Math 455: Homework 8 (due Friday, October 24, 2014)
Work in group of 4-5 people.

1. Define cubic B-Spline on the set of evenly distributed points $x_i = i \cdot h$, $h > 0$ by

$$B_i(x) = \frac{1}{6h^3} \begin{cases} (x - x_i)^3, & x_i \le x < x_{i+1} \\ h^3 + 3h^2(x - x_{i+1}) + 3h(x - x_{i+1})^2 - 3(x - x_{i+1})^3, & x_{i+1} \le x < x_{i+2} \\ h^3 + 3h^2(x_{i+3} - x) + 3h(x_{i+3} - x)^2 - 3(x_{i+3} - x)^3, & x_{i+2} \le x < x_{i+3} \\ (x_{i+4} - x)^3, & x_{i+3} \le x < x_{i+4} \\ 0, & \text{otherwise.} \end{cases}$$

Recall that we can interpolate f at $x_o, \dots, x_n$ using a cubic spline s given by

$$s(x) = \sum_{i=-3}^{n-1} \alpha_i B_i(x),$$

where two more conditions on the boundary are needed. There well-known options for boundary conditions are:

1. **Natural end conditions**
   $s''(x_0) = 0$ and $s''(x_n) = 0$.
2. **Hermite end conditions**
   $s'(x_0) = f'(x_0)$ and $s'(x_n) = f'(x_n)$.
3. **Periodic end conditions**
   $s'(x_0) = s'(x_n)$ and $s''(x_0) = s''(x_n)$.

Using the first two options, write MATLAB code to find α_i for $s(x)$ that interpolates $f(x) = e^x \cos(2\pi x)$ on $[x_0, x_n] = [0, 1]$ with $h = 1/10$.

Plot the graphs of f and s with title and legend. Label all points.

2. Show that the multistep method

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}), \quad n \ge p, \quad a_p \ne 0 \text{ or } b_p \ne 0$$

is consistent if and only if

$$\sum_{j=0}^{p} a_j = 1, \quad -\sum_{j=0}^{p} j a_j + \sum_{j=-1}^{p} b_j = 1.$$

3. Derive the Adams-Bashforth method and the Adams-Moulton method with $p = 2$. Find the truncation errors and orders of accuracy.
   i.e. find explicit and implicit three-step methods. For the Adam-Bashforth method, you need to use the values of f at x_n, x_{n-1}, x_{n-2}. For the Adams-Moulton method, you need to use the values of f at $x_{n+1}, x_n, x_{n-1}, x_{n-2}$.