Exercises 4.2 Selected solutions

(2) $f(x) = x^{2/3}, \quad [0,1]$

want to find c such that $f'(c) = \dfrac{f(1) - f(0)}{1 - 0}$

$f(1) = 1^{2/3} = 1, \quad f(0) = 0^{2/3} = 0$

$f'(x) = \frac{2}{3} x^{-\frac{1}{3}} = \dfrac{2}{3 x^{1/3}}$ so $f'(c) = \dfrac{2}{3c^{1/3}} = \dfrac{2}{3\sqrt[3]{c}}$

so, want to find c such that

$$\frac{2}{3\sqrt[3]{c}} = \frac{1-0}{1-0} = 1 \Rightarrow \frac{2}{3\sqrt[3]{c}} = \frac{1}{1}$$

$$\frac{3\sqrt[3]{c}}{2} = \frac{1}{1} \Rightarrow \sqrt[3]{c} = \frac{2}{3} \Rightarrow c = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

(4) $f(x) = \sqrt{x-1}, \quad x \in [1,3]$

$f(3) = \sqrt{3-1} = \sqrt{2}, \quad f(1) = \sqrt{1-1} = 0$

$f'(x) = \dfrac{1}{2\sqrt{x-1}} \Rightarrow f'(c) = \dfrac{1}{2\sqrt{c-1}}$

Want to find c such that

$$\frac{1}{2\sqrt{c-1}} = \frac{\sqrt{2}-0}{3-1}$$

$$\frac{1}{2\sqrt{c-1}} = \frac{\sqrt{2}}{3}$$

$$2\sqrt{c-1} = \frac{3}{\sqrt{2}}$$

$$\sqrt{c-1} = \frac{3}{2\sqrt{2}}$$

$$c-1 = \frac{9}{8}$$

$$c = 1 + \frac{9}{8}$$

$$c = \frac{17}{8}$$

(6) $f(x) = \ln(x-1)$, $x \in [2,4]$

$f(4) = \ln(4-1) = \ln 3$, $f(2) = \ln(2-1) = \ln 1 = 0$

$f'(x) = \dfrac{1}{x-1} \Rightarrow f'(c) = \dfrac{1}{c-1}$

Find c such that $\dfrac{1}{c-1} = \dfrac{\ln 3 - 0}{4-2}$

$\dfrac{1}{c-1} = \dfrac{\ln 3}{2}$

$\Rightarrow c-1 = \dfrac{2}{\ln 3} \Rightarrow c = 1 + \dfrac{2}{\ln 3}$

(8) $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$, $x \in [-2,2]$

$g(-2) = (-2)^3 = -8$, $g(2) = 2^2 = 4$

If $c \in (-2,0)$, $f'(c) = 3c^2$

If $c \in (0,2)$, $f'(c) = 2c$

Case 1 $c \in (-2,0) \Rightarrow 3c^2 = \dfrac{4-(-8)}{2-(-2)}$

$3c^2 = 3 \Rightarrow c = -1$

Case 2 $c \in (0,2) \Rightarrow 2c = \dfrac{4-(-8)}{2-(-2)}$

$2c = 3 \Rightarrow c = \dfrac{3}{2}$

So, $\boxed{c = -1 \text{ and } c = \frac{3}{2}}$

(10) $f(x) = x^{4/5}$, $x \in [0,1]$

1. $f = x^{4/5}$ is continuous on $[0,1]$

2. $f' = \frac{4}{5}x^{-\frac{1}{5}}$ is continuous of $(0,1)$.

so, f is differentiable on $(0,1)$

Therefore, the function satisfies the hypotheses.

(12) $f(x) = \begin{cases} \frac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$, $x \in [-\pi, 0]$

Note that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ but $f(0) = 0$, so f is not continuous at $x = 0$.

Therefore, f doesn't satify the hypotheses.

(14) $f(x) = \begin{cases} 2x - 3, & 0 \le x \le 2 \\ 6x - x^2 - 7, & 2 < x \le 3 \end{cases}$, $x \in [0, 3]$

1. $f(2) = 2(2) - 3 = 1$, $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (6x - x^2 - 7) = 12 - 4 - 7 = 1$,

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x - 3) = 4 - 3 = 1$. so, $\lim_{x \to 2} f(x) = 1 = f(2)$

Thus, f is continuous at 2 and on $[0,3]$.

2. $f'(x) = \begin{cases} 2, & 0 \le x \le 2 \\ 6 - 2x, & 2 < x \le 3 \end{cases}$ which is continuous at 2.

Thus, f is differentiable at 2 and on $(0,3)$

Therefore, f satifies the hypotheses.

(15) The function doesn't satisfy the hypotheses of the Rolle's theorem. We need to have f continuous on $[0,1]$. But f is not continuous at $x=1$ here.

(16)
$$f(x) = \begin{cases} 3 & x=0 \\ -x^2+3x+a & 0<x<1 \\ mx+b & 1 \le x \le 2 \end{cases}$$

$$f'(x) = \begin{cases} 0 & x=0 \\ -2x+3 & 0<x<1 \\ m & 1 \le x \le 2 \end{cases}$$

We need to have $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x)$

$$-(0)^2+3(0)+a = 3 \quad \Rightarrow \quad a = 3$$

We also need to have $\lim\limits_{x \to 1^+} f'(x) = \lim\limits_{x \to 1^-} f'(x)$

$$m = -2(1)+3$$

$$\Rightarrow \quad m = -1$$

Finally, we need to have $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} f(x)$

$$m = m(2)+b$$

$$-1 = (-1)(2)+b$$

$$\Rightarrow \quad b = 1$$

$$\boxed{a=3, \quad m=-1, \quad b=1}$$