Math 455: Homework 9 (due Friday, November 7, 2014)
Work in group of 4-5 people.

1. **Gauss-Legendre quadrature** is an approximation of $\int_{-1}^{1} f(x)\,dx$ using

$$\sum_{i=1}^{n} w_i f(x_i),$$

where $x_i\,, i = 1, \ldots, n$ are the roots of $n-$degree Legendre polynomials.

Use 16-point Gauss-Legendre quadrature to find $\tilde{p}$, the least square approximation of

$$f(x) = \sin(2\pi x^2 e^x), \quad x \in [-1, 1]$$

from P_6. You can find all w_i's and x_i's for the 16-point Gauss-Legendre quadrature from the course homepage. The Legendre polynomials of order upto six are given by:

$$\mathcal{L}_0(x) = 1,\ \mathcal{L}_1(x) = x,\ \mathcal{L}_2(x) = \frac{3x^2 - 1}{2},\ \mathcal{L}_3(x) = \frac{5x^3 - 3x}{2},\ \mathcal{L}_4(x) = \frac{35x^4 - 30x^2 + 3}{8},$$

$$\mathcal{L}_5(x) = \frac{63x^5 - 70x^3 + 15x}{8}, \quad \mathcal{L}_6(x) = \frac{231x^6 - 315x^4 + 105x^2 - 5}{16}.$$

Plot both $\tilde{p}$ and f on the same axes to compare. Include title, legend and labels.

2. Let $I_1 \ldots, I_{20}$ be the uniform subintervals of $[-1, 1]$. Use 16-point Gauss-Legendre quadrature (with the change of variable) to find the least square approximation $\tilde{p}_j$, $j = 1, \ldots, 20$, for the function

$$f(x) = \sin(2\pi x^2 e^x), \quad x \in I_j$$

from P_2. Plot f and all $\tilde{p}_j$ on the same axes. Include title, legend and labels.

3. Show that the Heun's method

$$\begin{aligned} y^* &= y_n + hf(x_n, y_n) \\ y_{n+1} &= y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y^*)\right] \end{aligned}$$

has order 2.

4. Implement the shooting method using forward Euler with $h = 1$ to solve the boundary value problem below by hand.

$$y'' - 2y' - y = x(x - 1), \quad y(0) = -12, \quad y(3) = -6.$$

Use $z_0 = 0$ and $z_0 = 10$ for your first two initial guesses. Keep decimals upto third place.