Hermitian matrix For $A \in \mathbb{C}^{n\times n}$, we define A^* to be its conjugate transpose. A is called Hermitian if

$$A = A^*.$$

Properties of a Hermitian matrix

1. Its eigenvalues are all real.

2. If all of its entries are real, it is symmetric.

3. It is normal matrix. (i.e. $AA^* = A^*A$.) Therefore, it is diagonalizable, and its eigenvectors form orthogonal basis of $\mathbb{C}^n$.

Rayleigh Quotient If A is Hermitian, we define

$$r(x) = \frac{x^* A x}{x^* x},$$

for all x in $\mathbb{C}^n$.

Note If x is an eigenvector of A, we have that $r(x) = ...$

Restriction to Real Symmetric Matrices In most application, we are interested in matrices with real entries. Therefore, we restrict ourselves to real symmetric matrices.

For real symmetric $A \in \mathbb{R}^{n \times n}$, the Rayleigh quotient is defined by

$$r(x) = \frac{x^T A x}{x^T x},$$

for $x \in \mathbb{R}^n$.

Note that if $\|x\|_2 = 1$, $r(x)$ reduces to $x^T A x$.

Stationary point We can consider r as a real-valued function of multivariable. An eigenvector of A is a stationary point of r.

Power Iteration

Choose $u^{(0)}$ such that $\|u^{(0)}\|_2 = 1$. Then, we repeat the following steps.

1. $w^{(k)} = Au^{(k-1)}$
2. $u^{(k)} = \dfrac{w^{(k)}}{\|w^{(k)}\|_2}$

Theorem Suppose $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$ and $v_1^T u^{(0)} \neq 0$.

Then the power iteration satisfies

1. the difference between $u^{(k)}$ and $|v_1|$ is of order $O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$

2. $|\lambda^{(k)} - \lambda_1| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right)$. Here, we define $\lambda^{(k)} = (u^{(k)})^T A u^{(k)}$.

Assume μ is not an eigenvalue of A. We have that if λ is an eigenvalue of A, then $(\lambda - \mu)^{-1}$ is an eigenvalue of $(A - \mu I)^{-1}$. If λ_i is closer to μ than any other eigenvalues of A, we have that $(\lambda_i - \mu)^{-1}$ is much lager than any other $(\lambda - \mu)^{-1}$.

If we apply the power iteration to the matrix $(A - \mu I)^{-1}$, the process will quickly converge to v_i, a unit eigenvector corresponding to λ_i. We call this...

Inverse Iteration

Choose $u^{(0)}$ such that $\|u^{(0)}\|_2 = 1$.

1. Solve for $w^{(k)}$ from $(A - \mu I)w^{(k)} = u^{(k-1)}$.
2. $u^{(k)} = \dfrac{w^{(k)}}{\|w^{(k)}\|_2}$

Rayleigh Quotient Iteration We incorporate the Rayleigh quotient to the inverse iteration to approximate both eigenvector and eigenvalue of A.

Choose $u^{(0)}$ such that $\|u^{(0)}\|_2 = 1$. Then, we have that $\lambda^{(0)} = (u^{(0)})^T A u^{(0)}$. Then, we repeat the following steps.

1. Solve for $w^{(k)}$ from $(A - \mu I)w^{(k)} = u^{(k-1)}$.
2. $u^{(k)} = \dfrac{w^{(k)}}{\|w^{(k)}\|_2}$
3. $\lambda^{(k)} = (u^{(k)})^T A u^{(k)}$

Theorem When $u^{(0)}$ is chosen so that the Rayleigh quotient iteration converges, the convergence rate is cubic. i.e.

$$\|\lambda^{(k+1)} - \lambda_i\|_2 = O(|\lambda^{(k)} - \lambda_i|^3).$$