Exercise 3.7

(2) $x^3 + y^3 = 18xy$

$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(18xy)$

$3x^2 + 3y^2\frac{dy}{dx} = 18\left(x\frac{dy}{dx} + y\right)$

$3x^2 + 3y^2\frac{dy}{dx} = 18x\frac{dy}{dx} + 18y$

$(3y^2 - 18x)\frac{dy}{dx} = 18y - 3x^2$

$\frac{dy}{dx} = \frac{18y - 3x^2}{3y^2 - 18x}$

(6) $(3xy+7)^2 = 6y$

$\frac{d}{dx}(3xy+7)^2 = 6\frac{dy}{dx}$

$2(3xy+7)(3x\frac{dy}{dx} + 3y) = 6\frac{dy}{dx}$

$2\left(9x^2y\frac{dy}{dx} + 9xy^2 + 21x\frac{dy}{dx} + 21y\right) = 6\frac{dy}{dx}$

$9x^2y\frac{dy}{dx} + 9xy^2 + 21x\frac{dy}{dy} + 21y = 3\frac{dy}{dx}$

$(9x^2y + 21x - 3)\frac{dy}{dx} = -9xy^2 - 21y$

$\frac{dy}{dx} = \frac{-9xy^2 - 21y}{9x^2y + 21x + 3}$

(4) $x^3 - xy + y^3 = 1$

$\frac{d}{dx}(x^3 - xy + y^3) = \frac{d1}{dx}$

$3x^2 - x\frac{dy}{dx} - y + 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx}(-x + 3y^2) = -3x^2 + y$

$\frac{dy}{dx} = \frac{-3x^2 + y}{-x + 3y^2}$

(8) $x^3 = \frac{2x - y}{x + 3y}$

$\frac{dx^3}{dx} = \frac{d}{dx}\left(\frac{2x-y}{x+3y}\right)$

$3x^2 = \frac{(x+3y)(2 - \frac{dy}{dx}) - (2x-y)(1 + 3\frac{dy}{dx})}{(x+3y)^2}$

$3x^2 = \frac{\cancel{2x} - x\frac{dy}{dx} + 6y - 3y\frac{dy}{dx} - \cancel{2x} - 6x\frac{dy}{dx} + y - 3y\frac{dy}{dx}}{(x+3y)^2}$

$3x^2 = \frac{7y + (-7x - 6y)\frac{dy}{dx}}{(x+3y)^2}$

$\frac{dy}{dx} = \frac{3x^2(x+3y)^2 - 7y}{(-7x - 6y)}$

(10) $xy = \cot(xy)$

$$\frac{d}{dx} xy = \frac{d}{dx}\cot(xy)$$

$$x\frac{dy}{dx} + y = \left[-\mathrm{cosec}^2(xy)\right]\left(x\frac{dy}{dx}+y\right)$$

$$x\frac{dy}{dx}+y = -x\,\mathrm{cosec}^2(xy)\frac{dy}{dx} - y\,\mathrm{cosec}^2(xy)$$

$$\frac{dy}{dx}\left(x + x\,\mathrm{cosec}^2(xy)\right) = -y\,\mathrm{cosec}^2(xy) - y$$

$$\frac{dy}{dx} = \frac{-y\,\mathrm{cosec}^2(xy)-y}{x+x\,\mathrm{cosec}^2(xy)}$$

(12) $x^4+\sin y = x^3y^2$

$$\frac{d}{dx}(x^4+\sin y) = \frac{d}{dx}(x^3y^2)$$

$$4x^3 + \cos y\frac{dy}{dx} = 2x^3y\frac{dy}{dx} + 3y^2x^2$$

$$\frac{dy}{dx}(\cos y - 2x^3y) = 3y^2x^2 - 4x^3$$

$$\frac{dy}{dx} = \frac{3y^2x^2-4x^3}{\cos y - 2x^3y}$$

(14) $x\cos(2x+3y) = y\sin x$

$$\frac{d}{dx}\left[x\cos(2x+3y)\right] = \frac{d}{dx} y\sin x$$

$$-x\sin(2x+3y)\left[2+3\frac{dy}{dx}\right] = y\cos x + \sin x\frac{dy}{dx}$$

$$-2x\sin(2x+3y) - 3x\sin(2x+3y)\frac{dy}{dx} = y\cos x + \sin x\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[-3x\sin(2x+3y) - \sin x\right] = y\cos x + 2x\sin(2x+3y)$$

$$\frac{dy}{dx} = \frac{y\cos x + 2x\sin(2x+3y)}{-3x\sin(2x+3y)-\sin x}$$

(16) $e^{x^2y} = 2x+2y$

$$\frac{d}{dx}e^{x^2y} = \frac{d}{dx}(2x+2y)$$

$$e^{x^2y}\left(x^2\frac{dy}{dx} + 2xy\right) = 2+2\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(x^2e^{x^2y}\right) + 2e^{x^2y}xy = 2+2\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(x^2e^{x^2y}-2\right) = 2-2e^{x^2y}xy$$

$$dy \quad = \quad 2-2e^{x^2y}xy$$

(18) $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$

$$\frac{d}{d\theta}(r-2\sqrt{\theta}) = \frac{d}{d\theta}\left(\frac{3}{2}\theta^{2/3}+\frac{4}{3}\theta^{3/4}\right)$$

$$\frac{dr}{d\theta} - 2\cdot\frac{1}{2\sqrt{\theta}} = \frac{3}{2}\cdot\frac{2}{3}\theta^{-1/3} + \frac{4}{3}\cdot\frac{3}{4}\theta^{-1/4}$$

$$\frac{dr}{d\theta} - \frac{1}{\sqrt{\theta}} = \theta^{-1/3}+\theta^{-1/4}$$

$$\frac{dr}{d\theta} = \theta^{-1/3}+\theta^{-1/4}+\frac{1}{\sqrt{\theta}}$$

(20) $\cos r + \cot\theta = e^{r\theta}$

$$\frac{d}{d\theta}(\cos r + \cot\theta) = \frac{d}{d\theta} e^{r\theta}$$

$$-\sin r \frac{dr}{d\theta} - \operatorname{cosec}^2\theta = e^{r\theta}\left(r + \theta\frac{dr}{d\theta}\right)$$

$$-\sin r \frac{dr}{d\theta} - \operatorname{cosec}^2\theta = re^{r\theta} + e^{r\theta}\theta\frac{dr}{d\theta}$$

$$\frac{dr}{d\theta}(-\sin r - e^{r\theta}\theta) = re^{r\theta} + \operatorname{cosec}^2\theta$$

$$\frac{dr}{d\theta} = \frac{re^{r\theta} + \operatorname{cosec}^2\theta}{-\sin r - e^{r\theta}\theta}$$

(24) $y^2 - 2x = 1 - 2y$

$$2y\frac{dy}{dx} - 2 = -2\frac{dy}{dx}$$

$$(2y+2)\frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{2}{2y+2} = \frac{1}{y+1}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{(y+1)^2}\cdot\frac{dy}{dx} = \frac{-1}{(y+1)^2}\cdot\frac{1}{(y+1)}$$

$$= \frac{-1}{(y+1)^3}$$

(22) $x^{2/3} + y^{2/3} = 1$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}} = \frac{-y^{1/3}}{x^{1/3}}$$

$$\frac{d^2y}{dx^2} = \frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\frac{dy}{dx}\right) + \left(y^{1/3}\cdot\frac{1}{3}x^{-2/3}\right)}{x^{2/3}}$$

$$\frac{d^2y}{dx^2} = \frac{x^{1/3}\left(-\frac{1}{3}y^{-2/3}\cdot\frac{-y^{1/3}}{x^{1/3}}\right) + \frac{1}{3}x^{-2/3}y^{1/3}}{x^{2/3}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{1}{3}y^{-1/3} + \frac{1}{3}x^{-2/3}y^{1/3}}{x^{2/3}}$$

(26) $xy + y^2 = 1$

$$x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$(x+2y)\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x+2y}$$

$$\frac{d^2y}{dx^2} = \frac{(x+2y)\left(-\frac{dy}{dx}\right) + y\left(1 + 2\frac{dy}{dx}\right)}{(x+2y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(x+2y)\left(\frac{y}{x+2y}\right) + y + 2y\left(\frac{-y}{x+2y}\right)}{(x+2y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2y - \frac{2y^2}{x+2y}}{(x+2y)^2}$$

(28) By (26), we have $\dfrac{d^2y}{dx^2} = \dfrac{2y - \frac{2y^2}{x+2y}}{(x+2y)^2}$

Then $\left.\dfrac{d^2y}{dx^2}\right|$ $= \dfrac{-2 - \frac{2}{-2}}{\ }= \dfrac{-2+1}{\ } = \dfrac{-1}{\ }$

(30) $(x^2+y^2)^2 = (x-y)^2$

$$2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 2(x-y)\left(1-\frac{dy}{dx}\right)$$

At $(1,0)$, $2(1)(2) = 2(1)\left(1-\frac{dy}{dx}\right)$

$$4 = 2 - 2\frac{dy}{dx}$$

$$\frac{dy}{dx} = -1$$

The slope of the curve at $(1,0)$ is -1.

At $(1,-1)$, $2(1+1)\left(2-2\frac{dy}{dx}\right) = 2(1+1)\left(1-\frac{dy}{dx}\right)$

$$8 - 8\frac{dy}{dx} = 4 - 4\frac{dy}{dx}$$

$$-4\frac{dy}{dx} = -4$$

$$\frac{dy}{dx} = 1$$

The slope of the curve at $(1,-1)$ is 1.

(32) $x^2+y^2 = 25$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

At $(3,-4)$, $\frac{dy}{dx} = \frac{3}{4}$

$\therefore$ The slope of the tangent line is $\frac{3}{4}$

Tangent line : $y+4 = \frac{3}{4}(x-3)$

The slope of the normal line is $-\frac{4}{3}$.

Normal line : $y+4 = -\frac{4}{3}(x-3)$

(34) $y^2 - 2x - 4y - 1 = 0$

$$2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} = 0$$

$$(2y-4)\frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{1}{y-2}$$

At $(-2,1)$, $\frac{dy}{dx} = -1$

The slope of the tangent line is -1.

Tangent line : $y - 1 = -(x+2)$

The slope of the normal line is 1.

Normal line : $y - 1 = x + 2$

(38) $x\sin 2y = y\cos 2x$

$$2x\cos 2y\frac{dy}{dx} + \sin 2y = -2y\sin 2x + \cos 2x\frac{dy}{dx}$$

At $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$,

$$2\cdot\frac{\pi}{4}(-1)\frac{dy}{dx} + 0 = -2\left(\frac{\pi}{2}\right)(1) + 0$$

$$-\frac{\pi}{2}\frac{dy}{dx} = -\pi$$

$$\frac{dy}{dx} = 2$$

The slope of the tangent line is 2.

Tangent line : $y - \frac{\pi}{2} = 2\left(x - \frac{\pi}{4}\right)$

The slope of the normal line is $-\frac{1}{2}$

Normal line : $y - \frac{\pi}{2} = -\frac{1}{2}\left(x - \frac{\pi}{4}\right)$

(36) $x^2 - \sqrt{3}xy + 2y^2 = 5$

$$2x - \sqrt{3}\left(x\frac{dy}{dx} + y\right) + 4y\frac{dy}{dx} = 0$$

At $(\sqrt{3}, 2)$,

$$2\sqrt{3} - \sqrt{3}\left(\sqrt{3}\frac{dy}{dx} + 2\right) + 4(2)\frac{dy}{dx} = 0$$

$$2\sqrt{3} - 3\frac{dy}{dx} - 2\sqrt{3} + 8\frac{dy}{dx} = 0$$

$$5\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 0$$

The slope of the tangent line is 0.

Tangent line : $y = 2$

Normal line : $x = \sqrt{3}$

(40) $x^2\cos^2 y - \sin y = 0$

$$-2x^2\cos y\sin y\frac{dy}{dx} + 2x\cos^2 y - \cos y\frac{dy}{dx} = 0$$

At $(0, \pi)$,

$$-\cos\pi\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 0$$

The slope of the tangent line is 0

Tangent line : $y = \pi$

Normal line : $x = 0$