\sin t) + \cos t + \cos t}{\cos t} = \frac{0 + 1 + 1}{1} = 2$$

(26) $\lim\limits_{x \to \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x \right) \tan x$ $(0 \cdot \infty \text{ form})$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\pi}{2} - x}{\frac{1}{\tan x}} = \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\pi}{2} - x}{\cot x} \quad \left(\frac{0}{0} \text{ form} \right)$$

$$\overset{\text{L.R.}}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{-1}{-\csc^2 x} = \frac{1}{\csc^2 \frac{\pi}{2}} = 1$$

(28) $\lim\limits_{\theta \to 0} \dfrac{(\frac{1}{2})^\theta - 1}{\theta}$ $\left(\frac{0}{0} \text{ form} \right)$ $\overset{\text{L.R.}}{=} \lim\limits_{\theta \to 0} \dfrac{(\frac{1}{2})^\theta \cdot \ln(\frac{1}{2})}{1}$

$$= \frac{(\frac{1}{2})^0 \cdot \ln(\frac{1}{2})}{1} = \ln\left(\frac{1}{2}\right) = -\ln 2$$

(30) $\lim_{x \to 0} \dfrac{3^x - 1}{2^x - 1}$ $\left(\frac{0}{0} \text{ form}\right)$ $\overset{\text{L.R.}}{=} \lim_{x \to 0} \dfrac{3^x \ln 3}{2^x \ln 2}$

$= \dfrac{3^0 \ln 3}{2^0 \ln 2} = \dfrac{\ln 3}{\ln 2}$.

(32) $\lim_{x \to \infty} \dfrac{\log_2 x}{\log_3 (x+3)}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$ $\overset{\text{L.R.}}{=} \lim_{x \to \infty} \dfrac{\frac{1}{x \ln 2}}{\frac{1}{(x+3)\ln 3}}$

$= \lim_{x \to \infty} \dfrac{(x+3)\ln 3}{x \ln 2}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$ $\overset{\text{L.R.}}{=} \lim_{x \to \infty} \dfrac{\ln 3}{\ln 2} = \dfrac{\ln 3}{\ln 2}$

(34) $\lim_{x \to 0^+} \dfrac{\ln(e^x - 1)}{\ln x}$ $\left(\frac{-\infty}{-\infty} \text{ form}\right)$ $\overset{\text{L.R.}}{=} \lim_{x \to 0^+} \dfrac{\frac{1}{e^x - 1} \cdot e^x}{\frac{1}{x}}$

$= \lim_{x \to 0^+} \dfrac{x e^x}{e^x - 1}$ $\left(\frac{0}{0} \text{ form}\right)$ $\overset{\text{L.R.}}{=} \lim_{x \to 0^+} \dfrac{x e^x + e^x}{e^x}$

$= \lim_{x \to 0^+} \dfrac{e^x (x+1)}{e^x} = \lim_{x \to 0^+} (x+1) = 0 + 1 = 1$

(36) $\lim_{y \to 0} \dfrac{\sqrt{ay + a^2} - a}{y}$, $a > 0$ $\left(\frac{0}{0} \text{ form}\right)$

$\overset{\text{L.R.}}{=} \lim_{y \to 0} \dfrac{\frac{1}{2\sqrt{ay + a^2}} \cdot a}{1} = \lim_{y \to 0} \dfrac{a}{2\sqrt{ay + a^2}} = \dfrac{a}{2\sqrt{0 + a^2}}$

$= \dfrac{a}{2\sqrt{a^2}} = \dfrac{a}{2a}$ (because $a > 0$) $= \dfrac{1}{2}$

(38) $\lim_{x \to 0^+} (\ln x - \ln \sin x)$ $\quad (-\infty - (-\infty) = -\infty + \infty$ form$)$

$$= \lim_{x \to 0^+} \ln \frac{x}{\sin x} = \ln 1 = 0$$

(because $\lim_{x \to 0^+} \frac{x}{\sin x} = \lim_{x \to 0^+} \frac{1}{\cos x} = \frac{1}{\cos(0)} = \frac{1}{1} = 1$.)

(40) $\lim_{x \to 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x} \right)$ $\quad (\infty - \infty$ form$)$

$$= \lim_{x \to 0^+} \frac{\sin x\,(3x+1) - x}{x \sin x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

L.R.

$$= \lim_{x \to 0^+} \frac{\sin x \cdot 3 + (3x+1) \cdot \cos x - 1}{x \cos x + \sin x \cdot 1}$$

$$= \lim_{x \to 0^+} \frac{3 \sin x + 3x \cos x + \cos x - 1}{x \cos x + \sin x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

L.R.

$$= \lim_{x \to 0^+} \frac{3 \cos x + 3x(-\sin x) + \cos x \cdot 3 - \sin x}{x(-\sin x) + \cos x \cdot 1 + \cos x}$$

$$= \lim_{x \to 0^+} \frac{3 \cos x - 3x \sin x + 3 \cos x - \sin x}{-x \sin x + \cos x + \cos x}$$

$$= \frac{3 - 0 + 3 - 0}{0 + 1 + 1} = \frac{6}{2} = 3$$

(42) $\lim_{x \to 0^+} (\csc x - \cot x + \cos x) = \lim_{x \to 0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} + \cos x \right)$

$$= \lim_{x \to 0^+} \left(\frac{1 - \cos x}{\sin x} \right) + \lim_{x \to 0^+} \cos x \overset{\text{L.R.}}{=} \lim_{x \to 0^+} \frac{\sin x}{\cos x} + \lim_{x \to 0^+} \cos x$$

$$= 0 + 1 = 1$$

(44) $\lim_{h \to 0} \frac{e^h - (1+h)}{h^2}$ ($\frac{0}{0}$ form) $\overset{L.R.}{=} \lim_{h \to 0} \frac{e^h - 1}{2h}$ ($\frac{0}{0}$ form)

$\overset{L.R.}{=} \lim_{h \to 0} \frac{e^h}{2} = \frac{e^0}{2} = \frac{1}{2}$

(46) $\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x}$ ($\frac{\infty}{\infty}$ form) $\overset{L.R.}{=} \lim_{x \to \infty} \frac{2x}{e^x}$ ($\frac{\infty}{\infty}$)

$(\infty \cdot 0)$

$\overset{L.R.}{=} \lim_{x \to \infty} \frac{2}{e^x} = 0$

(48) $\lim_{x \to 0} \frac{(e^x - 1)^2}{x \sin x}$ ($\frac{0}{0}$ form) $\overset{L.R.}{=} \lim_{x \to 0} \frac{2(e^x - 1)(e^x)}{x \cos x + \sin x}$ ($\frac{0}{0}$ form)

$\overset{L.R.}{=} \lim_{x \to 0} \frac{2(e^x - 1)(e^x) + 2(e^x)(e^x)}{x(-\sin x) + \cos x + \cos x} = \frac{2(e^0 - 1) \cdot e^0 + 2(e^0)(e^0)}{0(-\sin 0) + \cos 0 + \cos 0}$

$= \frac{0 + 2}{0 + 1 + 1} = 1$

(50) $\lim_{x \to 0} \frac{\sin 3x - 3x + x^2}{\sin x \sin 2x}$ ($\frac{0}{0}$ form)

$\overset{L.R.}{=} \lim_{x \to 0} \frac{3\cos 3x - 3 + 2x}{\sin x \cos 2x \cdot 2 + \sin 2x \cdot \cos x}$ ($\frac{0}{0}$ form)

$\overset{L.R.}{=} \lim_{x \to 0} \frac{-3 \sin 3x \cdot 3 + 2}{[\sin x(-\sin 2x) \cdot 2 + \cos 2x \cdot \cos x] \cdot 2 + \sin 2x(-\sin x) + \cos x \cos 2x \cdot 2}$

$= \frac{0 + 2}{[0 + 1] \cdot 2 + 0 + 2} = \frac{1}{2}$

(52) $\lim_{x \to 1^+} x^{\frac{1}{x-1}}$ (1^∞ form)

Let $y = x^{\frac{1}{x-1}}$, so $\ln y = \frac{1}{x-1} \ln x$

$\lim_{x \to 1^+} \ln y = \lim_{x \to 1^+} \frac{1}{x-1} \ln x = \lim_{x \to 1^+} \frac{\ln x}{x-1}$ ($\frac{0}{0}$ form)

$\overset{L.R.}{=} \lim_{x \to 1^+} \frac{\frac{1}{x}}{1} = \frac{\frac{1}{1}}{1} = 1$

Since $\lim_{x \to 1^+} \ln y = 1$, we have $\lim_{x \to 1^+} y = e^1 = e$

(54) $\lim_{x \to e^+} (\ln x)^{\frac{1}{x-e}}$ (1^∞ form)

Let $y = (\ln x)^{\frac{1}{x-e}}$, so $\ln y = \frac{1}{x-e} \ln(\ln x)$

$\lim_{x \to e^+} \ln y = \lim_{x \to e^+} \frac{\ln(\ln x)}{x-e}$ ($\frac{0}{0}$ form)

$\overset{L.R.}{=} \lim_{x \to e^+} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{1} = \frac{\frac{1}{\ln e} \cdot \frac{1}{e}}{1} = \frac{1}{e}$

Since $\lim_{x \to e^+} \ln y = \frac{1}{e}$, we have $\lim_{x \to e^+} y = e^{\frac{1}{e}}$

(56) $\lim_{x \to \infty} x^{1/\ln x}$ (∞^0 form)

Let $y = x^{\frac{1}{\ln x}}$ so, $\ln y = \frac{1}{\ln x} \cdot \ln x = 1$

$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} 1 = 1$, so $\lim_{x \to \infty} y = e$.

(58) $\lim\limits_{x \to 0} (e^x + x)^{\frac{1}{x}}$ $\quad (1^{\infty} \text{ form})$

Let $y = (e^x + x)^{\frac{1}{x}}$, so $\ln y = \frac{1}{x} \ln(e^x + x) = \frac{\ln(e^x + x)}{x}$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(e^x + x)}{x} \quad \left(\tfrac{0}{0} \text{ form}\right) \overset{\text{L.R.}}{=} \lim_{x \to 0} \frac{\frac{1}{e^x + x} \cdot (e^x + 1)}{1}$$

$$= \lim_{x \to 0} \frac{e^x + 1}{e^x + x} = \frac{e^0 + 1}{e^0 + 0} = \frac{1+1}{1+0} = 2, \quad \text{so } \lim_{x \to 0} y = e^2$$

(60) $\lim\limits_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x$ $\quad (\infty^0 \text{ form})$

Let $y = \left(1 + \frac{1}{x}\right)^x$, so $\ln y = x \ln\left(1 + \frac{1}{x}\right)$

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} x \ln\left(1 + \frac{1}{x}\right) \quad (0 \cdot (-\infty) \text{ form})$$

$$= \lim_{x \to 0^+} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\overset{\text{L.R.}}{=} \lim_{x \to 0^+} \frac{\frac{1}{1 + \frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x \to 0^+} \frac{1}{1 + \frac{1}{x}} \cdot \left(\frac{x}{x}\right)$$

$$= \lim_{x \to 0^+} \frac{x}{x+1} = \frac{0}{0+1} = 0$$

so, $\lim\limits_{x \to 0^+} y = e^0 = 1$

(62) $\lim_{x\to\infty} \left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}}$ (∞^0 form)

Let $y = \left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}}$, so $\ln y = \frac{1}{x} \ln\left(\frac{x^2+1}{x+2}\right) = \frac{\ln\left(\frac{x^2+1}{x+2}\right)}{x}$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(\frac{x^2+1}{x+2}\right)}{x} = \lim_{x\to\infty} \frac{\ln(x^2+1)}{x} - \lim_{x\to\infty} \frac{\ln(x+2)}{x}$

$\overset{L.R.}{=} \lim_{x\to\infty} \frac{\frac{1}{x^2+1}\cdot(2x)}{1} - \lim_{x\to\infty} \frac{\frac{1}{x+2}\cdot 1}{1}$

$= \lim_{x\to\infty} \frac{2x}{x^2+1} - \lim_{x\to\infty} \frac{1}{x+2} = 0 - 0 = 0.$

so, $\lim_{x\to\infty} y = e^0 = 1$

(64) $\lim_{x\to 0^+} x(\ln x)^2$ ($0\cdot\infty$ form)

$= \lim_{x\to 0^+} \frac{(\ln x)^2}{\frac{1}{x}}$ ($\frac{\infty}{\infty}$ form) $\overset{L.R.}{=} \lim_{x\to 0^+} \frac{2\ln x\cdot\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim_{x\to 0^+} -2x\ln x$ ($0\cdot(-\infty)$ form) $= \lim_{x\to 0^+} \frac{-2\ln x}{\frac{1}{x}}$ ($\frac{-\infty}{\infty}$)

$\overset{L.R.}{=} \lim_{x\to 0^+} \frac{-2\cdot\frac{1}{x}}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} 2x = 0$

(66) $\lim_{x\to 0^+} \sin x\cdot\ln x$ ($0\cdot(-\infty)$ form) $= \lim_{x\to 0^+} \frac{\ln x}{\csc x}$ ($\frac{-\infty}{\infty}$ form)

$\overset{L.R.}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc x\cot x} = \lim_{x\to 0^+} \frac{-\sin x\tan x}{x}$ ($\frac{0}{0}$ form)

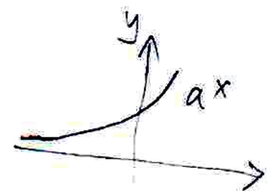
$= \lim_{x\to 0} \frac{-\sin x\sec^2 x - \tan x\cos x}{1} = \frac{-0-0}{1} = 0$

(68) $\lim\limits_{x \to 0^+} \dfrac{\sqrt{x}}{\sqrt{\sin x}} = \sqrt{1} = 1$ (because we know $\lim\limits_{x \to 0} \dfrac{x}{\sin x} =$
$\lim\limits_{x \to 0^+} \dfrac{1}{\cos x}$ (L.R.) $= \dfrac{1}{\cos 0} = 1$)

(70) $\lim\limits_{x \to 0^+} \dfrac{\cot x}{\csc x} = \lim\limits_{x \to 0^+} \dfrac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} = \lim\limits_{x \to 0^+} \cos x = \cos 0 = 1$

(72) $\lim\limits_{x \to -\infty} \dfrac{2^x + 4^x}{5^x - 2^x} = \lim\limits_{x \to -\infty} \dfrac{(2^x + 4^x) \cdot \frac{1}{2^x}}{(5^x - 2^x) \cdot \frac{1}{2^x}} = \lim\limits_{x \to -\infty} \dfrac{1 + \left(\frac{4}{2}\right)^x}{\left(\frac{5}{2}\right)^x - 1}$

$= \lim\limits_{x \to -\infty} \dfrac{1 + 2^x}{(2.5)^x - 1} = \dfrac{1 + 0}{0 - 1} = -1$

(Note that if $a > 1$, than $\lim\limits_{x \to -\infty} a^x = 0$ 

)

(74) $\lim\limits_{x \to 0^+} \dfrac{x}{e^{-\frac{1}{x}}} = \lim\limits_{x \to 0^+} x e^{\frac{1}{x}} = \lim\limits_{x \to 0^+} \dfrac{e^{\frac{1}{x}}}{\frac{1}{x}}$ ($\frac{\infty}{\infty}$ form)

$\overset{L.R.}{=} \lim\limits_{x \to 0^+} \dfrac{e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim\limits_{x \to 0^+} e^{\frac{1}{x}} = \infty.$

(76) (a) is wrong because. $\lim\limits_{x \to 0} \dfrac{2x - 2}{2x - \cos x}$ is NOT $\frac{0}{0}$, $\frac{\infty}{\infty}$ form
so we can't use L.R. $\Rightarrow \lim\limits_{x \to 0} \dfrac{2x - 2}{2x - \cos x} = \dfrac{0 - 2}{0 - \cos 0} = 2.$

(b) is right,