Heat equation

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0 \tag{1a}$$

$$u(0,t) = u(1,t) = 0, \quad t > 0 \tag{1b}$$

$$u(x,0) = g(x), \quad 0 \le x \le 1. \tag{1c}$$

Forward difference in time

$$u_{j,k} = \lambda u_{j+1,k} + (1-2\lambda) u_{j,k} + \lambda u_{j+1,k},$$

where $\lambda = \dfrac{\Delta t}{h^2}$.

Vector form

$$\vec{u}_{k+1} = A\vec{u}_k,$$

where

$$A = \begin{bmatrix} 1-2\lambda & \lambda & 0 & \dots & 0 \\ \lambda & 1-2\lambda & \lambda & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \lambda & 1-2\lambda & \lambda \\ 0 & \dots & 0 & \lambda & 1-2\lambda \end{bmatrix}$$

Stability

$$w_{k+1} = \kappa w_k,$$

where $\kappa = 1 - 4\lambda \sin^2\left(\frac{h}{2}\right)$ is the amplification factor for this method. The stability condition for this method is $|\kappa| \leq 1$, which holds when

$$\lambda \leq \frac{1}{2}.$$

Backward difference in time

Replace u_t in (1a) with the backward difference

$$\frac{u_{j,k} - u_{j,k-1}}{\Delta t},$$

we get the scheme

$$-\lambda u_{j+1,k} + (1 + 2\lambda)u_{j,k} + u_{j-1,k} = u_{j,k-1}$$

Vector form

$$B\vec{u}_k = \vec{u}_{k-1},$$

where

$$B = \begin{bmatrix} 1+2\lambda & -\lambda & 0 & \dots & 0 \\ -\lambda & 1+2\lambda & -\lambda & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & -\lambda & 1+2\lambda & -\lambda \\ 0 & \dots & 0 & -\lambda & 1+2\lambda \end{bmatrix}$$

Stability For this implicit method, the amplification factor is given by

$$\kappa = \frac{1}{1 + 4\lambda \sin^2\left(\frac{h}{2}\right)}$$

Crank-Nicolson method Vector form

$$(B+I)\vec{u}_{k+1} = (A+I)\vec{u}_k$$

Method of line From equation (1a), we replace u_{xx} by the centered difference but leave the variable t continuous. This reduces the PDE to IVP problem

$$\frac{d}{dt}\vec{u}(t) = C\vec{u}(t), \quad t > 0,$$

$$\vec{u}(0) = \vec{g},$$

where $C = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 \end{bmatrix}.$

Advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0 \tag{2a}$$

$$u(x,0) = g(x), \tag{2b}$$

where $a > 0$.

Method of Characteristics

The idea is to transform the variable x and t into s and r so that

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial t} + a\frac{\partial}{\partial s},$$

which gives

$$\frac{\partial u}{\partial r} = 0.$$

Upwind scheme We use backward difference in space to approximate u_x.

$$\frac{u_{j,k+1} - u_{j,k}}{\Delta t} + a\frac{u_{j,k} - u_{j-1,k}}{h} = 0.$$

Solving for $u_{j,k+1}$, we get

$$u_{j,k+1} = (1-\lambda)u_{j,k} + \lambda u_{j-1,k},$$

where $\lambda = \dfrac{a\Delta t}{h}$.

Downwind scheme We use forward difference in space to approximate u_x.

$$\frac{u_{j,k+1} - u_{j,k}}{\Delta t} + a\frac{u_{j+1,k} - u_{j,k}}{h} = 0.$$

Solving for $u_{j,k+1}$, we get

$$u_{j,k+1} = -\lambda u_{j+1,k} + (1+\lambda)u_{j,k},$$

where $\lambda = \dfrac{a\Delta t}{h}$.

Numerical domain of dependence The grid points along the $x-$axis that contribute to the approximation of the solution at the point (x_j, t_k) are called numerical domain of dependence.

On the other hand, for the advection equation, the exact value of u at the point (x_j, t_k) comes from the value of u at the point $(\bar{x}_0, 0)$ where $\bar{x}_0 =$ .....

CFL condition (Courant-Friedrichs-Lewy) The numerical domain of dependence must bound, or contain, the domain of dependence for the problem.

Stability Similar to heat equation, we assume

$$u_{j,k} = w_k e^{ix_j},$$

and can show that, for the upwind method

$$w_k = \kappa^k w_0, \quad \text{where,}$$

$$\kappa = 1 - \lambda + \lambda \cos(h) - i\lambda \sin(h).$$

The stability condition for the upwind method is

$$|\kappa| \leq 1,$$

which holds when $\lambda \leq 1$. This agrees with the CFL condition.

Centered difference in space We try using centered difference in space to approximate u_x.

$$\frac{u_{j,k+1} - u_{j,k}}{\Delta t} + a\frac{u_{j+1,k} - u_{j-1,k}}{2h} = 0.$$

Solving for $u_{j,k+1}$, we get

$$u_{j,k+1} = -\frac{\lambda}{2}u_{j,k} + u_{j,k} + \frac{\lambda}{2}u_{j-1,k},$$

where $\lambda = \dfrac{a\Delta t}{h}$.

We can show that the CFL condition for this method is $\lambda \leq 1$. However, one can show that the amplifying factor for this method is

$$\kappa = 1 - i\lambda \sin(h),$$

which implies that the method is unstable because .....