Laplace's equation

$$\nabla^2 u = 0, \quad (x, y) \in D, \tag{1a}$$

$$u = g(x, y), \quad (x, y) \in \partial D, \tag{1b}$$

Finite difference approximation

Let $D \cup \partial D = [0, a] \times [0, b]$. We discretize x and y into

$$x_j = j\Delta x, \quad j = 0, \ldots, N+1,$$

$$y_k = k\Delta y, \quad k = 0, \ldots, M+1,$$

where $\Delta x = a/(N+1)$ and $\Delta y = b/(M+1)$.

Approximate the second derivatives in (1a) with the centered differences to get the scheme

$$-\lambda^2 u_{j+1,k} + 2(1+\lambda^2)u_{j,k} - \lambda^2 u_{j-1,k} - u_{j,k-1} - u_{j,k-1}, \tag{2}$$

for $j = 1, \ldots, N$ and $k = 1, \ldots, M$. Here, $\lambda = \dfrac{\Delta y}{\Delta x}$.

The boundary condition (1b) gives the values for the points on the boundary: $u_{0,k}$, $u_{N+1,k}$, $u_{j,0}$, and $u_{j,M+1}$.

Vector form Let g be defined such that $g := g_L, g_R, g_T, g_B$ on the left, right, top, and bottom of ∂D respectively. We can then write the scheme (2) in vector-matrix form as:

$$Av = b,$$

where A is an $MN \times MN$ matrix. The vector $\vec{v}$ is defined by $\vec{v} = [\vec{v}_1, \ldots, \vec{v}_M]^T$, where $\vec{v}_k = [u_{1,k}, \ldots u_{N,k}]^T$. Because the matrix A is large, we need a numerical method to approximate A^{-1}.

Matrix eigenvalue problems

Definitions We define $\mathbb{C}^{n\times n}$ to be the set of all $n \times n$ complex matrices. The set of complex vectors of size $n \times 1$ is called $\mathbb{C}^n$.

1. Let A be in $\mathbb{C}^n$, we say $\lambda \in \mathbb{C}$ is an **eigenvalue** of A if there is a non-zero vector $x \in \mathbb{C}^n$ such that

$$Ax = \lambda x. \tag{3}$$

2. The vector x satisfying (3) is called **eigenvector** of A associated with λ. We denote the subspace of $\mathbb{C}^n$ that contains all eigenvectors associated with λ by E_λ. It is called **eigenspace**.

3. The **geometric multiplicity** of an eigenvalue λ is the dimension of the eigenspace E_λ.

4. The set of all eigenvalues of A is called **spectrum** of A. It is denoted by $\sigma(A)$.

5. The characteristic polynomial of A, denoted p_A, is the $n-$degree polynomial

$$p_A(z) = \det(zI - A).$$

Note λ is an eigenvalue of A if and only if λ is a root of p_A.

6. The **algebraic multiplicity** of an eigenvalue λ is the multiplicity of λ as a root of p_A.

Note The algebraic multiplicity of λ is greater than or equal to the geometric multiplicity of λ.

Examples

1. $A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Applications of eigenvalue

1. **System of ODE**

   Let $v = [y, z]^T$. Consider

$$v' = Av,\ v(0) = [-4, 2]^T \quad \text{where } A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}.$$

2. **Eigenvalue decomposition** A matrix $A \in \mathbb{C}^{n \times n}$ is called **diagonalizable** if it can be written as

$$A = X \Gamma X^{-1},$$

where $\Gamma = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix whose entries are eigenvalues of A, and $X = [x_1, \ldots, x_n]$ is a matrix whose columns are associated eigenvectors.

Gerschgorin Theorem All of the eigenvalues of A must lie in the union of the Gerschgorin disks

$$K_i = \{z \in \mathbb{C}\,,\ |z - a_{i,i}| \leq r_i\},$$

where the radii are

$$r_i = \sum_{j \neq i} |a_{i,j}|.$$