Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \ell, \quad t > 0 \tag{1a}$$

$$u(0,t) = u(\ell,t) = 0, \quad t > 0 \tag{1b}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le \ell. \tag{1c}$$

d'Alembert Representation If we assume $x \in (-\infty, \infty)$, then we can show that

$$u(x,t) = F(x+ct) + G(x-ct),$$

is a solution to (1a) for any F and G that are twice differentiable. To satisfy the initial condition (1c), the solution u takes the form

$$u(x,t) = \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\, dz.$$

Example Consider the problem with $g = 0$ and f is given by

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the solution at time t looks like:

Domain of dependence The solution of the wave equation at (x, t) depends on the initial value at.....

Explicit method We approximate the second derivatives in (1a) with the centered differences (in time and space) to get

$$\frac{u_{j,k+1} - 2u_{j,k} + u_{j,k-1}}{(\Delta t)^2} = c^2 \frac{u_{j+1,k} - 2u_{j,k} + u_{j-1,k}}{h^2}$$

we get the scheme

$$u_{j,k+1} = \lambda^2 u_{j+1,k} + 2(1 - \lambda^2) u_{j,k} + \lambda^2 u_{j-1,k} - u_{j,k-1}, \qquad (2)$$

for $j = 1, \ldots, N$ and $k = 1, \ldots, M - 1$. Here, $\lambda = \dfrac{c\Delta t}{h}$.

Note We still need to approximate $u_{j,1}$ for $j = 1, \ldots, N$.

CFL condition We can show that the CFL condition for the scheme (2) is $\lambda \leq 1$.

Stability We claim without proving that the stability condition for the scheme (2) is $\lambda \leq 1$.

Laplace's equation

$$\nabla^2 u = 0, \quad (x, y) \in D, \tag{3a}$$

$$u = g(x, y), \quad (x, y) \in \partial D, \tag{3b}$$

where ∇ denotes $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$.

Maximum and Minimum principles The maximum and minimum of the solution occurs on the boundary. It follows that the solution is unique and the problem is stable.

Finite difference approximation For a rectangular domain $D \cup \partial D = [0, a] \times [0, b]$, we discretize x and y into

$$x_j = j\Delta x, \quad j = 0, \ldots, N+1,$$

$$y_k = k\Delta y, \quad k = 0, \ldots, M+1,$$

where $\Delta x = a/(N+1)$ and $\Delta y = b/(M+1)$.

As before, we approximate the second derivatives in (3a) with the centered differences to get

$$\frac{u_{j+1,k} - 2u_{j,k} + u_{j-1,k}}{(\Delta x)^2} + \frac{u_{j,k+1} - 2u_{j,k} + u_{j,k-1}}{(\Delta y)^2} = 0.$$

Let $\lambda = \dfrac{\Delta y}{\Delta x}$, we get the scheme

$$-\lambda^2 u_{j+1,k} + 2(1+\lambda^2)u_{j,k} - \lambda^2 u_{j-1,k} - u_{j,k-1} - u_{j,k-1}, \qquad (4)$$

for $j = 1, \ldots, N$ and $k = 1, \ldots, M$. Here, $\lambda = \dfrac{\Delta y}{\Delta x}$.

The boundary condition (3b) gives the values for the points on the boundary: $u_{0,k}$, $u_{N+1,k}$, $u_{j,0}$, and $u_{j,M+1}$.

Vector form Let $N = M = 3$, and let g be defined such that $g := g_L, g_R, g_T, g_B$ on the left, right, top, and bottom of ∂D respectively. We can then write the scheme (4) in vector-matrix form as: