Math 455: Homework 6 Due Tuesday, September 23
(Work in group of 5-6 students)

1. Given n data points $(x_0, y_0), \ldots, (x_n, y_n)$, write a MATLAB program called *Interpolation* which returns the interpolating polynomial value at a certain point x.

   The program should take the vectors $xi = (x_0, \ldots, x_n)$, $yi = (y_0, \ldots, y_n)$, and x as inputs. It returns px, the value of the interpolating polynomial at x, as an output. Write your code so that x can be a vector.

   Example Consider the case with two data points: $(1, 2), (3, 4)$. We have that $xi = [1, 3]$ and $yi = [2, 4]$. We also know that the polynomial which interpolates these two data points is given by $p(x) = x + 1$. Therefore, your function will return $px = x + 1$, where x can be any point or vector. For example,

```
>> px = Interpolation( [1,3], [2,4], 1.5 )

px =

      2.5

>> px = Interpolation( [1,3], [2,4], [0,1,2,3] )

px =

      1 2 3 4
```

   (a) Use this function to plot p and f where p is the polynomial that interpolates $f(x) = e^x \cos(x)$ at the points $x = 0.2, 0.5, 0.8, 0.9$. Use $x = 0 : 1/1000 : 1$. Label the graphs and all the interpolating points.

   (b) Compute the error $|f(x) - p(x)|$ at the points $x = 0, 0.1, 0.2, \ldots, 1$.

   Send all codes via e-mail before class on Tuesday, September 23 (please CC to everyone in the group), and print out the results (graphs and table of errors) to submit in class.

2. Determine the best approximation of $f \in (C[0, 1], \|\cdot\|_\infty)$ from P_1.

   (a) $f(x) = \cos(2\pi x) + x$

   (b) $f(x) = \min(5x - 2x^2, 22(1 - x)^2)$

3. Find the best approximation $\tilde{p} \in P_2$ of $f(x) = x|x|$ where $f \in (C[-1, 1], \|\cdot\|_\infty)$.

4. Let $f(x) = e^x \in C[-1, 1]$. Find the best approximation of f from P_n, $n = 0, 1, 2$, with respect to $\|\cdot\|_2$

   (a) using the normal equation;

   (b) by the expansion of f in Legendre polynomials.

5. In Statistics, we seek a linear function $ax + b$ to best fit the data $(x_1, y_1), \dots, (x_n, y_n)$ in the least square sense. i.e. we want to find a, b such that the error

$$E(a,b) = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2$$

is minimal. Use the normal equation to show that such a, b satisfy the system of equations:

$$\begin{aligned} \left(\sum_{i=1}^{n} x_i\right) a + nb &= \sum_{i=1}^{n} y_i, \\ \left(\sum_{i=1}^{n} x_i^2\right) a + \left(\sum_{i=1}^{n} x_i\right) b &= \sum_{i=1}^{n} x_i y_i. \end{aligned}$$

Hint: Think of this fitting problem as a problem of finding the best approximation of $\mathbf{Y} = [y_1, y_2, \dots, y_n]^T$ in $(\mathbb{R}^n, \|\cdot\|_2)$ from $\operatorname{span}(\mathbf{1}, \mathbf{X})$, where $\mathbf{1} = [1, 1, \dots, 1]^T, \mathbf{X} = [x_1, x_2, \dots, x_n]^T$. Here, we define $\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i=1}^{n} x_i y_i$ and $\|\mathbf{X}\|^2 = \sum_{i=1}^{n} x_i^2$.