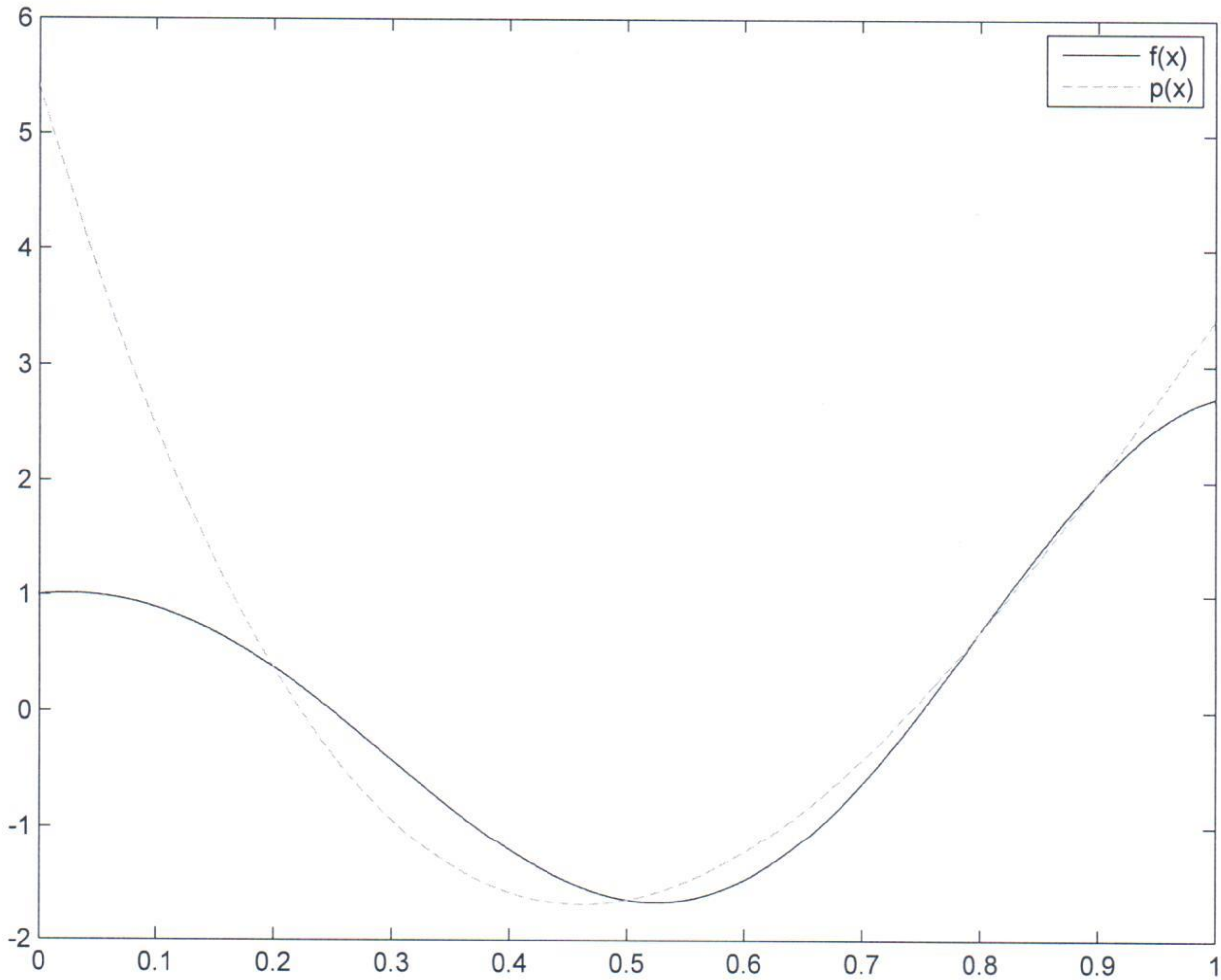
```
function [ px ] = Interpolation( xi, yi, x )
%Interpolation Return value of p at x
%   where p interpolates the data points xi and yi

coef = NewtonDD( xi, yi );

n = length(xi);

px = coef(1)*ones(size(x));
for r=2:n
    temp = ones(size(x));
    for t=1:r-1
        temp = temp.*(x-xi(t));
    end
    px = px + coef(r)*temp;
end

end
```

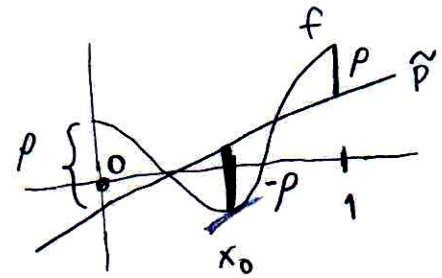
```
% Plot p(x) and f(x), and compute error at 0:0.1:1.

xi=[.2, .5, .8, .9 ];
yi = exp(xi).*cos(2*pi*xi);
x=0:1/1000:1;
[ px ] = Interpolation( xi, yi, x );
plot(x,exp(x).*cos(2*pi*x),'-k',x,px,'--k',xi,yi,'ok');
legend('f(x)','p(x)');
a=0:.1:1;
b=exp(a).*cos(2*pi*a);
[ px ] = Interpolation( xi, yi, a );
err=abs(b-px);
```


f(x)
p(x)
6
5
4
3
2
1
0
-1
-2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

```
abs(b-px)'

ans =

    4.4266
    1.5743
         0
    0.5249
    0.3786
         0
    0.2470
    0.2058
         0
    0.0000
    0.6760

format long
abs(b-px)'

ans =

   4.426602048290546
   1.574339966474080
                   0
   0.524896609039618
   0.378631369562300
                   0
   0.246959516382485
   0.205803507921478
                   0
   0.000000000000001
   0.676025579709470
```

② Find best approx. of $f \in (C[0,1], \|\cdot\|_\infty)$ form P_1.

(a) $f(x) = \cos(2\pi x) + x$



<u>Method 1</u>

Let $\tilde{p}(x) = a + bx$

From graph, we can solve for a, b, x_0, ρ from the following equations:

(i) $f(0) - \tilde{p}(0) = \rho \quad \Rightarrow \quad 1 - a = \rho$

(ii) $f(x_0) - \tilde{p}(x_0) = -\rho \quad \Rightarrow \quad \cos(2\pi x_0) + x_0 - (a + b x_0) = -\rho$

(iii) $f(1) - \tilde{p}(1) = \rho \quad \Rightarrow \quad 2 - (a+b) = \rho$

(iv) $f'(x_0) = \tilde{p}'(x_0) \quad \Rightarrow \quad -2\pi \sin(2\pi x_0) + 1 = b$

Solving all of these equations, we get

$a = 0, \quad b = 1, \quad x_0 = \frac{1}{2}, \quad \rho = 1.$

Therefore, $\boxed{\tilde{p}(x) = x}$

<u>Method 2</u> If you can find $\tilde{p}(x) \in P_n$ and $x_1, \dots, x_n$ in $[0,1]$ such that

(i) $|f(x_i) - \tilde{p}(x_i)| = \|f - \tilde{p}\|_\infty, \quad i = 1, \dots, n+2$

(ii) $f(x_i) - \tilde{p}(x_i) = (-1)\left(f(x_{i+1}) - \tilde{p}(x_{i+1})\right), \quad i = 1, \dots, n+1,$

then we're done.

Here, we claim that $\tilde{p}(x) = x$ and $x_1, x_2, x_3 = 0, \frac{1}{2}, 1$. It's easy to verify that both (i) and (ii) are satisfied. Therefore, $\boxed{\tilde{p}(x) = x}$.

(b) $f(x) = \min(5x - 2x^2, \ 22(1-x)^2)$

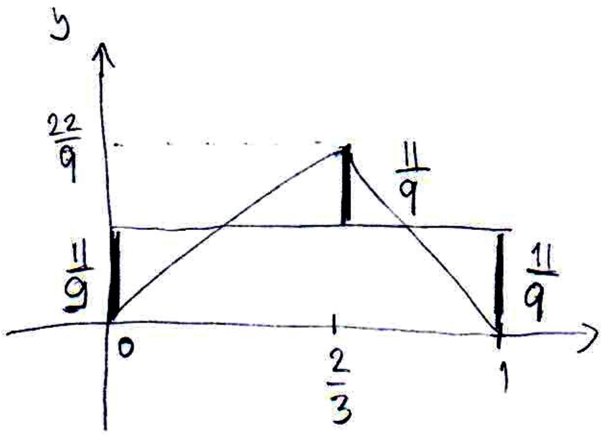
<u>Solution</u> We first sketch the graph of f by finding the intersection of $5x - 2x^2$ and $22(1-x)^2$.

Set $5x - 2x^2 = 22(1-x)^2$

We get
$$5x - 2x^2 = 22 - 44x + 22x^2$$
$$0 = 22 - 49x + 24x^2$$
$$0 = (8x - 11)(3x - 2)$$

Because $x \in [0,1]$, we get $x = \frac{2}{3}$.

$$f(\tfrac{2}{3}) = 22(1 - \tfrac{2}{3})^2 = \frac{22}{9}$$

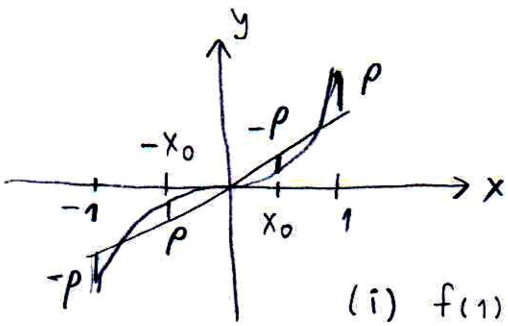


From graph, it is obvious that $\tilde{p}(x) = \frac{11}{9}$ (half of $\frac{22}{9}$) because it satisfies conditions (i) and (ii) from part (a), with $x_1, x_2, x_3 = 0, \frac{2}{3}, 1$.

So, $\boxed{\tilde{p}(x) = \frac{11}{9}}$

③ $f(x) = x|x|$, $f \in (C[-1,1], \|\cdot\|_\infty)$.

<u>Solution</u> Let $\tilde{p}(x) = a + bx + cx^2$



From graph, we know that x_0 and $-x_0$ are the two points where the maximum errors occur.

We can solve for a, b, c, x_0, ρ from

(i) $f(1) - \tilde{p}(1) = \rho \Rightarrow 1 - (a+b+c) = \rho$

(ii) $f(x_0) - \tilde{p}(x_0) = -\rho \Rightarrow x_0^2 - (a + bx_0 + cx_0^2) = -\rho$

(iii) $f(-x_0) - \tilde{p}(-x_0) = \rho \Rightarrow -x_0^2 - (a - bx_0 + cx_0^2) = \rho$

(iv) $f(-1) - \tilde{p}(-1) = -\rho \Rightarrow -1 - (a - b + c) = -\rho$

(v) $f'(x_0) = \tilde{p}'(x_0) \Rightarrow 2x_0 = b + 2cx_0$

<u>Method 1</u>

Solving (i) - (v), we get $a = c = 0$, $b = -2 + 2\sqrt{2}$, $\rho = 3 - 2\sqrt{2}$ and $x_0 = -1 + \sqrt{2}$

so, $\tilde{p}(x) = (-2 + 2\sqrt{2})x$

<u>Method 2</u> Note that $f(x)$ is an odd function ($f(-x) = -f(x)$).

From equations (i) - (v) (and from graph), we have that $\tilde{p}(x)$ also has to be odd function.

(i.e. we want $\tilde{p}(-x) = -\tilde{p}(x)$, for $x = 1, x_0$ here.)

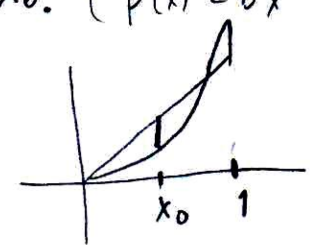
Therefore, we automatically have that $c = 0$ and $a = 0$.

This reduces to solving only for b, ρ, x_0. ($\tilde{p}(x) = bx$)

(i) $f(1) - \tilde{p}(1) = \rho \Rightarrow 1 - b = \rho$

(ii) $f(x_0) - \tilde{p}(x_0) = -\rho \Rightarrow x_0^2 - bx_0 = -\rho$

(iii) $f'(x_0) = \tilde{p}'(x_0) \Rightarrow 2x_0 = b$



solving (i)-(iii) to get $x_0 = -1 + \sqrt{2}$, $b = -2 + 2\sqrt{2}$, $\rho = 3 - 2\sqrt{2}$.

so, $\tilde{p}(x) = (-2 + 2\sqrt{2})x$

④ $f(x) = e^x \in C[-1,1]$

$\boxed{N=0}$ (a) $\tilde{p}(x) = a_0 \cdot 1 = a_0$

Normal eqn: $a_0 \langle 1,1 \rangle = \langle f, 1 \rangle$

so, $a_0 = \dfrac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \dfrac{\int_{-1}^{1} e^x \, dx}{\int_{-1}^{1} 1 \, dx}$

$= \dfrac{-\frac{1}{e} + e}{2}$

so, $\boxed{\tilde{p}(x) = -\frac{1}{2e} + \frac{e}{2}}$

(b) $\tilde{p}(x) = b_0 \mathcal{L}_0(x)$

Because $\mathcal{L}_0(x) = 1$, we get the same result:

$\boxed{\tilde{p}(x) = -\frac{1}{2e} + \frac{e}{2}} \approx 1.1752$

$\boxed{N=1}$ (a) $\tilde{p}(x) = a_0 \cdot 1 + a_1 \cdot x$

Normal eqn: $\begin{cases} a_0 \langle 1,1 \rangle + a_1 \langle x, 1 \rangle = \langle f, 1 \rangle \\ a_0 \langle 1, x \rangle + a_1 \langle x, x \rangle = \langle f, x \rangle \end{cases}$

$\Rightarrow \begin{cases} 2a_0 + 0 = -\frac{1}{e} + e \\ 0 + \frac{2}{3} a_1 = \frac{2}{e} \end{cases}$

so, $a_0 = -\frac{1}{2e} + \frac{e}{2}$, $a_1 = \frac{3}{e}$

$\boxed{\tilde{p}(x) = \left(-\frac{1}{2e} + \frac{e}{2}\right) + \frac{3}{e} x}$

(b) $\tilde{p}(x) = b_0 \mathcal{L}_0(x) + b_1 \mathcal{L}_1(x)$, where $\mathcal{L}_0(x) = 1$, $\mathcal{L}_1(x) = x$.

We get $b_0 = \dfrac{\langle f, \mathcal{L}_0 \rangle}{\langle \mathcal{L}_0, \mathcal{L}_0 \rangle} = \dfrac{-\frac{1}{e} + e}{2} = -\frac{1}{2e} + \frac{e}{2}$

$b_1 = \dfrac{\langle f, \mathcal{L}_1 \rangle}{\langle \mathcal{L}_1, \mathcal{L}_1 \rangle} = \dfrac{\frac{2}{e}}{\frac{2}{3}} = \frac{3}{e}$

So, $\boxed{\tilde{p}(x) = \left(-\frac{1}{2e} + \frac{e}{2}\right) + \frac{3}{e} x}$ $\approx 1.1752 + 1.104\,x$

$\boxed{N=3}$ (a) $\tilde{p}(x) = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2$

Normal eqn $\begin{cases} a_0 \langle 1,1\rangle + a_1 \langle x,1\rangle + a_2 \langle x^2,1\rangle = \langle f,1\rangle \\ a_0 \langle 1,x\rangle + a_1 \langle x,x\rangle + a_2 \langle x^2,x\rangle = \langle f,x\rangle \\ a_0 \langle 1,x^2\rangle + a_1 \langle x,x^2\rangle + a_2 \langle x^2,x^2\rangle = \langle f,x^2\rangle \end{cases}$

$$\Rightarrow \begin{cases} 2a_0 + 0 + \frac{2}{3}a_2 = -\frac{1}{e} + e \\ 0 + \frac{2}{3}a_1 + 0 = \frac{2}{e} \\ \frac{2}{3}a_0 + 0 + \frac{2}{5}a_2 = -\frac{5}{e} + e \end{cases}$$

Solving the above system of equations, we get

$$a_0 = \frac{33}{4e} - \frac{3e}{4} \approx 0.996, \quad a_1 = \frac{3}{e} \approx 1.104$$

$$a_2 = -\frac{105}{4e} + \frac{15}{4}e \approx 0.537$$

So, $\tilde{p}(x) = \left(\frac{33}{4e} - \frac{3e}{4}\right) + \frac{3}{e}x + \left(-\frac{105}{4e} + \frac{15}{4}e\right)x^2$

$$\approx 0.996 + 1.104x + 0.537x^2$$

(b) $\tilde{p}(x) = b_0 \mathcal{L}_0(x) + b_1 \mathcal{L}_1(x) + b_2 \mathcal{L}_2(x)$, $\mathcal{L}_2(x) = \frac{3x^2-1}{2}$

$$\left.\begin{aligned} b_0 &= \frac{\langle \mathcal{L}_0, f\rangle}{\langle \mathcal{L}_0, \mathcal{L}_0\rangle} = \frac{-\frac{1}{e}+e}{2} = -\frac{1}{2e} + \frac{e}{2} \\ b_1 &= \frac{\langle \mathcal{L}_1, f\rangle}{\langle \mathcal{L}_1, \mathcal{L}_1\rangle} = \frac{\frac{2}{e}}{\frac{2}{3}} = \frac{3}{e} \end{aligned}\right\} \text{same as before}$$

$$b_2 = \frac{\langle \mathcal{L}_2, f\rangle}{\langle \mathcal{L}_2, \mathcal{L}_2\rangle} = \frac{-\frac{7}{e}+e}{\frac{2}{5}} = -\frac{35}{2e} + \frac{5e}{2}$$

So, $\tilde{p}(x) = \left(-\frac{1}{2e} + \frac{e}{2}\right) + \frac{3}{e}x + \left(-\frac{35}{2e} + \frac{5e}{2}\right)\left(\frac{3x^2-1}{2}\right)$

After some simplification, you get the same answer as above.

⑤ $(V, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_2)$, $U = \text{Span}(1, X)$.

Want to find best approximation of Y from U.

Sol'n Let $\tilde{p}$ be the best approx., then we can write

$$\tilde{p} = aX + b = b \cdot 1 + a \cdot X$$

By normal equation, we get

$$b\langle 1,1\rangle + a\langle X, 1\rangle = \langle Y, 1\rangle$$

$$b\langle 1, X\rangle + a\langle X, X\rangle = \langle Y, X\rangle$$

Using definition of $\langle \cdot, \cdot \rangle$, we get

$$b\left(\sum_{i=1}^{n} 1\right) + a\left(\sum_{i=1}^{n} x_i \cdot 1\right) = \sum_{i=1}^{n} y_i \cdot 1$$

$$b\left(\sum_{i=1}^{n} 1 \cdot x_i\right) + a\left(\sum_{i=1}^{n} x_i \cdot x_i\right) = \sum_{i=1}^{n} y_i \cdot x_i$$

Simplify the above equations to get

$$nb + a\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} y_i$$

$$b\left(\sum_{i=1}^{n} x_i\right) + a\left(\sum_{i=1}^{n} x_i^2\right) = \sum_{i=1}^{n} x_i y_i$$

as needed.