Iterative methods

1. Derive Composite simpson's $\frac{3}{8}$ rule.

2. Consider the following two methods for solving the initial value problem for the differential equation $x' = f(t, x)$, where $h_n = t_{n+1} - t_n$:

   Method 1
   $$x_{n+1} = x_n + \frac{1}{2}h_n[f(t_n, x_n) + f(t_{n+1}, x_{n+1})];$$

   Method 2
   $$\begin{aligned} x_{n+1}^{(1)} &= x_n + h_n f(t_n, x_n), \\ x_{n+1} &= x_n + \frac{1}{2}h_n[f(t_n, x_n) + f(t_{n+1}, x_{n+1}^{(1)})]. \end{aligned}$$

   (i) Assume that $0 < h_n < H$. What can you say about
   $$|x_n - x(t_n)| \text{ for } 0 < t_n < T$$
   as a function of H, for Method 1?
   (ii) Assume that $0 < h_n < H$. What can you say about
   $$|x_n - x(t_n)| \text{ for } 0 < t_n < T$$
   as a function of H, for Method 2?
   (iii) Prove your assertion for Method 2.

3. Let $f : R^3 \to R^2$ be given by
   $$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 x_3 - 2x_1x_2 + 2x_2x_3 \\ x_1x_2 - x_2x_3 + x_1x_3 \end{bmatrix}.$$

   (i) Calculate $f(e)$ and $f'(e)$, for $e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
   (ii) The equation $f(x) = f(e)$ may be expected to define a curve in R^3 passing through e, since this equation is a system of 2 scalar equations in 3 unknowns. Outline an algorithm for finding a point on this curve near e. Sketch a proof that the algorithm works.

Numerical Methods:BVP and PDE

1. Consider the boundary value problem

$$-u'' + e^u = 0, u(0) = u(1) = 0, \quad 0 < x < 1. \tag{1}$$

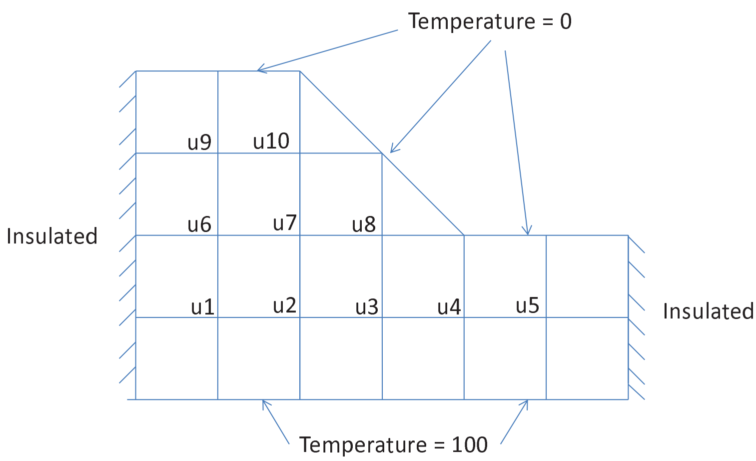
Discretize the problem with a **finite element method** using continuous, piecewise linear functions on a uniform partition of $[0,1] : x_i = ih, i = 0, 1, 2, \ldots, n, h = \frac{1}{n}$. Quadratire is to be done with the trapezoidal rule. Write the method in the form

$$AU_h + F_h(U_h) = 0,$$

where $U_h \in R^m$ denoted the vector of unknown nodal values of the approximate solution, A is an $m \times m$ matrix whose elements are independent of the discretization parameter h, and $F_h : R^m \to R^m$ is a nonlinear vector-valued function.

2. Consider the heat transfer problem on the irregular region shown in figure below. The mathematical statement of this problem is as follows:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \quad x \in \Omega, && (2) \\ \frac{\partial u}{\partial x} &= 0, \quad x \in \partial\Omega_1, && (3) \\ u &= 0, \quad x \in \partial\Omega_2, && (4) \\ u &= 100, \quad x \in \partial\Omega_3, && (5) \end{aligned}$$



Here is the partial derivative $\frac{\partial u}{\partial x}$ can be approximated by a divided-difference formula. In this problem, the insulated boundaries act like mirrors so that we can assume the temperature is the same as at an adjacent interior grid point. Determine the associate linear system to for solve the temperature u_i with $1 \le i \le 10$. (Do not solve for u_i)