Matrix eigenvalue problems

Recall

1. $A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}$

$$\Rightarrow \sigma(A) = \{2, 4\}, \quad v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

2. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Applications of eigenvalue

1. **System of ODE**

   Let $v = [y, z]^T$. Consider

$$v' = Av,\ v(0) = [-4, 2]^T \quad \text{where } A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}.$$

2. **Eigenvalue decomposition** A matrix $A \in \mathbb{C}^{n\times n}$ is called **diagonalizable** if it can be written as

$$A = X\Gamma X^{-1},$$

where $\Gamma = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix whose entries are eigenvalues of A, and $X = [x_1, \ldots, x_n]$ is a matrix whose columns are associated eigenvectors.

Gerschgorin Theorem All of the eigenvalues of A must lie in the union of the Gerschgorin disks

$$K_i = \{z \in \mathbb{C}\,,\ |z - a_{i,i}| \leq r_i\},$$

where the radii are

$$r_i = \sum_{j \neq i} |a_{i,j}|.$$

Example

Consider the matrix $A = \begin{bmatrix} 1+0.5i & 0.5 & 0.1 \\ 0.3 & 1-0.5i & 0.5 \\ 0.4 & 0 & -0.5 \end{bmatrix}$.

Remark

Because the set of eigenvalues of A is the same as that of the transpose of $A \in \mathbb{C}^{n \times n}$, it follows that....

The power method

Sometimes, we may need to compute only the largest eigenvalue (in terms of absolute value) instead of all of them. Suppose the eigenvalues of A are arranged in descending order

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|,$$

and that the associated eigenvectors

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

make up a basis of $\mathbb{C}^n$. We can write $u^{(0)} \in \mathbb{C}^n$ as

$$u^{(0)} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n.$$

Define the iteration process to be

$$u^{(k+1)} = Au^{(k)}, k = 0, 1, 2, \ldots$$

we have that $u^{(1)} =$ .....

Assume $u^{(0)}$ is chosen so that $a_1 \neq 0$, and that $|\lambda_1| > |\lambda_2|$, we have

$$\lim_{k\to\infty} \frac{u^{(k)}}{\lambda_1^k} = ...$$

Thus, the quotients $q_i^{(k)} = \dfrac{u_i^{(k)}}{u_i^{(k-1)}}, \quad u_i^{(k-1)} \neq 0$, satisfies

$$\lim_{k\to\infty} q_i^{(k)} = ...$$

given that the ith component of $\vec{v_1}$ is not zero.