Home Work 7

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Make two exam problems with solution for chapter 2. : Approximation for functions.

1) Let $f(x) = x^2+3 \in C[-1,1]$. Find the best appoximation of f from P_2 with respect to $\|\cdot\|_2$ by the expansion of f in Legendre polynomials.

Sol$^{\underline{n}}$

$$\text{Let } \tilde{p}(x) = b_0\mathcal{L}_0(x) + b_1\mathcal{L}_1(x) + b_2\mathcal{L}_2(x)$$

$$\text{from } \mathcal{L}_n = \frac{(-1)^n}{2^n n!}\cdot\frac{d^n}{dx^n}\left[(1-x^2)^n\right]$$

$$\mathcal{L}_0(x) = 1, \quad \mathcal{L}_1(x) = x, \quad \mathcal{L}_2(x) = \frac{3x^2-1}{2}$$

$$b_0\langle 1,1\rangle + b_1\langle x,1\rangle + b_2\left\langle \frac{3x^2-1}{2},1\right\rangle = \langle f,1\rangle$$

$$b_0\langle 1,x\rangle + b_1\langle x,x\rangle + b_2\left\langle \frac{3x^2-1}{2},x\right\rangle = \langle f,x\rangle$$

$$b_0\left\langle 1,\frac{3x^2-1}{2}\right\rangle + b_1\left\langle x,\frac{3x^2-1}{2}\right\rangle + b_2\left\langle \frac{3x^2-1}{2},\frac{3x^2-1}{2}\right\rangle = \left\langle f,\frac{3x^2-1}{2}\right\rangle$$

$$\begin{pmatrix} 2 ✓ & 0 & 0 \\ 0 & 2/3 ✓ & 0 \\ 0 & 0 & 2/3 \end{pmatrix} ✗ \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{20}{3} ✓ \\ 0 ✓ \\ \frac{4}{15} ✓ \end{pmatrix}$$

We know that $\langle \mathcal{L}_n, \mathcal{L}_n\rangle = \frac{2}{2n+1}$

so, $\langle \mathcal{L}_2, \mathcal{L}_2\rangle = \frac{2}{5}$

$$\text{from } b_i = \langle f,\mathcal{L}_i\rangle / \langle \mathcal{L}_i,\mathcal{L}_i\rangle$$

$$b_0 = \langle f,1\rangle / \langle 1,1\rangle = \frac{40}{6} = \frac{20}{3}$$

$$b_1 = \langle f,x\rangle / \langle x,x\rangle = \frac{0}{\frac{2}{3}} = 0$$

$$b_2 = \left\langle f,\frac{3x^2-1}{2}\right\rangle \Big/ \left\langle \frac{3x^2-1}{2},\frac{3x^2-1}{2}\right\rangle = \frac{4}{15}\Big/\frac{2}{5} = \frac{4}{15}\cdot\frac{5}{2} = \frac{2}{3}$$

$$\therefore \quad \tilde{p}(x) = \frac{20}{3} + \frac{\cancel{2}}{3}\left(\frac{3x^2-1}{\cancel{2}}\right) = \frac{20}{3} + x^2 - \frac{1}{3} = x^2 + \frac{19}{3} \quad ✗$$

You should get $\tilde{p} = f$ because f is also from P_2.

15/15

540510594
สุทธิศักดิ์ ธีรเศรษฐวงษา

1. $(C[-1,1], \|\cdot\|_2)$

Let $A(x)=1$, $B(x)=\begin{cases} -1 & (x<0) \\ 1 & (x\geq 0)\end{cases}$

1.1 Show that $A(x)$, $B(x)$ are orthogonal, with respect to $\langle f,g\rangle = \int_{-1}^{1} fg\, dx$

Proof $\langle A,B\rangle = \int_{-1}^{1} A(x)B(x)\,dx$

$= \int_{-1}^{0} -1\,dx + \int_{0}^{1} 1\,dx$

$= (-1) + 1 = 0$ ✓

-1

It is true that $A \perp B$.
However, B is not in $C[-1,1]$.

1.2 For $f(x)=e^x$ on $[-1,1]$, using $A(x),B(x)$ to find $p(x)\in \text{Span}(A(x),B(x))$ that be the best approximation of $f(x)$.

This set is not a subset of $C[-1,1]$;

Sol^n $P(x) = a_1 A(x) + b_1 B(x)$ for some $a,b\in\mathbb{R}$

by A,B are Orthogonal.

$a_1 = \frac{\langle A,f\rangle}{\langle A,A\rangle}$, $b_1 = \frac{\langle B,f\rangle}{\langle B,B\rangle}$

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$\langle A,f\rangle = \int_{-1}^{1} 1\cdot e^x dx = e-e^{-1}$, $\langle B,f\rangle = \int_{-1}^{0} -e^x dx + \int_0^1 e^x dx = -1+e^{-1}+e-1 = e+e^{-1}-2$

$\langle A,A\rangle = \int_{-1}^{1}(1)(1)dx = 2$, $\langle B,B\rangle = \int_{-1}^{0}(-1)(-1)dx + \int_0^1 (1)(1)dx = 2$

$P(x) = \left(\frac{e-e^{-1}}{2}\right)A(x) + \left(\frac{e+e^{-1}-2}{2}\right)B(x)$ O.K.

2. John is a Numerical Analysis teacher. He loves to give homework. He made a rule for his student. "Anyone who sends homework after the deadline will lose the point" But he is not heartless. He plans to give point to every student who send the home work no matter how long after deadline so he use:

This is from ch 3, not ch 2, but o.k.

$\frac{ds}{dt} = C\cdot S\cdot\left(-\frac{1}{2}\right)$ when s is the score after dead line t day(s).

C is point if the home work has been ~~send~~ sent on time.

Sammy is one of John's best students and he's also John's mother's friend's son. He was sick on the deadline day. After that was Saturday and Sunday. More over Monday was Christmas day, another holiday. Totally, he can send homework 4 days after the deadline. Your job is help Sammy calculate what % that he will lose from his work after his long weekend nap. Using Euler's method and time step = 1 day.

Sol^n

t	0	1	2	3	4
r	1	0.5 ✓	0.25 ✓	0.125 ✓	0.0625 ✓

Sammy will get $0.0625\times 100\%$ of his work

It's mean he lose $100-6.25 = 93.75\%$ ✓

That's too mean of John!

สกนบรรณ ศรีสุริยะกาดา

No. : 590510597

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14/15

1. Let $f(x)$ be a function in $[0,1]$.
Let $y = ax+b$ be a linear minimax approximation of $f(x)$.
By your intuition (you don't have to calculate, just give reason.), can you find linear minimax approximation of $-f(x)$?

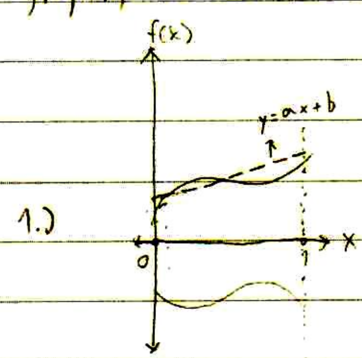
- By the minimax approximation, we get $\text{Max}(|f(x) - (ax+b)|) = P$, $P \geq 0$, P is minimax error
From absolute property, we know that $|a| = |-a|$
$\therefore$ $\text{Max}(|f(x) - (ax+b)|) = \text{Max}(|-(f(x) - (ax+b))|) = \text{Max}(|-f(x) - (-ax-b)|) = P$
Thus $-(ax+b)$ should be a linear minimax approximation of $-f(x)$

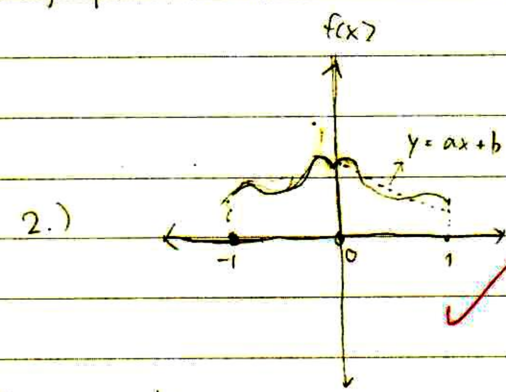
2. Let $f(x)$ be a function in $[0,1]$.
Let $y = ax+b$ be a linear minimax approximation of $f(x)$
Let $f_1(x)$ be a function in $[-1,0]$, $f_1(x) = f(-x)$
By your intuition, can you find linear minimax approximation of $f_1(x)$?

These two questions are similar.

- From $f_1(x) = f(-x)$, $x \in [-1,0]$, we will get $f_1(-x) = f(x)$, $0 \leq x \leq 1$
From $y = ax+b$ be a linear minimax approximation of ~~x~~, we get $\text{Max}(|f(x) - (ax+b)|) = P$, $P \geq 0$
P is minimax error, $x \in [0,1]$
Thus, we should get $a(-x)+b$, $x \in [-1,0]$ be a linear minimax approximation of f_1
because $a(-x)+b = y$ that is a linear minimax approximation of $f(x)$.

By graphing $f(x)$ of 1 and 2, we get the graph like this.





By reflexive property of graph, we can get the answer too.

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Numerical Analysis (H.W.)

1.) Prove that for a Pre-Hilbert space V and a subspace U of V, where $u_1, \dots, u_n$ are basis of U, for $v \in V$ the best approx. $\hat{u}$ of v from U is unique.

Pf. Since $\hat{u}$ is the best approx. of v it must satisfies

$\langle v-\hat{u}, u \rangle = 0 \quad \forall u \in U$, Let $\bar{u}$ be another best approx., then

$\langle v-\bar{u}, u \rangle = 0 \quad \forall u \in U$.

$\therefore \langle v, u \rangle = \langle \hat{u}, u \rangle$ and $\langle v, u \rangle = \langle \bar{u}, u \rangle \quad \forall u \in U$

$\therefore \quad \langle \hat{u}, u \rangle = \langle \bar{u}, u \rangle \Rightarrow \langle \hat{u}-\bar{u}, u \rangle = 0 \quad \forall u \in U$

Since both $\hat{u}$ and $\bar{u}$ belong to U, then $\hat{u}-\bar{u} \in U$

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$\therefore \quad \langle \hat{u}-\bar{u}, \hat{u}-\bar{u} \rangle = 0$

$\because \quad \| \hat{u}-\bar{u} \| = \sqrt{\langle \hat{u}-\bar{u}, \hat{u}-\bar{u} \rangle} = 0$

$\therefore \quad \hat{u} = \bar{u}$ thus $\hat{u}$ is unique QED.

2.) Show that, without using the 2 conditions (i.e. $|(f-\hat{p})(x_i)| = \|f-\hat{p}\|_\infty$; $i=1,\dots,(n+2)$ and $(-1)(f-\hat{p})(x_{i+1}) = (f-\hat{p})(x_i)$; $i=1,\dots,n+1$), that the best approx. $\hat{p}$ of $f \in C[-1,1]$ from P_1 is $\hat{p}(x) = (1/2)\left[\max_{x\in[-1,1]} f(x) + \min_{x\in[-1,1]} f(x)\right]$ if f is such a function that $f(-1) = f(1) = \min_{x\in[-1,1]} f(x)$.

Pf. Let $\hat{p}$ a constant function $\hat{p}(x) = \frac{1}{2}\left[\max\limits_{x\in[-1,1]} f(x) + \min\limits_{x\in[-1,1]} f(x)\right]$ for all $x \in [-1,1]$. We will show that for any $p \in P_1$ such that $p \neq \hat{p}$, $\|f-p\|_\infty > \|f-\hat{p}\|_\infty$. Let $\bar{c}$ be defined by $\bar{c} = \frac{1}{2}\left[\max\limits_{x\in[-1,1]} f(x) + \min\limits_{x\in[-1,1]} f(x)\right]$, then $\hat{p}(x) = \bar{c}$ and $p(x) = ax+b$. Let ξ be the point in $[-1,1]$ such that $f(\xi) = \max\limits_{x\in[-1,1]} f(x)$

if $\xi = 1$ or -1 then $\max f = \min f$ thus f is a constant function, which makes it obvious that $\hat{p}$ indeed is the best approximation.

Let $\xi \in (-1,1)$, since $\|f-\hat{p}\|_\infty = \bar{c}/2$ $p(x)$ must satisfies $p(-1), p(1) \leq f(1) + \bar{c}/2$ otherwise $\|f-p\|_\infty$ would be greater that $\bar{c}/2$, making it a worse approximation that $\hat{p}$. However, since $p(\xi) = \left(\frac{1-\xi}{2}\right)p(-1) + \left(\frac{\xi+1}{2}\right)p(1) \leq \max\{p(-1), p(1)\}$

$$\therefore \quad f(\xi) - p(\xi) \geq f(\xi) - \max\{p(-1), p(1)\} \geq f(\xi) - f(\xi) + \bar{c}/2$$

$$\Rightarrow \quad f(\xi) - p(\xi) \geq \bar{c}/2$$

The only case where $f(\xi) - p(\xi) = \bar{c}/2$ is when $p(-1) = p(1) = f(1)$ i.e. $p = \hat{p}$.

$\therefore$ $\hat{p} = \bar{c}$ is indeed the best approximation of f #

Ⅱ Express $f(x)$ in terms of Legendre's polynomials where $f(x) = x^3 + 2x^2 - x - 3$, $I_i = P_i$

Sol^n

$P_0(x) = 1$

$\therefore 1 = P_0(x)$

$P_1(x) = x$

$\therefore x = P_1(x)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$2P_3(x) = 5x^3 - 3x$

$2P_3(x) + 3x = 5x^3$

$2P_3(x) + 3P_1(x) = 5x^3$

Good idea.

$\therefore x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$2P_2(x) = (3x^2 - 1)$

$2P_2(x) + 1 = 3x^2$

$2P_2(x) + P_0(x) = 3x^2$

$\therefore x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$

คราวหน้าให้
ห้ามแยกกันบนกระดาษ 2 แผ่น
ไม่ใช้หน้าหลัง
เพื่อสะดวกต่อการสแกน

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So $f(x) = x^3 + 2x^2 - x - 3$

$= \frac{2}{5}P_3(x) + \frac{3}{3}P_1(x) + \frac{4}{2}P_2(x) + \frac{2}{3}P_0(x) - P_1(x) - 3P_0(x)$

ฆษภาภรณ์ ดาวศรีปู่

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Homework 7 Numerical Analysis

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(1) Approximation problem ใช้หลักการใดในการหา best approximation ของฟังก์ชัน พร้อมยกตัวอย่างการหา best Approximation มา 1 ตัวอย่าง

sol^n หลักการหา best Approximation

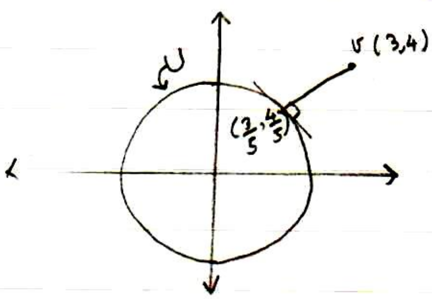
จะให้ $(V, \|\cdot\|)$ เป็น นอร์มลิเนียร์สเปซ และ $U \subset V$ ✓

$\tilde{u}$ จะเป็น best Approximation ของ $v \in V$

ถ้า $\|v - \tilde{u}\| \leq \|v - u\|$ สำหรับทุก $u \in U$ ✓

ตัวอย่างการหา best Approximation

ให้ $V = \mathbb{R}^2$ และ U = unit circle และให้ $\|\cdot\| = \|\cdot\|_2$ ✓



จะหา $\tilde{u}$ ซึ่งเป็น best Approximation ของ $v = (3,4) \in \mathbb{R}^2$

พิจารณา $\forall u \in U$, $\tilde{u}$ ที่ทำให้ $\|v - \tilde{u}\| \leq \|v - u\|$ สำหรับ $\forall u \in U$

ก็คือ $\tilde{u} = (\frac{3}{5}, \frac{4}{5})$ ✓ ใช้ความรู้พื้นฐานอธิบาย ระยะทางที่ใกล้ที่สุดระหว่างจุด v กับเซต U คือระยะของเส้นที่ลากจากจุด v มาตั้งฉากกับเส้นรอบวงของวงกลม 1 หน่วย นั่นคือ เส้นที่ลากจากจุด $(\frac{3}{5}, \frac{4}{5})$ ไปถึง v

ดังนั้น $\|v - \tilde{u}\| \leq \|v - u\|$ $\forall u \in U$

ดังนั้น $\tilde{u} = (\frac{3}{5}, \frac{4}{5})$ เป็น best Approximation ของ v ✓

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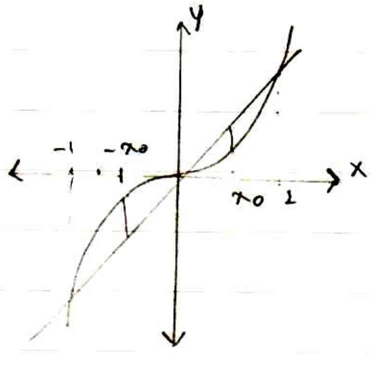
(2) ในการหา Polynomial เพื่อประมาณค่าของฟังก์ชันโดยใช้ Uniform norm $(\|\cdot\|_\infty)$ จะนิยมใช้ในการประมาณ P_0 และ P_1 ในกรณีที่ P_n, $n \geq 2$ การประมาณโดยวิธีนี้อาจเกิดความยุ่งยาก เพราะจะติดตัวแปรหลายค่า แต่ว่ามีบางกรณีที่ P_n, $n = 2$ สามารถใช้วิธีนี้ในการหา best Approximation ได้ ถามว่าคุณสมบัติของฟังก์ชันที่ทำให้เราสามารถประมาณค่าของฟังก์ชัน (Polynomial ดีกรี 2) โดยใช้ Uniform norm $(\|\cdot\|_\infty)$ คืออะไร พร้อมยกตัวอย่าง

ตอบ ความสมมาตรกับแกน Y

เช่น $f = x|x|$, $x \in [-1, 1]$



จากภาพจะเห็นว่า f สมมาตรกับแกน y

เมื่อประมาณค่า function โดยใช้ Uniform norm จะได้สมการต่อไปนี้

$$f(1) - \tilde{p}(1) = \rho$$
$$f(x_0) - \tilde{p}(x_0) = -\rho$$
$$f(-x_0) - \tilde{p}(-x_0) = \rho$$
$$f(-1) - \tilde{p}(-1) = -\rho$$
$$f'(x_0) = \tilde{p}'(x_0)$$

เมื่อทำการแก้ระบบสมการ จะทำให้เราสามารถหา $px = a_0 + a_1 x + a_2 x^2$ ซึ่งเป็น Polynomial ที่ประมาณค่า f ได้ ✓

จากคุณสมบัติสมมาตรของ f, เราสามารถบอกอะไรเกี่ยวกับ $\tilde{p}$ ได้บ้าง ?

Homework 7

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1. Find the least squares approximation to $\frac{1}{x}$ on $[1, 4]$.

You have to indicate the order of $\tilde{p}(x)$. For example, here you use $\tilde{p}(x)$ from P_2.

Sol^{n} Use normal equation directly

Let $\tilde{p}(x) = a_0 + a_1 x + a_2 x^2$

$$a_0 \int_1^4 1\,dx + a_1 \int_1^4 x\,dx + a_2 \int_1^4 x^2\,dx = \int_1^4 \frac{1}{x}\,dx$$

$$a_0 \int_1^4 x\,dx + a_1 \int_1^4 x^2\,dx + a_2 \int_1^4 x^3\,dx = \int_1^4 x \cdot \frac{1}{x}\,dx$$

$$a_0 \int_1^4 x^2\,dx + a_1 \int_1^4 x^3\,dx + a_2 \int_1^4 x^4\,dx = \int_1^4 x^2 \cdot \frac{1}{x}\,dx$$

So,

$$3\begin{bmatrix} 1 & \frac{5}{2} & 7 \\ \frac{5}{2} & 7 & \frac{85}{4} \\ 7 & \frac{85}{4} & \frac{341}{5} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = 3\begin{bmatrix} \frac{1}{3}\ln 4 \\ 1 \\ \frac{5}{2} \end{bmatrix}$$

Which yields the following solution to four decimal places :

$$a_2 = 0.0923 \quad , a_1 = -0.6685 \quad , a_0 = 1.4873$$

$$\therefore \tilde{p}(x) = 0.0923 + 0.6685x + 1.4873x^2 \quad \#$$

$$1.4873 - .6685x + 0.0923x^2$$

2. Find the least square approximation function of $f(x) = x^2$ where $x \in [0, 1]$, by using Chebyshev polynomial T_1

You can use Chebyshev for f defined on $[-1, 1]$ only. Otherwise, you need the change of variable.

Sol^{n} We want to find $\tilde{p}(x) = a_0 T_0 + a_1 T_1$

since $T_n(x) = \cos(n\cos^{-1}(x)), n \geq 0 \Rightarrow T_0(x) = 1$ and $T_1(x) = x$

$$\Rightarrow a_0 = \frac{\langle T_0, f\rangle_w}{\langle T_0, T_0\rangle_w} = \frac{\langle 1, x^2\rangle_w}{\langle 1, 1\rangle_w} = \frac{\int_0^1 1\,dx}{\int_0^1 \frac{1}{x^2}\,dx} = \frac{[x]_0^1}{[-\frac{1}{x}]_0^1} = \frac{1-0}{-1-0} = -1$$

$$a_1 = \frac{\langle T_1, f\rangle_w}{\langle T_1, T_1\rangle_w} = \frac{\langle x, x^2\rangle_w}{\langle x, x\rangle_w} = \frac{\int_0^1 x\,dx}{\int_0^1 \frac{x^2}{x^2}\,dx} = \frac{[\frac{x^2}{2}]_0^1}{[x]_0^1} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

Therefore, $\tilde{p}(x) = -1(1) + \frac{1}{2}x = \frac{x}{2} - 1 \quad \#$

Here, $\langle T_n, f\rangle$ means $\int_{-1}^{1} \frac{T_n(x) f(x)}{\sqrt{1-x^2}}\,dx$

$\frac{14}{15}$

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Date: " Numer 455 "

Homework 7

① Find the best approximation $\tilde{p} \in P_2$ of $f(x) = \sin(x)$ where $x \in [0, \frac{\pi}{2}]$ by using the normal equation with respect to $\|\cdot\|_2$.

Sol. We want to find $\tilde{p} \in P_1$. Given $\boxed{\tilde{p} = ax+b} \in \text{span}(1, x)$

From normal equation : (we get two equations)

① $a\langle x,1\rangle + b\langle 1,1\rangle = \langle f,1\rangle$

$$\Rightarrow a\int_0^{\pi/2} x\cdot 1\,dx + b\int_0^{\pi/2} 1\cdot 1\,dx = \int_0^{\pi/2}\sin(x)\cdot 1\,dx$$

$$a\left[\frac{x^2}{2}\right]_0^{\pi/2} + b[x]_0^{\pi/2} = [-\cos(x)]_0^{\pi/2}$$

$$a\cdot\frac{\pi^2}{8} + b\cdot\frac{\pi}{2} = 1 \quad \text{③}$$

② $a\langle x,x\rangle + b\langle 1,x\rangle = \langle f,x\rangle$

$u = x$, $dv = \sin(x)dx$

$$a\int_0^{\pi/2} x\cdot x\,dx + b\int_0^{\pi/2} 1\cdot x\,dx = \int_0^{\pi/2}\sin(x)\cdot x\,dx$$

$$a\left[\frac{x^3}{3}\right]_0^{\pi/2} + b\left[\frac{x^2}{2}\right]_0^{\pi/2} = [-x\cancel{\cos(x)}]_0^{\pi/2} + \int_0^{\pi/2}\cos(x)\,dx$$

$$a\cdot\frac{\pi^3}{24} + b\cdot\frac{\pi^2}{8} = [\sin(x)]_0^{\pi/2} = 1 \quad \text{④}$$

Consider ④ $- \frac{\pi}{4}\times$③ : $a\pi^3\left(\frac{1}{24} - \frac{1}{32}\right) = 1 - \frac{\pi}{4}$ $\Rightarrow a\cdot\frac{8\pi^3}{24\cdot 32} = \frac{4-\pi}{4}$ $\Rightarrow \{a = 96 - 24\pi\}$ ⑤ ✗ π^3

⑤ & ③ : $(96-24\pi)\cdot\frac{\pi^2}{8} + b\frac{\pi}{2} = 1$ $\Rightarrow 12\pi^2 - 3\pi^3 + b\frac{\pi}{2} = 1 \Rightarrow \{b = (1 - 12\pi^2 + 3\pi^3)\cdot\frac{2}{\pi}\}$ ✗

Therefore, the best approximation is $\tilde{p}(x) = (96-24\pi)x + (1 - 12\pi^2 + 3\pi^3)\cdot\frac{2}{\pi}$ #

π^3 ✗

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② Find the least square approximation function of $f(x) = \sqrt{1-x^2}$ where $x \in [-1,1]$, by using Chebyshew polynomials T_1.

Sol. We want to find $\boxed{\tilde{p}(x) = a_0T_0 + a_1T_1}$ ✓

$\sin^{-1}(-1) = -\frac{\pi}{2}$

Since $T_n(x) = \cos(n\cos^{-1}(x))$, $n \geq 0$ $\Rightarrow T_0(x) = 1$ and $T_1(x) = x$

▸ $$a_0 = \frac{\langle T_0, f\rangle_w}{\langle T_0, T_0\rangle_w} = \frac{\langle 1, \sqrt{1-x^2}\rangle_w}{\langle 1, 1\rangle_w} = \frac{\int_{-1}^{1} 1\,dx}{\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}dx} = \frac{[x]_{-1}^{1}}{[\sin^{-1}(x)]_{-1}^{1}} = \frac{2}{\frac{\pi}{2} - \frac{3\pi}{2}} = \frac{-4}{\pi}$$ ✗

▸ $$a_1 = \frac{\langle T_1, f\rangle_w}{\langle T_1, T_1\rangle_w} = \frac{\langle x, \sqrt{1-x^2}\rangle_w}{\langle x, x\rangle_w} = \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}dx} = \frac{\left[\frac{x^2}{2}\right]_{-1}^{1}}{\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}dx} = 0$$ ✓

Therefore, $\tilde{p}(x) = \frac{-4}{\pi}(1) = \frac{-4}{\pi}$ is approximation of $f(x) = \sqrt{1-x^2}$ #

✗

$\frac{15}{15}$

นางสาวชลิดา คงสนุ่น
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Homework 7

1. Find least square approximation of $f(x) = e^{5x}$, $x \in [0,1]$ from P_1 by using $\|\cdot\|_2$.

sol$^{\underline{n}}$. Let $\tilde{p}(x) = a + bx \in \text{span}(1, x)$

From normal equation : $a\langle 1,1\rangle + b\langle x,1\rangle = \langle f,1\rangle$ — ①

$a\langle 1,x\rangle + b\langle x,x\rangle = \langle f,x\rangle$ — ② ✓

We get ① ; $a\int_0^1 1\cdot 1\,dx + b\int_0^1 x\cdot 1\,dx = \int_0^1 e^{5x}\cdot 1\,dx$

$a + b\left[\frac{x^2}{2}\right]_0^1 = \left[\frac{e^{5x}}{5}\right]_0^1$

$a + \frac{1}{2}b = \frac{e^5 - 1}{5}$ ✓

$\frac{14}{15}$

② ; $a\int_0^1 1\cdot x\,dx + b\int_0^1 x\cdot x\,dx = \int_0^1 e^{5x}\cdot x\,dx$

$a\left[\frac{x^2}{2}\right]_0^1 + b\left[\frac{x^3}{3}\right]_0^1 = \left[x\left(\frac{e^{5x}}{5}\right)\right]_0^1 - \int_0^1 \frac{e^{5x}}{5}\,dx$

$\frac{1}{2}a + \frac{1}{3}b = \frac{e^5}{5} - \frac{1}{5}e^5 + \frac{1}{5} = \frac{1}{5}$ ✗

②×2 ; $a + \frac{2}{3}b = \frac{2}{5}$ — ③

③ - ① ; $\frac{1}{6}b = \frac{3 - e^5}{5} \Rightarrow b = \frac{18 - 6e^5}{5}$

$a + \frac{1}{2}\left(\frac{18 - 6e^5}{5}\right) = \frac{e^5 - 1}{5} \Rightarrow a = \frac{e^5 - 1 - 9 + 3e^5}{5} = \frac{4e^5 - 10}{5}$

$\therefore \tilde{p}(x) = \left(\frac{4e^5 - 10}{5}\right) + \left(\frac{18 - 6e^5}{5}\right)x$ ✗ Ans

ควรใช้ฟังก์ชันต่างกับข้อ 1.
ไม่อย่างนั้นจะถือว่าเป็นข้อเดียวกัน

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2. Find Legendre polynomials ($\mathcal{L}_1$) approximation of $f(x) = e^{5x}$, $x \in [0,1]$

sol$^{\underline{n}}$. $\tilde{p}(x) = b_0\mathcal{L}_0(x) + b_1\mathcal{L}_1(x)$, where $\mathcal{L}_0(x) = 1$, $\mathcal{L}_1(x) = x$,

$b_0 = \frac{\langle f, \mathcal{L}_0\rangle}{\langle \mathcal{L}_0, \mathcal{L}_0\rangle} = \frac{1}{2}\int_0^1 e^{5x}\,dx = \frac{1}{2}\left(\frac{e^5 - 1}{5}\right) = \frac{e^5 - 1}{10}$

$b_1 = \frac{\langle f, \mathcal{L}_1\rangle}{\langle \mathcal{L}_1, \mathcal{L}_1\rangle} = \frac{3}{2}\int_0^1 e^{5x}\cdot x\,dx = \frac{3}{2}\left[\left(\frac{xe^{5x}}{5}\right)_0^1 - \int_0^1 \frac{e^{5x}}{5}\,dx\right]$

$= \frac{3}{2}\left(\frac{1}{5}\right) = \frac{3}{10}$

การใช้ $\mathcal{L}_n$ จะต้องอยู่บนช่วง $[-1,1]$ เท่านั้น
หรือต้องเปลี่ยนตัวแปร จาก $x \in [0,1]$ ไปเป็น $t \in [-1,1]$

$\therefore \tilde{p}(x) = \left(\frac{e^5 - 1}{10}\right) + \left(\frac{3}{10}\right)x$ ✗ Ans

NO.

DATE :

นายเอกชัย ตานะกุล 540510733

Math 455 : Homework 7

Make two exam problems with solutions for Chapter 2 ; Approximation of functions.

1. Find least square approximation of $f(x) = x^2$, $x \in [-1,1]$ from P_2 by Legendre polynomials

15/15

Sol^n Use Legendre polynomials.

let $\tilde{p}(x) = b_0\mathcal{L}_0(x) + b_1\mathcal{L}_1(x) + b_2\mathcal{L}_2(x)$

$\mathcal{L}_0(x) = 1$, $\mathcal{L}_1(x) = x$, $\mathcal{L}_2(x) = \frac{(3x^2-1)}{2}$

$$b_0\langle\mathcal{L}_0,\mathcal{L}_0\rangle + b_1\langle\mathcal{L}_1,\mathcal{L}_0\rangle + b_2\langle\mathcal{L}_2,\mathcal{L}_0\rangle = \langle f,\mathcal{L}_0\rangle$$
$$b_0\langle\mathcal{L}_0,\mathcal{L}_1\rangle + b_1\langle\mathcal{L}_1,\mathcal{L}_1\rangle + b_2\langle\mathcal{L}_2,\mathcal{L}_1\rangle = \langle f,\mathcal{L}_1\rangle$$
$$b_0\langle\mathcal{L}_0,\mathcal{L}_2\rangle + b_1\langle\mathcal{L}_1,\mathcal{L}_2\rangle + b_2\langle\mathcal{L}_2,\mathcal{L}_2\rangle = \langle f,\mathcal{L}_2\rangle$$

$\langle\mathcal{L}_0,\mathcal{L}_0\rangle = 2$; $\int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}$

$\langle\mathcal{L}_1,\mathcal{L}_1\rangle = \frac{2}{3}$; $\int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\Big|_{-1}^{1} = \frac{1}{4} - (-\frac{1}{4}) = \frac{2}{4} = \frac{1}{2}$

$\langle\mathcal{L}_2,\mathcal{L}_2\rangle = \frac{2}{5}$; $\int_{-1}^{1} \frac{3x^3-x}{2}\,dx = \frac{x^4}{2} - \frac{x^2}{4}\Big|_{-1}^{1} = (\cancel{\frac{1}{2}} - \cancel{\frac{1}{4}}) - (\cancel{\frac{1}{2}} - \cancel{\frac{1}{4}}) = 0$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

✓ ✓ ✓ ✓ ✗ ✗

$b_0 = \frac{\langle f,\mathcal{L}_0\rangle}{\langle\mathcal{L}_0,\mathcal{L}_0\rangle} = \frac{\frac{2}{3}}{2} = \frac{1}{3}$

$b_1 = \frac{\langle f,\mathcal{L}_1\rangle}{\langle\mathcal{L}_1,\mathcal{L}_1\rangle} = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}$

$b_2 = \frac{\langle f,\mathcal{L}_2\rangle}{\langle\mathcal{L}_2,\mathcal{L}_2\rangle} = \frac{0}{\frac{2}{5}} = 0$

we should get $\tilde{p} = f$ because $f \in P_2$

so, $\tilde{p}(x) = \frac{1}{3}(1) + \frac{3}{4}(x) + 0\left(\frac{3x^2-1}{2}\right) = \frac{3}{4}x + \frac{1}{3}$ ✗

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HOMEWORK 7. DueDate 30/9/57

Give 2 Example and Solution by Youself.

Example 1 Find Least square with
$f(x) = x^2+10,\ x \in [-1, 1]$ and $n=1$

15/15

Sol^n We have $f(x) = x^2+10;\ x \in [-1, 1]$

From $n=1$, So $\tilde{p}(x) = a + bx$

And Defination of Least square ($\|f\|_2^2$)

$$\|f - \tilde{p}(x)\|_2^2 = \int_{-1}^{1} [(x^2+10) - (a+bx)]^2 dx \quad ✓$$

We want to minimize $F(a,b)$

$$F_a = \int_{-1}^{1} 2[\overset{x^2+10}{x^2} - (a+bx)](-1)\, dx \overset{set}{=} 0 \quad ✗$$

$$F_b = \int_{-1}^{1} 2[\overset{x^2+10}{x^2} - (a+bx)](-x)\, dx \overset{set}{=} 0 \quad ✗$$

Then

$$\int_{-1}^{1} (x^2+10) = \int a+bx$$

$$\int (x^2+10)x = \int (a+bx)x$$

$$\int_{-1}^{1} -x^2 + a + bx\, dx = 0$$

$$\int_{-1}^{1} -x^3 + ax + bx^2\, dx = 0$$

$$-\frac{x^3}{3}\Big|_{-1}^{1} + ax\Big|_{-1}^{1} + \frac{bx^2}{2}\Big|_{-1}^{1} = 0$$

$$-\frac{x^4}{4}\Big|_{-1}^{1} + \frac{ax^2}{2}\Big|_{-1}^{1} + \frac{bx^3}{3}\Big|_{-1}^{1} = 0$$

$$-\left(\frac{1}{3} - \frac{(-1)^3}{3}\right) + a(1-(-1)) + \frac{b}{2}(1^2 - (-1)^2) = 0$$

$$-\frac{1}{4}(1^4 - (-1)^4) + \frac{a}{2}(1-(-1)^2) + \frac{b}{3}(1^3 - (-1)^3) = 0$$

$$-\frac{2}{3} + 2a + 0 = 0 \rightarrow a = \frac{3}{4}$$

$$0 + 0 + \frac{2}{3}b = 0 \rightarrow b = 0$$

So, The solution for Example 1 is

$$y = ax + b = \frac{3}{4}x + 0 \quad \#$$

✗

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การบ้าน ๗

linear approx.

① Find the least squares, straight line approximation to $x^{1/2}$ on $[0,1]$, find the $\psi(x) \in P_1$ that best fits $x^{1/2}$ on $[0,1]$.

15/15

Solⁿ

First choose an orthogonal basis for P_1:

$$\phi_0(x) = 1 \quad \text{and} \quad \phi_1(x) = \left(x - \tfrac{1}{2}\right)$$

These form an orthogonal basis for P_1 since

$$\int_0^1 \phi_0\phi_1\,dx = \int_0^1 \left(x - \tfrac{1}{2}\right)dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{2}x\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{2} = 0 \;\checkmark$$

Now construct $\psi = c_0\phi_0 + c_1\phi_1 = c_0 + c_1\left(x - \tfrac{1}{2}\right)$

To find the ψ which satisfies $\|f - \psi\| \leq \|f - p\|$

We solve for the c_i as follows

คราวหน้าให้ทำมาในกระดาษสองแผ่น ไม่ใช้หน้าและหลัง เพื่อให้ง่ายต่อการสแกน

$i = 0$:

$$c_0 = \frac{\langle f, \phi_0\rangle}{\langle \phi_0, \phi_0\rangle}$$

- $\langle f, \phi_0\rangle = \langle x^{1/2}, 1\rangle = \int_0^1 x^{1/2}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^1 = \tfrac{2}{3}$ ✓
- $\langle \phi_0, \phi_0\rangle = \langle 1, 1\rangle = \int_0^1 1^2\,dx = 1$ ✓

$$c_0 = \tfrac{2}{3} \;\checkmark$$

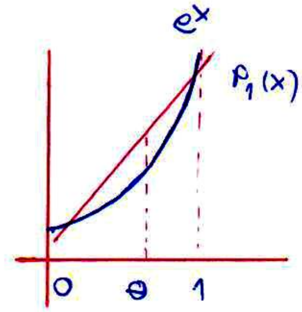
$i = 1$:

$$c_1 = \frac{\langle f, \phi_1\rangle}{\langle \phi_1, \phi_1\rangle}$$

- $\langle f, \phi_1\rangle = \langle x^{1/2}, x - \tfrac{1}{2}\rangle = \int_0^1 x^{1/2}\left(x - \tfrac{1}{2}\right)dx = \tfrac{1}{15}$ ✓
- $\langle \phi_1, \phi_1\rangle = \langle x - \tfrac{1}{2}, x - \tfrac{1}{2}\rangle = \int_0^1 \left(x - \tfrac{1}{2}\right)^2 dx = \tfrac{1}{12}$ ✓

② Find the best straight line fit ($\tilde{p} \in P_1$) to $f(x) = e^x$ in the interval $[0,1]$

Minimax or uniform approximation



Sol^n We want to find the straight line fit, hence we let $\tilde{p} = mx + c$

$$\| f(x) - \tilde{p} \|_\infty = \| e^x - (mx+c) \|_\infty$$

$x = 0$: $e^0 - (0 + c) = \rho$ —— ①

$x = \theta$: $e^\theta - (m\theta + c) = -\rho$ —— ②

$x = 1$: $e^1 - (m + c) = \rho$ —— ③

also, the error at $x = \theta$ has a turning point, so that

$$\frac{d}{dx}\left(e^x - (mx+c)\right)\Big|_{x=\theta} = 0$$

$$e^\theta - m = 0$$

$$m = e^\theta$$

$$\theta = \log_e m$$

① = ③

$$1 - c = \rho = e - m - c$$

$$m = e - 1$$

$$\approx 1.7183$$

$$\Rightarrow \quad \theta = \log_e (1.7183).$$

② = ③

$$e^\theta - e - m\theta - c - m = 0$$

$$c = \frac{1}{2}\left[m + e - m\theta - m\right]$$

$$\approx 0.8941.$$

Hence the straight line is given by $1.7183x + 0.8941$.

นาย[illegible] พ่วงพี 540510577

Make two exam problems with solution for Chapter 2 : Approximation of functions.

Problem 1 : A truncated Fourier Series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\pi) + b_n \sin(n\pi) \right)$$

is a least squares approximation of $f(x)$ for any $f(x)$ in the interval $[-\pi, \pi]$. Use the normal equation to show that such coefficients a_0, a_n and b_n satisfy :

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx \; , \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx$$

15/15

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx$$

Proof: Assume $f(x)$ be functions spanned by the basis

$$\phi_0 = 1 \, , \quad \phi_1 = \cos(x) \, , \quad \phi_2 = \sin(x) \, ,$$
$$\phi_3 = \cos(2x) \, , \quad \phi_4 = \sin(2x) \, ,$$
$$\vdots \qquad \vdots$$
$$\phi_{2n-1} = \cos(nx) \text{ and } \phi_{2n} = \sin(nx)$$

Since

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = \frac{1}{n}\left[\sin(nx)\right]_{-\pi}^{\pi} = 0$$

and

$$\int_{-\pi}^{\pi} \sin(nx)\,dx = \frac{1}{n}\left[-\cos(nx)\right]_{-\pi}^{\pi} = 0$$

Moreover we knows

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0 \quad \text{for all } n = 1, 2, \ldots$$

and
$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0$$

where $n \neq m \neq 0$

Then, these basis form an orthogonal set of functions because $\langle \phi_i , \phi_j \rangle = 0$ for $i \neq j$

we can write a least square approximation of $f(x)$

$\tilde{p}(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + \ldots + a_n \cos(nx) + b_n \cos(nx)$

and find scalars from

$$a_0 = \frac{\langle f, \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle} = \int_{-\pi}^{\pi} f(x)\cdot 1\,dx \Big/ \int_{-\pi}^{\pi} 1\cdot 1\,dx$$

$$\therefore \quad a_0 = \int_{-\pi}^{\pi} f(x)\,dx \Big/ 2\pi$$

$$a_n = \frac{\langle f, \phi_{2n-1} \rangle}{\langle \phi_{2n-1}, \phi_{2n-1} \rangle} = \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \Big/ \int_{-\pi}^{\pi} \cos^2(nx)\,dx$$

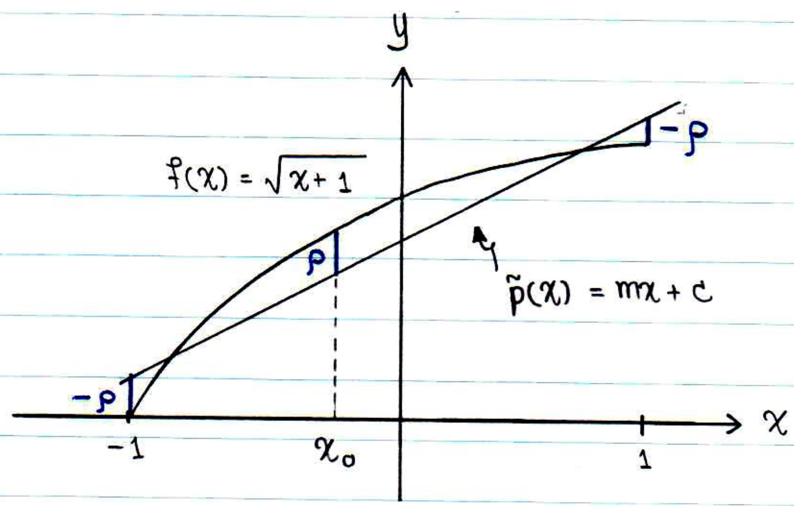
$$b_n = \frac{\langle f, \phi_{2n} \rangle}{\langle f, \phi_{2n} \rangle} = \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \Big/ \int_{-\pi}^{\pi} \sin^2(nx)\,dx$$

It is easy to determine

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi$$

$$\therefore \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx , \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Problem 2 : Construct the best approximation, straight line approximation $\tilde{p}(x) \in P_1$ of $f(x) = \sqrt{x+1}$ on $[-1, 1]$



Solution Let $\tilde{p}(x) = mx + c \in P_1$

So, we have ① $f(-1) - \tilde{p}(-1) = -\rho$

② $f(x_0) - \tilde{p}(x_0) = \rho$

③ $f(1) - \tilde{p}(1) = -\rho$

④ $f'(x_0) = \tilde{p}'(x_0)$

Then, $0 - (-m + c) = -\rho \quad \therefore \quad m - c = -\rho$ ...... ①

$\sqrt{x_0 + 1} - (mx_0 + c) = \rho$

$\sqrt{2} - (m + c) = -\rho \quad \therefore \quad \sqrt{2} - m - c = -\rho$ ...... ②

$\frac{1}{2\sqrt{x_0+1}} = m \Rightarrow 4m^2(x_0 + 1) = 1$

From ① & ② ; $2m = +\sqrt{2} \quad \therefore \quad m = +\frac{1}{\sqrt{2}}$

$\therefore \quad x_0 + 1 = \frac{1}{2} \quad \therefore \quad x_0 = -\frac{1}{2}$

We have
$(f-p)(-1) = -\frac{1}{4\sqrt{2}}$
$(f-p)(-\frac{1}{2}) = \frac{1}{4\sqrt{2}}$
But $(f-p)(1) \neq -\frac{1}{4\sqrt{2}}$

Then, $\rho = \frac{1}{4\sqrt{2}}$ and $c = \frac{5}{4\sqrt{2}}$

Hence; the best approximation is $\tilde{p}(x) = \frac{1}{\sqrt{2}}x + \frac{5}{4\sqrt{2}}$

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No. 540510693

Date ....../....../......

Hw.7

1. Find a linear least-squares approximation of $f(x)=e^x$ using Chebyshev polynomials.

Sol^n Here $p_1(x) = a_0\phi_0(x) + a_1\phi_1(x) = a_0T_0(x) + a_1T_1(x)$

$$= a_0 + a_1 x,$$

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{e^x dx}{\sqrt{1-x^2}}$$

$$\approx 1.2660$$

$$a_1 = \frac{2}{\pi}\int_{-1}^{1} \frac{xe^x}{\sqrt{1-x^2}}$$

$$\approx 1.1303$$

Thus, $P_1(x) = 1.2660 + 1.1303x$

Accuracy Check :

$$P_1(0.5) = 1.2660 + 1.1303(0.5)$$

$$= 1.8312 \;;$$

$$e^{0.5} = 1.6487$$

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No. ....
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Find linear and quadratic least-squares polynomial approximation to $f(x) = x^2 + 5x + 6$ in $[0, 1]$;

Solⁿ Linear Fit: $P_1(x) = a_0 + a_1 x$

$$s_0 = \int_0^1 dx = 1$$
$$s_1 = \int_0^1 x\,dx = \frac{1}{2}$$
$$s_2 = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$b_0 = \int_0^1 (x^2 + 5x + 6)\,dx = \frac{1}{3} + \frac{5}{2} + 6 = \frac{53}{6}$$

$$b_1 = \int_0^1 x(x^2 + 5x + 6)\,dx = \int_0^1 (x^3 + 5x^2 + 6)\,dx = \frac{1}{4} + \frac{5}{3} + \frac{6}{2} = \frac{59}{12}$$

$= \frac{35}{6}$

The normal equations are.

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \frac{53}{6} \\ \frac{59}{12} \end{pmatrix} \Rightarrow \begin{array}{l} a_0 = 5.8333 \\ a_1 = 6 \end{array}$$

The linear least squares polynomial $P_1(x) = 5.8333 + 6x$

Accuracy Check:

Exact Value: $f(0.5) = 8.75$; $P_1(0.5) = 8.833$

Relative error: $\frac{|8.833 - 0.75|}{|8.75|} = 0.0095$

Quadratic Least-Square Approximation: $P_2(x) = a_0 + a_1 x + a_2 x^2$

$$S = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

$b_0 = \frac{53}{6}$, $b_1 = \frac{59}{12}$

$$b_2 = \int_0^1 x^2(x^2 + 5x + 6)\,dx = \int_0^1 (x^4 + 5x^3 + 6x^2)\,dx = \frac{1}{5} + \frac{5}{4} + \frac{6}{3} = \frac{69}{20}.$$

The solution of the linear system is: $a_0 = 6$, $a_1 = 5$, $a_2 = 1$

$$P_2(x) = 6 + 5x + x^2 \quad (\text{Exact})$$

We get exact approximation because f is in P_2.

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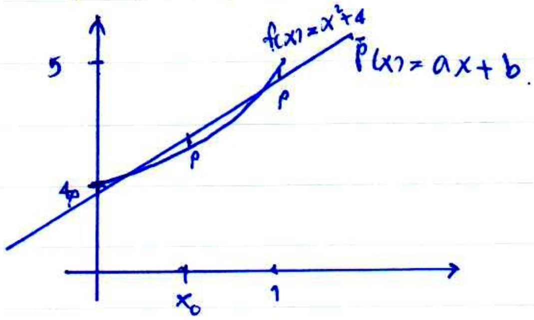
NO.

นายจักรกฤษณ์ แดงแสน. 540510574

1. Determine the best approximation of $f \in (C[0,1], \|\cdot\|_\infty)$ form P_1

$f(x) = x^2 + 4$



equations

1) $f(0) - \bar{p}(0) = \rho \rightarrow (4)-(b) = \rho$

2) $f(x_0) - \bar{p}(x_0) = -\rho \rightarrow (x_0^2+4)-(ax_0+b) = -\rho$

3) $f(1) - \bar{p}(1) = \rho \rightarrow (1+4)-(a+b) = \rho$

4) $\bar{p}'(x_0) = f'(x_0) \rightarrow a = 2x_0$

จะได้ว่า

① $4-b = \rho$

② $x_0^2 + 4 - ax_0 - b = -\rho$

③ $5-a-b = \rho$

④ $a = 2x_0$

③ − ① $1-a = 0$

$\boxed{a=1}$ แทนใน ④

$\boxed{x_0 = \frac{1}{2}}$

15/15

จาก ② $\frac{1}{4} + 4 - \frac{1}{2} - b = -\rho$

$\frac{1+16-2}{4} - b = -\rho$

$\frac{15}{4} - b = -\rho$ — ⑤

⑤+① $\frac{15}{4} + 4 = 2b$

$\left(\frac{15+16}{4}\right)\left(\frac{1}{2}\right) = b \Rightarrow b = \frac{31}{8}$ แทนใน ①

$\therefore \rho = 4 - \frac{31}{8} = \frac{32-31}{8} = \frac{1}{8}$.

$\therefore \bar{p}(x) = x + \frac{31}{8}$ โดยที่ $\|\bar{p}(x) - f(x)\|_\infty = \frac{1}{8}$

NO. ....

DATE / /

2. Let $f(x) = 6x^2$, $x \in [0,1]$ find least square approximation of f from P_1

วิธีทำ Let $\tilde{p}(x) = a + bx \in \text{span}(1, x)$

$$a\langle 1,1\rangle + b\langle x,1\rangle = \langle f,1\rangle$$

$$a\langle 1,x\rangle + b\langle x,x\rangle = \langle f,x\rangle$$

We get

$$\Rightarrow a\int_0^1 1.1\,dx + b\int_0^1 x.1\,dx = \int_0^1 6x^2 \cdot 1\,dx$$

$$a\int_0^1 1.x\,dx + b\int_0^1 x.x\,dx = \int_0^1 6x^2 \cdot x\,dx$$

ได้ว่า $\Rightarrow a\left[x\right]_0^1 + b\left[\frac{x^2}{2}\right]_0^1 = 2\cancel{6}\left[\frac{x^3}{\cancel{3}}\right]_0^1$

$$a\left[\frac{x^2}{2}\right]_0^1 + b\left[\frac{x^3}{3}\right]_0^1 = 3\cancel{6}\left[\frac{x^4}{\cancel{4}_2}\right]_0^1$$

$$\Rightarrow a(1-0) + b\left(\frac{1}{2}-0\right) = 2(1-0)$$

$$\frac{a}{2}(1-0) + \frac{b}{3}(1-0) = \frac{3}{2}(1-0)$$

$$\Rightarrow a + \frac{b}{2} = 2 \quad —①$$

$$\frac{a}{2} + \frac{b}{3} = \frac{3}{2} \quad —②$$

$① \times \frac{1}{2} \Rightarrow \frac{a}{2} + \frac{b}{2} = 1 \quad —③$

$③ - ①$ $\quad \frac{b}{2} - \frac{b}{3} = 1 - \frac{3}{2}$

$$\frac{b}{6} = -\frac{1}{2}$$

$b = -3$ แทนค่า ①

$a = 5$

$\Rightarrow \tilde{p}(x) = 5 - 3x$ #

นางสาวจรินทิพย์ จันทร์หอกลอง

540510629

(1) Let $f(x) = x^4 + 2x^2$, $x \in [0,1]$ Find least approximation of f from P_n, $n=0$

Sol^n $\tilde{P}(x) = a + bx \in \text{span}(1, x)$

$a\langle 1,1\rangle + b\langle x,1\rangle = \langle f,1\rangle$

$a\langle 1,x\rangle + b\langle x,x\rangle = \langle f,x\rangle$

From normal equation

we get $a\int_0^1 1\cdot 1\,dx + b\int_0^1 x\cdot 1\,dx = \int_0^1 (x^4+2x^2)\cdot 1\,dx$ ——(1)

$a\int_0^1 1\cdot x\,dx + b\int_0^1 x\cdot x\,dx = \int_0^1 (x^4+2x^2)\cdot x\,dx$ ——(2)

$\frac{15}{15}$

consider (1) : $a\int_0^1 1\,dx + b\int_0^1 x\,dx = \int_0^1 (x^4+2x^2)\,dx$

$a[x]_0^1 + b\left[\frac{x^2}{2}\right]_0^1 = \left[\frac{x^5}{5} + \frac{2x^3}{3}\right]_0^1$

$a[1-0] + b\left[\frac{1}{2} - 0\right] = \left[\frac{1}{5} + \frac{2}{3} - 0\right]$

$a + \frac{1}{2}b = \frac{13}{15}$ ——(3)

consider (2) : $a\int_0^1 x\,dx + b\int_0^1 x^2\,dx = \int_0^1 (x^5 + 2x^3)\,dx$

$a\left[\frac{x^2}{2}\right]_0^1 + b\left[\frac{x^3}{3}\right]_0^1 = \left[\frac{x^6}{6} + \frac{2x^4}{4_2}\right]_0^1$

$a\left[\frac{1}{2} - 0\right] + b\left[\frac{1}{3} - 0\right] = \left[\frac{1}{6} + \frac{1}{2} - 0\right]$

$\frac{1}{2}a + \frac{1}{3}b = \frac{2}{3}$ ——(4)

solving the system of equation

we get $a = -\frac{8}{15}$, $b = \frac{14}{5}$

$\therefore \tilde{P}(x) = -\frac{8}{15} + \frac{14}{5}x$

(2.) Find least square approximation of $f(x) = x^2 + 2x$

$x \in [-1,1]$ from P_2

Sol^n $\tilde{p}(x) = b_0 \mathcal{L}_0(x) + b_1 \mathcal{L}_1(x) + b_2 \mathcal{L}_2(x)$

$\mathcal{L}_0 = 1 \quad , \quad \mathcal{L}_1 = x \quad , \quad \mathcal{L}_2 = \dfrac{3x^2 - 1}{2}$

$$b_0 \langle \mathcal{L}_0, \mathcal{L}_0 \rangle + b_1 \langle \mathcal{L}_1, \mathcal{L}_0 \rangle + b_2 \langle \mathcal{L}_2, \mathcal{L}_0 \rangle = \langle f, \mathcal{L}_0 \rangle$$

$$b_0 \langle \mathcal{L}_0, \mathcal{L}_1 \rangle + b_1 \langle \mathcal{L}_1, \mathcal{L}_1 \rangle + b_2 \langle \mathcal{L}_2, \mathcal{L}_1 \rangle = \langle f, \mathcal{L}_1 \rangle$$

$$b_0 \langle \mathcal{L}_0, \mathcal{L}_2 \rangle + b_1 \langle \mathcal{L}_1, \mathcal{L}_2 \rangle + b_2 \langle \mathcal{L}_2, \mathcal{L}_2 \rangle = \langle f, \mathcal{L}_2 \rangle$$

consider $\langle f, \mathcal{L}_0 \rangle = \langle x^2 + 2x, 1 \rangle = \int_{-1}^{1} (x^2 + 2x)\,dx = \left[\frac{x^3}{3} + \frac{2x^2}{2} \right]_{-1}^{1} = \left[\frac{1}{3} + 1 - \left(-\frac{1}{3} + 1 \right) \right]$

$= \left[-\frac{1}{3} + 1 + \frac{1}{3} - 1 \right] = 0$

consider $\langle f, \mathcal{L}_1 \rangle = \langle x^2 + 2x, x \rangle = \int_{-1}^{1} (x^2 + 2x) \cdot x\,dx = \int_{-1}^{1} (x^3 + 2x^2)\,dx$

$= \left[\frac{x^4}{4} + \frac{2}{3} x^3 \right]_{-1}^{1} = \left[\frac{1}{4} + \frac{2}{3} - \left(\frac{1}{4} - \frac{2}{3} \right) \right]$

$= \cancel{\frac{1}{4}} + \frac{2}{3} - \cancel{\frac{1}{4}} + \frac{2}{3} = \frac{4}{3}$

consider $\langle f, \mathcal{L}_2 \rangle = \langle x^2 + 2x, \frac{3x^2 - 1}{2} \rangle = \int_{-1}^{1} (x^2 + 2x) \cdot \left(\frac{3x^2 - 1}{2} \right) dx$

$= \int_{-1}^{1} \left(\frac{3}{2} x^4 + 3x^3 - \frac{1}{2} x^2 - x \right) dx$

$= \left[\frac{3}{2} \frac{x^5}{5} + \frac{3}{4} x^4 - \frac{1}{2} \frac{x^3}{3} - \frac{x^2}{2} \right]_{-1}^{1}$

$= \left[\frac{3x^5}{10} + \frac{3}{4} x^4 - \frac{1}{6} x^3 - \frac{x^2}{2} \right]_{-1}^{1}$

$= \left[\frac{3}{10} + \frac{3}{4} - \frac{1}{6} - \frac{1}{2} - \left(-\frac{3}{10} + \frac{3}{4} + \frac{1}{6} - \frac{1}{2} \right) \right]$

$= \left[\frac{3}{10} + \cancel{\frac{3}{4}} - \frac{1}{6} - \cancel{\frac{1}{2}} + \frac{3}{10} - \cancel{\frac{3}{4}} - \frac{1}{6} + \cancel{\frac{1}{2}} \right] = \left[\frac{3}{10} - \frac{1}{6} + \frac{3}{10} - \frac{1}{6} \right]$

$= \frac{9 - 5 + 9 - 5}{30} = \frac{18 - 10}{30} = \frac{\cancel{8}^{\,4}}{\cancel{30}_{\,15}}$

we get $\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{4}{3} \\ \frac{4}{15} \end{bmatrix}$

$\therefore b_0 = 0 \quad , \quad b_1 = 2 \quad , \quad b_2 = \frac{2}{3}$

$\therefore \tilde{p}(x) = 2x + \frac{2}{3} \left(\frac{3x^2 - 1}{2} \right) = 2x + \frac{6x^2}{6} - \frac{2}{6} = 2x + x^2 - \frac{1}{3}$

$\therefore \tilde{p}(x) = x^2 + 2x - \frac{1}{3}$

You should get $\tilde{p}(x) = f$ because f is from P_2.

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① Let $f(x) = x$, $x \in [0,1]$ Find least ~~sq~~ square approximation

$\tilde{p}(x) = a + bx \in \text{span}(1, x)$

$a\langle 1, 1\rangle + b\langle x, 1\rangle = \langle f, 1\rangle$ ✓

$a\langle 1, x\rangle + b\langle x, x\rangle = \langle f, x\rangle$ ✓

From normal eqn we get

$a\int_0^1 1\cdot 1\,dx + b\int_0^1 x\cdot 1\,dx = \int_0^1 x\cdot 1\,dx$

$a\int_0^1 1\cdot x\,dx + b\int_0^1 x\cdot x\,dx = \int_0^1 x\cdot x\,dx$

$a + \frac{1}{2}b = \frac{1}{2}$ — (1)

$\frac{1}{2}a + \frac{1}{3}b = \frac{1}{3}$ — (2)

ได้ $a = 1$, $b = 1$ $a = 0, b = 1$

$\therefore \tilde{p}(x) = 1 + x$ ✗

You should get $\tilde{p} = f$ because $f \in P_1$.

$\frac{14}{15}$

(2) Find least square approximation of $f(x) = 3x+5$, $x \in [-1,1]$ from P_2

Let $\tilde{p}(x) = b_0 \mathcal{L}_0(x) + b_1 \mathcal{L}_1(x) + b_2 \mathcal{L}_2(x)$

$\mathcal{L}_0 = 1$, $\mathcal{L}_1 = x$, $\mathcal{L}_2 = \frac{3x^2-1}{2}$

$$b_0 \langle \mathcal{L}_0, \mathcal{L}_0 \rangle + b_1 \langle \mathcal{L}_1, \mathcal{L}_0 \rangle + b_2 \langle \mathcal{L}_2, \mathcal{L}_0 \rangle = \langle f, \mathcal{L}_0 \rangle$$
$$b_0 \langle \mathcal{L}_0, \mathcal{L}_1 \rangle + b_1 \langle \mathcal{L}_1, \mathcal{L}_1 \rangle + b_2 \langle \mathcal{L}_2, \mathcal{L}_1 \rangle = \langle f, \mathcal{L}_1 \rangle$$
$$b_0 \langle \mathcal{L}_0, \mathcal{L}_2 \rangle + b_1 \langle \mathcal{L}_1, \mathcal{L}_2 \rangle + b_2 \langle \mathcal{L}_2, \mathcal{L}_2 \rangle = \langle f, \mathcal{L}_2 \rangle$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{pmatrix} 10 \\ 2 \\ 4 \end{pmatrix}$$

i.e $b_0 = \langle f, \mathcal{L}_0 \rangle / \langle \mathcal{L}_0, \mathcal{L}_0 \rangle = \frac{10}{2} = 5$

$b_1 = \langle f, \mathcal{L}_1 \rangle / \langle \mathcal{L}_1, \mathcal{L}_1 \rangle = \frac{6}{2} = 3$

$b_2 = \langle f, \mathcal{L}_2 \rangle / \langle \mathcal{L}_2, \mathcal{L}_2 \rangle = \frac{20}{2} = 10$

$\tilde{p}(x) = 5 + 3x + 15x^2 - 5$

$= 15x^2 + 3x$

same as before, you should get $\tilde{p}(x) = 3x+5$

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SUBJECT: NO: DATE: / /

HW. 7.

1. Find least square approximation of $f(x) = x^2+2$, $x \in [0,1]$ from P_1

Sol^n Let $\tilde{p}(x) = a_0 + a_1 x \in \operatorname{span}(1, x)$

by normal equation

we get $a_0\langle 1, 1\rangle + a_1\langle x, 1\rangle = \langle f, 1\rangle$

$a_0\langle 1, x\rangle + a_1\langle x, x\rangle = \langle f, x\rangle$

ดูนี่นี้ $\langle 1, x\rangle = \int_0^1 x\,dx$

$= \frac{1}{2}x^2\Big|_0^1$

$= \frac{1}{2}$

$$\therefore \begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5/4 \end{bmatrix}$$

we have $a_0 = \frac{7}{6}$, $a_1 = \frac{15}{8}$

$\therefore \tilde{p}(x) = \frac{7}{6} + \frac{15}{8}x$ #

14/15

SUBJECT: .... NO: .... DATE: ..../..../....

2. Find $(B_n f)(x)$ from function $f(x) = x + 3$

Sol$^{\underline{n}}$ From $(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$

$$\therefore (B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} + 3\right) x^k (1-x)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}\left(\frac{k}{n}\right) x^k (1-x)^{n-k} + \sum_{k=0}^{n} \binom{n}{k} (3) x^k (1-x)^{n-k}$$

$k=1 \Rightarrow$

$$k=1 \Rightarrow \quad = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{k}{n}\right) x^k (1-x)^{n-k} + 3$$

$$k=1 \Rightarrow \quad = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{k}{n} x^k (1-x)^{n-k} + 3$$

$$= x \sum_{k=0}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{(n-1)-(k-1)} + 3$$

$k=1$ $k=0$

$$= x(1) + 3$$

$$\therefore (B_n f)(x) = x + 3 \quad ✱$$