Exercises 3.11 Solutions to selected problems.

(2) $f(x) = \sqrt{x^2+9}$, $a = -4$

$f'(x) = \dfrac{1}{2\sqrt{x^2+9}} \cdot (2x) = \dfrac{x}{\sqrt{x^2+9}}$

$f(-4) = \sqrt{(-4)^2 + 9} = \sqrt{16+9} = \sqrt{25} = 5$

$f'(-4) = \dfrac{-4}{\sqrt{(-4)^2+9}} = -\dfrac{4}{5}$

$L(x) = f(-4) + f'(-4)(x-(-4))$

$L(x) = 5 + \left(-\frac{4}{5}\right)(x+4)$

$L(x) = 5 - \frac{4}{5}(x+4) = 5 - \frac{4}{5}x - \frac{16}{5} = \frac{9}{5} - \frac{4}{5}x$

(4) $f(x) = \sqrt[3]{x}$, $a = -8$

$f'(x) = \frac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$

$f(-8) = \sqrt[3]{-8} = -2$

$f'(-8) = \dfrac{1}{3(-8)^{2/3}} = \dfrac{1}{3\left(\sqrt[3]{(-8)}\right)^2} = \dfrac{1}{3(-2)^2} = \dfrac{1}{12}$

$L(x) = f(-8) + f'(-8)(x-(-8))$

$L(x) = -2 + \frac{1}{12}(x+8)$

$L(x) = -2 + \frac{x}{12} + \frac{2}{3} = \frac{x}{12} - \frac{4}{3}$

(6) (a) $f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = \cos(0) = 1$

$L(x) = 0 + 1(x-0) = x$

(b) $f(x) = \cos x \qquad f(0) = \cos(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = -\sin(0) = 0$

$L(x) = 1 + 0(x-0) = 1$

(c) $f(x) = \tan x \qquad f(0) = \tan(0) = 0$

$f'(x) = \sec^2 x \qquad f'(0) = \sec^2(0) = 1$

$L(x) = 0 + 1(x-0) = x$

(d) $f(x) = e^x \qquad f(0) = e^0 = 1$

$f'(x) = e^x \qquad f'(0) = e^0 = 1$

$L(x) = 1 + 1(x-0) = 1 + x$

(e) $f(x) = \ln(1+x) \qquad f(0) = \ln(1+0) = \ln 1 = 0$

$f'(x) = \frac{1}{1+x} \qquad f'(0) = \frac{1}{1+0} = 1$

$L(x) = 0 + 1(x-0) = x$

Note that $\sin x$, $\tan x$, $\ln(1+x)$ have the same linearization at $x = 0$.

(8) $f(x) = x^{-1}$, $x_0 = 0.9 \quad \Rightarrow$ Choose $a = 1$

$f'(x) = (-1)x^{-2} = -\frac{1}{x^2}$

$f(a) = f(1) = 1^{-1} = 1$

$f'(a) = f'(1) = \frac{-1}{1^2} = 1$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = 1 + 1(x-1)$

$L(x) = x$

(10) $f(x) = 1 + x, \quad x_0 = 8.1 \Rightarrow$ Choose $a = 8$

$f'(x) = 1 \qquad f(8) = 1 + 8 = 9, \quad f'(8) = 1$

$L(x) = 9 + 1(x-8) = 1 + x$

(12) $f(x) = \dfrac{x}{x+1}, \quad x_0 = 1.3 \Rightarrow$ Choose $a = 1$

$f'(x) = \dfrac{(x+1)\cdot 1 - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$

$f(1) = \dfrac{1}{1+1} = \dfrac{1}{2}, \qquad f'(1) = \dfrac{1}{(1+1)^2} = \dfrac{1}{4}$

$L(x) = \dfrac{1}{2} + \dfrac{1}{4}(x-1) = \dfrac{1}{4}x + \dfrac{1}{4}$

(14) $f(x) = \sin^{-1} x, \qquad x_0 = \dfrac{\pi}{12} \approx 0.262 \Rightarrow$ Choose $a = 0$.

$f'(x) = \dfrac{1}{\sqrt{1-x^2}}, \qquad f(0) = \sin^{-1}(0) = 0, \quad f'(0) = \dfrac{1}{\sqrt{1-0^2}} = 1$

$L(x) = 0 + 1(x-0) = x$

(16) Know $(1+x)^k \approx 1 + kx$ where x is near 0.

(a) $f(x) = (1-x)^6 = (1 + (-1)x)^6 \approx 1 + (-1)6x$

$= 1 - 6x$

(b) $f(x) = \dfrac{2}{1-x} = 2(1-x)^{-1} \approx 2(1 - (-1)x) = 2(1+x)$

(c) $f(x) = \dfrac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}x$

(d) $f(x) = \sqrt{2+x^2} = (2+x^2)^{\frac{1}{2}} = \sqrt{2}\left(1 + \frac{x^2}{2}\right)^{\frac{1}{2}} \approx \sqrt{2}\left(1 + \frac{1}{2}\cdot\frac{x^2}{2}\right)$

(e) $f(x) = (4+3x)^{1/3} = 4^{1/3}\left(1 + \frac{3}{4}x\right)^{\frac{1}{3}} \approx 4^{1/3}\left(1 + \frac{1}{3}\cdot\frac{3}{4}x\right) = 4^{\frac{1}{3}}\left(1 + \frac{1}{4}x\right)$

(18) $f(x) = \sqrt{x+1} + \sin x$, $a = 0$

$f'(x) = \dfrac{1}{2\sqrt{x+1}} + \cos x$

$f(0) = \sqrt{0+1} + \sin 0 = 1$, $f'(0) = \dfrac{1}{2\sqrt{0+1}} + \cos(0) = \dfrac{3}{2}$

$L(x) = 1 + \dfrac{3}{2}(x-0) = 1 + \dfrac{3}{2}x$

If $f_1(x) = \sqrt{x+1}$ then $f'(x) = \dfrac{1}{2\sqrt{x+1}}$

$f_1(0) = \sqrt{0+1} = 1$, $f'(0) = \dfrac{1}{2\sqrt{0+1}} = \dfrac{1}{2}$

so, $L_1(x) = 1 + \frac{1}{2}(x-0) = 1 + \frac{1}{2}x$

If $f_2(x) = \sin x$ then $f_2'(x) = \cos x$

$f_2(0) = \sin(0) = 0$, $f_2'(0) = \cos(0) = 1$

so, $L_2(x) = 0 + 1(x-0) = x$

Thus, if $f = f_1 + f_2$, we have $L = L_1 + L_2$

because $L = 1 + \frac{3}{2}x$

$L_1 + L_2 = \left(1 + \frac{1}{2}x\right) + x = 1 + \frac{3}{2}x = L.$

(20) $y = x\sqrt{1-x^2} = f(x)$

$f'(x) = x \cdot \dfrac{1}{2\sqrt{1-x^2}}(-2x) + \sqrt{1-x^2}$

$= \dfrac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} = \dfrac{(-x^2) + (1-x^2)}{\sqrt{1-x^2}} = \dfrac{1}{\sqrt{1-x^2}}$

$dy = f'(x)\,dx = \dfrac{1}{\sqrt{1-x^2}}\,dx$

(22) $y = \dfrac{2\sqrt{x}}{3(1+\sqrt{x})} = f(x)$

$$f'(x) = \frac{3(1+\sqrt{x})(2\sqrt{x})' - (2\sqrt{x})(3(1+\sqrt{x}))'}{[3(1+\sqrt{x})]^2}$$

$$= \frac{3(1+\sqrt{x})\left(\frac{1}{\sqrt{x}}\right) - (2\sqrt{x})\left(\frac{3}{2\sqrt{x}}\right)}{[3(1+\sqrt{x})]^2}$$

$$= \frac{\frac{3}{\sqrt{x}} + 3 - 3}{9(1+\sqrt{x})^2} = \frac{1}{3\sqrt{x}(1+\sqrt{x})^2}$$

so, $dy = f'(x)\,dx = \dfrac{1}{3\sqrt{x}(1+\sqrt{x})^2}$

(24) $xy^2 - 4x^{3/2} - y = 0$

Implicit differentiation

$$x\cdot(y^2)' + y^2(x)' - 4(x^{3/2})' - (y)' = 0$$

$$x\cdot 2y\,y' + y^2\cdot 1 - 4\cdot\tfrac{3}{2}x^{\frac{1}{2}} - y' = 0$$

$$2xy\,y' - y' = 6x^{1/2} - y^2$$

$$(2xy-1)\,y' = 6x^{1/2} - y^2$$

$$y' = \frac{6x^{1/2} - y^2}{2xy-1}$$

so, $dy = \dfrac{6x^{1/2} - y^2}{2xy-1}\,dx$