C^*-algebras and Non-commutative Geometry.

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- Introduction.

Introduction 1.

The aim of this elementary talk is to give a very partial and simplified panorama of what is going on in a broad area of mathematics and in its applications (to physics and beyond . . .).

The field of mathematics we are talking about is traditionally called **operator algebras** and is a very abstract and technical part of **functional analysis** that in the last few years, under the name of **non-commutative geometry**, has undergone incredible developments that, in our opinion, are going to change forever our basic image of mathematics and its impact on fundamental physics.

Unfortunately these great achievements are taking place very quickly and at a high technical level, preventig most of the potentially interested people from entering into the subject.

Introduction 2.

The first part of this talk is based on a very expanded version of a manuscript that has been used for a short seminar in Cha Am in 1997, an internal workshop in Thammasat University in 1998 and notes written as a very partial attempt to present some basic ideas of operator algebras and non-commutative geometry as introduction and motivation for a Thammasat University workshop by Prof. Laszlo Zsido in 1999.

Introduction 3.

The second part of talk will be inspired by ongoing joint research projects

- "Modular Spectral Triples" and
- "Categorical Non-commutative Geometry"

in cooperation with Dr. Roberto Conti and Assist. Prof. Wicharn Lewkeeratityutkul and partially supported by the Thai Research Fund.

The material is not yet in final form and since I am still myself in the process of learning this subject, I hope you will excuse me for the poor exposition of most of the ideas.

Mathematics Subject Classification.

MSC-2000 1.

In order to get some initial idea about this new area of mathematics and especially to relate it to other well known fields, it can be useful to give a look at the relevant entries that already appear in the Mathematics Subject Classification (MSC-2000).

46 **Functional Analysis**

- 46L Selfadjoint Operator Algebras (C^*-Algebras, W^*-Algebras etc.)
  - 46L51 Non-commutative Measure and Integration (*)
  - 46L52 Non-commutative Function Spaces
  - 46L53 Non-commutative Probability and Statistics
  - 46L55 Non-commutative Dynamical Systems
  - 46L85 Non-commutative Topology
  - 46L87 Non-commutative Differential Geometry (*)

58 **Global Analysis, Analysis on Manifolds**

- 58B Infinite Dimensional Manifolds
  - 58B34 Non-commutative Geometry (a la Connes)
- 58J Partial Differential Equations on Manifolds
  - 58J42 Non-commutative Global Analysis

MSC-2000 2.

14 **Algebraic Geometry**

- 14A Foundations
  - 14A22 Non-commutative Algebraic Geometry

81 **Quantum Theory**

- 81R Groups and Algebras in Quantum Theory
  - 81R60 Non-commutative Geometry (*)
- 81T Quantum Field Theory
  - 81T75 Non-commutative Geometry Methods (*)

83 **Relativity and Gravitational Theory**

- 83C General Relativity
  - 83C65 Methods of Non-commutative Geometry (*)

MSC-2000 3.

Other already strictly related fields are the following:

18 **Category Theory and Homological Algebra**

- 18F Categories and Geometry (*)

19 **K-Theory**

- 19K K-Theory and Operator Algebras

46 **Functional Analysis**

- 46L Selfadjoint Operator Algebras (C^*-Algebras, W^*-Algebras etc.)
  - 46L60 Applications of Selfadjoint Operator Algebras to Physics
  - 46L65 Quantizations, Deformations
  - 46L80 K-Theory and Operator Algebras (including Cyclic Theory)
- 46M Methods of Category Theory in Functional Analysis
  - 46M15 Categories and Functors (*)

55 **Algebraic Topology**

MSC-2000 4.

- 58 **Global Analysis, Analysis on Manifolds**
  - 58H Pseudogroups, Differentiable Groupoids and General Structures on Manifolds
  - 58J Partial Differential Equations on Manifolds, Differential Operators
    - 58J22 Exotic Index Theories
- 81 **Quantum Theory**
  - 81T Quantum Field Theory
    - 81T05 Axiomatic Quantum Field Theory, Operator Algebras (*)

Historical Remarks.

Historical Remarks 1.

Let's start with a very brief historical sketch.

The theory of operator algebras has a very different history compared with most of the other branches of mathematics. Operator algebras is a very recent subject and we can easily identify its birthdate in 1929 in a work of J. Von Neumann.

A few years later, J. Von Neumann and F. Murray already developed the basic theory of the now called **Von Neumann algebras**. The original motivation, was not coming from concrete problem, but by an incredible intuition that these algebras, would have been useful in the theory of group representations and in the general foundations of quantum mechanics.

Historical Remarks 2.

Actually, the very first examples of **non commutative mathematics** had already appeared in the formalism of **quantum mechanics** as developed in algebraic form by W. Heisenberg (1925) [and also by M. Born, P. Jordan, E. Schrödinger, W. Pauli, P.A.M. Dirac, J. Von Neumann].

In this sense, we can say that non commutative mathematics is the mathematics of quantum theory or that it is the **quantization of mathematics**.
In fact, what we mean here by "quantization", is simply the replacement of commuting quantities by non-commuting ones.

Historical Remarks 3.

I.M. Gel'fand and M.A. Naĭmark developed the abstract theory of **C*-Algebras** in 1943. Most of the classical work in operator algebras in the subsequent years, was simply dedicated to refinements of the works of Von Neumann and Gel'fand.

The use of C*-Algebras techniques in quantum field theory has been first advocated by I.E. Segal in 1957.

In 1964, in a fundamental paper, R. Haag, D. Kastler, H. Araki started to apply C*-Algebras to the study of the foundations of quantum field theory, an area of research now called **algebraic quantum field theory**.

Historical Remarks 4.

At the end of the sixties, a deep revolution in the basic techniques occurred thanks to the creation of the **modular theory of Tomita-Takesaki**.

In the same period, R. Haag, N.M. Hugenholtz and P. Winnik found an important link between Tomita-Takesaki modular theory and statistical mechanics.

In the seventies, the research in operator algebras expanded at an exponential rate. The genius of Alain Connes solved some structural problems and at the end of the eighties laid the foundations of the so called **non-commutative differential geometry** that is an approach to geometry completely based on operator algebras, where the same notion of space with its points is eliminated from the formulation.

- Quantum Mathematics.

Quantum Topology = C^*-algebras.

Vector Spaces 1.

A **vector space** V over the complex numbers $\mathbb{C}$ is by definition a set V equipped with two binary operations

$$+ : V \times V \to V, \quad \cdot : \mathbb{C} \times V \to V$$

called respectively addition and multiplication by scalars, such that:

$$v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3, \quad \forall v_1, v_2, v_3 \in V,$$
$$v_1 + v_2 = v_2 + v_1 \quad \forall v_1, v_2 \in V,$$
$$\exists 0_V \in V, \quad : \quad \forall v \in V, \quad v + 0_V = v,$$
$$\forall v \in V, \quad \exists(-v) \in V, \quad : \quad v + (-v) = 0_V,$$
$$\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v, \quad \forall \alpha, \beta \in \mathbb{C}, \quad \forall v \in V,$$
$$\alpha \cdot (v_1 + v_2) = \alpha \cdot v_1 + \beta \cdot v_2, \quad \forall \alpha \in \mathbb{C}, \quad \forall v_1, v_2 \in V,$$
$$(\alpha + \beta) \cdot v = (\alpha \cdot v) + (\beta \cdot v), \quad \forall \alpha, \beta \in \mathbb{C}, \quad \forall v \in V,$$
$$\forall v \in V, \quad 1 \cdot v = v.$$

Vector Spaces 2.

As an example, consider the space

$$\mathbb{C}^2 := \mathbb{C} \times \mathbb{C}$$

with the following operations:

$$(a_1, b_1) + (a_2, b_2) := (a_1 + a_2, b_1 + b_2);$$

$$\alpha \cdot (a, b) := (\alpha a, \alpha b).$$

Vector Spaces 3.

A complex **inner product space** (also called a pre-Hilbert space) is a complex vector space V equipped with an operation

$$(\mid) : V \times V \to \mathbb{C},$$

called inner product such that:

$$(v_1|\alpha v_2 + \beta v_3) = \alpha(v_1|v_2) + \beta(v_1|v_3), \quad \forall \alpha, \beta \in \mathbb{C}, \quad \forall v_1, v_2, v_3 \in V,$$
$$(v_1|v_2) = \overline{(v_2|v_1)}, \quad \forall v_1, v_2 \in V,$$
$$(v|v) \geq 0, \quad \forall v \in V,$$
$$(v|v) = 0 \Rightarrow v = 0_V.$$

As an example, consider again $\mathbb{C}^2$ with inner product given by:

$$\forall a_1, a_2, b_1, b_2 \in \mathbb{C},$$
$$((a_1, b_1)|(a_2, b_2)) := \overline{a_1}a_2 + \overline{b_1}b_2.$$

Associative Unital Algebras 1.

An **algebra** over the complex numbers is by definition a complex vector space A equipped with a binary operation (called product)

$$\cdot : A \times A \to A, \quad \text{that is bilinear i.e.:}$$

$$(\alpha a + \beta b) \cdot c = \alpha(a \cdot c) + \beta(b \cdot c), \quad \forall \alpha, \beta \in \mathbb{C}, \quad \forall a, b, c \in A,$$

$$a \cdot (\beta b + \gamma c) = \beta(a \cdot b) + \gamma(a \cdot c) \quad \forall \beta, \gamma \in \mathbb{C}, \quad \forall a, b, c \in A.$$

The algebra is called:

associative if $\quad a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad \forall a, b, c \in A,$

commutative if $\quad a \cdot b = b \cdot a, \quad \forall a, b \in \mathcal{A}.$

The algebra is **unital** if:

$$\exists 1_A \in A, \quad : \quad \forall a \in A, \quad a \cdot 1_A = 1_A \cdot a = a.$$

Associative Unital Algebras 2.

A **homomorphism** between two algebras $\mathcal{A}, \mathcal{B}$ is by definition a function $\phi : \mathcal{A} \to \mathcal{B}$ such that:

$$\phi(a_1 + a_2) = \phi(a_1) + \phi(a_2), \quad \forall a_1, a_2 \in \mathcal{A};$$
$$\phi(\lambda a) = \lambda \phi(a), \quad \forall \lambda \in \mathbb{C}, \ \forall a \in \mathcal{A};$$
$$\phi(a_1 a_2) = \phi(a_1)\phi(a_2) \quad \forall a_1, a_2 \in \mathcal{A}.$$

An **isomorphism** is a bijective homomorphism and an **automorphism** is an isomorphism $\phi : \mathcal{A} \to \mathcal{A}$ from an algebra $\mathcal{A}$ to itself.
A **unital homomorphism** between unital algebras is a homomorphism such that:

$$\phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}.$$

Involutive Algebras 1.

The algebra is called **involutive** if it is equipped with a function

$$
\begin{gathered}
* : A \longrightarrow A, \quad \text{such that:} \\
(a^*)^* = a, \quad \forall a \in A, \\
(a \cdot b)^* = b^* \cdot a^*, \quad \forall a, b \in A, \\
(\alpha a + \beta b)^* = \overline{\alpha}(a^*) + \overline{\beta}(b^*), \quad \forall \alpha, \beta \in \mathbb{C}, \quad \forall a, b \in A.
\end{gathered}
$$

A $*$-**morphism** $\phi : \mathcal{A} \to \mathcal{B}$ between (unital) involutive algebras is a (unital) homomorphism such that:

$$\phi(a^*) = (\phi(a))^*, \quad \forall a \in \mathcal{A}.$$

Involutive Algebras 2.

As an example consider $\mathbb{M}_{2,2}(\mathbb{C})$, the set of two by two matrices with complex entries where the operations are defined as follows:

- $\forall A, B \in \mathbb{M}_{2,2}(\mathbb{C})$, $A + B$ is the sum of matrices;
- $\forall \alpha \in \mathbb{C}$, $\forall A \in \mathbb{M}_{2,2}(\mathbb{C})$, αA is the product of the matrix A be the complex number α;
- $\forall A, B \in \mathbb{M}_{2,2}(\mathbb{C})$, $A \cdot B$ is the "line by column" product of the matrices;
- $\forall A \in \mathbb{M}_{2,2}(\mathbb{C})$, A^* is the transpose conjugate matrix of A.

Topology 1.

In order to give a meaning to the convergence of sequences or to expressions involving the sum of infinite terms (series) it is necessary to equip vector spaces with a topology[1]

A **topological space** is a set X equipped with a family of subsets $\mathcal{T}$ (called the **open** sets) such that:

- $\varnothing, X \in \mathcal{T}$,
- If $A_1, A_2 \in \mathcal{T}$ then $A_1 \cap A_2 \in \mathcal{T}$,
- If $\mathcal{A} \subset \mathcal{T}$ then $\bigcup_{A \in \mathcal{A}} A \in \mathcal{T}$.

A set is **closed** if its complement is open.

A set U is called a **neighborhood** of a point x if it contains an open set containing x.

[1] For infinite dimensional vector spaces there are actually several inequivalent ways to do so making continuous the operations of addition and multiplication.

Topology 2.

A function $F : X \to Y$ between to topological spaces is said to be **continuous** in the point $x_0 \in X$ if for all the neghborhoods U of the point $F(x_0) \in Y$, it is possible to find a neighborhood V_U of x_0 such that $F(V_U) \subset U$.

A sequence $n \mapsto x_n$ is said to be a **convergent sequence** if there exist a point $l \in X$ such that for all the neigborhoods U of l, there exists a number N_U such that $x_n \in U$, when $n > N_U$.

It is a well known theorem that a function $F : X \to Y$ is continuous if and only if for all the open sets A of Y, $F^{-1}(A)$ is open in X.

A **homeomorphism** between two topological spaces is a bijective continuous function whose inverse is also continuous.

Homeomorphic topological spaces are regarded as essentially the same.

Topology 3.

A topological space X is said to be **Hausdorff** if given two arbitrary different points $x, y \in X$ it is possible to separate them using two neighborhoods i.e. there exist U neighborhood of x and V neighborhood of y such that $U \cap V = \varnothing$.
A set in a topological space is said to be **compact** if from every covering with open sets it is possible to extract a subcovering that contains only a finite number of open sets. A topological space is **locally compact** if every point has a compact neighborhood.
A **metric space** is by definition a set M equipped with a function $d : M \times M \to \mathbb{R}$ (called metric or distance) such that:

$$d(x, y) = 0 \Leftrightarrow x = y,$$
$$d(x, y) = d(y, x), \quad \forall x, y \in M,$$
$$d(x, z) \leq d(x, y) + d(y, z), \quad \forall x, y, z \in M.$$

Topology 4.

A **ball with center** $x_0 \in M$ **and radius** $\epsilon > 0$ is the set:

$$B(x_0, \epsilon) := \{x \mid d(x, x_0) < \epsilon\}.$$

It is a well known fact that a metric space is always a topological space where the open sets are given by arbitrary unions of families of balls. [But not all the topological (vector) spaces are metrizable!]

A sequence $n \mapsto x_n \in M$ in a metric space is said to be a **Cauchy sequence** if given an arbitrary positive number ϵ, it is possible to find an index N_ϵ such that for all $n, m > N_\epsilon$, $d(x_n, x_m) < \epsilon$.

It is a known fact that a convergent sequence is always a Cauchy sequence, but unfortunately, not all the Cauchy sequences are convergent.

A metric space where the Cauchy sequences are all convergent sequences is called **complete.**

Banach and Hilbert Spaces 1.

Functional Analysis is essentially the marriage of algebra and general topology. A typical example of this is given by the definition of Banach and Hilbert space that are actually the infinite dimensional generalizations of the finite dimensional normed and inner product spaces.
A **normed space** is a vector space $(V, +, \cdot)$ equipped with a function

$$\| \ \| : V \to \mathbb{R}, \quad \text{such that :}$$
$$\|a+b\| \leq \|a\| + \|b\|, \quad \forall a, b \in V,$$
$$\|\alpha a\| = |\alpha| \cdot \|a\|, \quad \forall \alpha \in \mathbb{C}, \forall a \in V,$$
$$\|a\| = 0 \Rightarrow a = 0_V.$$

Banach and Hilbert Spaces 2.

It is a well known theorem that all inner product spaces become normed spaces defining:

$$\|a\| := \sqrt{(a \mid a)}.$$

It is also a well know result that all normed spaces become metric spaces defining:

$$d(a, b) := \|a - b\|.$$

[It is not so well known that every metric space is a subset of a normed space!]
A **Banach space** is a normed space that, with the above defined metric, is a complete metric space.
A **Hilbert space** is an inner product space that is a complete metric space with the metric arising from the norm associated to the inner product [Not every Banach space is a Hilbert space].

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Banach and Hilbert Spaces 3.

A linear function between two Hilbert spaces $U : \mathcal{H}_1 \to \mathcal{H}_2$ is said to preserve the scalar product if:

$$(U\xi|U\eta) = (\xi|\eta), \quad \forall \xi, \eta \in \mathcal{H}_1.$$

If such a U is also surjective, then it is called a **unitary** operator and in this case the two Hilbert spaces are essentially considered the same. (In this case it is possible to show that U is an isomorphism of vector spaces that preserve the scalar product).

C^*-algebras 1.

Operator algebras are simply generalizations of the algebras of matrices on a complex vector space. It is possible to think about them as algebras of matrices, but allowing the matrices to be infinite dimensional. It is the study of "linear algebra" of Banach and Hilbert spaces.

A **normed algebra** is by definition an algebra that is at the same time a normed space with a norm that satisfies the following "submultiplicativity" property:

$$\|a \cdot b\| \leq \|a\| \cdot \|b\|, \quad \forall a, b \in A.$$

A **Banach algebra** is a complete normed algebra i.e. it is an algebra that is a Banach space with the submultiplicativity property of the norm.

C*-algebras 2.

A **C*-algebra** is an involutive Banach algebra with the property:

$$\|a^* \cdot a\| = \|a\|^2, \quad \forall a \in A.$$

In other terms, a C*-algebra is an involutive algebra that is at the same time a Banach space with the properties:

$$\|a \cdot b\| \leq \|a\| \cdot \|b\|, \quad \forall a, b \in A,$$
$$\|a^* \cdot a\| = \|a\|^2, \quad \forall a \in A.$$

C*-algebras 3.

As an example, let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{B}(\mathcal{H})$ be the set of all the continuous linear functions $L : \mathcal{H} \to \mathcal{H}$. Then $\mathcal{B}(\mathcal{H})$ is C*-algebra. To see this, recall that:

- A linear map $L : V \to V$ in a vector space is a function such that: $L(\alpha a + \beta b) = \alpha L(a) + \beta L(b), \quad \forall \alpha, \beta \in \mathbb{C}, \quad \forall a, b \in V$.
- A linear map $L : V \to V$ in a normed space is continuous if and only if: $\exists k \in \mathbb{R}$, such that $\forall a \in V, \quad \|L(a)\| \leq k\|a\|$.
- The sum of two linear maps L_1, L_2 defined by:
  $(L_1 + L_2)(a) := L_1(a) + L_2(a)$
  is linear and is continuous when L_1, L_2 are continuous.
- The product of a linear map L with a complex number α defined by: $(\alpha L)(a) := \alpha(L(a))$
  is linear and is continuous if L is.

C*-algebras 4.

- The product of two linear maps L_1, L_2 defined by:
  $(L_1 \cdot L_2)(a) := L_1(L_2(a))$
  is linear and continuous if L_1 and L_2 are continuous.
- In a Hilbert space, to a linear continuous map $L : \mathcal{H} \to \mathcal{H}$, we can associate a unique operator, called the adjoint of L, such that:
  $(b \mid L(a)) = (L^*(b) \mid a), \quad \forall a, b \in \mathcal{H}.$
- In a normed space, the norm of a linear continuous function $L : V \to V$, defined by:
  $\|L\|_{\mathcal{B}(V)} := \sup\{\|L(a)\| \mid \|a\| \leq 1\},$
  gives a norm on the space $\mathcal{B}(V)$.
- If V is a Banach space, then $\mathcal{B}(V)$ with the previous norm $\| \, \|_{\mathcal{B}(V)}$ is a Banach space.

C*-algebras 5.

As a second example: let X be a compact Hausdorff topological space. The set $C(X;\mathbb{C})$ of the complex valued continuous functions on X, is a commutative C*-algebra.
To see this recall that:

- If $f, g \in C(X;\mathbb{C})$, $f+g$ is defined by: $(f+g)(x) := f(x)+g(x)$.
- If $\alpha \in \mathbb{C}$ and $f \in C(X;\mathbb{C})$, αf is defined by: $(\alpha f)(x) := \alpha(f(x))$.
- If $f, g \in C(X;\mathbb{C})$ then $f \cdot g$ is defined by: $(f \cdot g)(x) := f(x)g(x)$.
- If $f \in C(X;\mathbb{C})$, then f^* is defined by: $f^*(x) := \overline{f(x)}$.
- If $f \in C(X;\mathbb{C})$, then $\|f\|$ is defined by: $\|f\| := \sup\{|f(x)| \mid x \in X\}$.

C*-algebras 6.

$\mathcal{B}(\mathcal{H})$, as all the matrix algebras, is a non-commutative algebra (iff $\dim \mathcal{H} > 1$). If we have a C*-algebra, any subset that is closed under the algebraic operations and closed (complete) from the topological point of view, is again a C*-algebra.
Let us introduce some useful terminology about the elements of a C*-algebra. A C*-algebra is said to be **unital** if it contains an element $1_\mathcal{A}$ such that:

$$a 1_\mathcal{A} = 1_\mathcal{A} a = a, \quad \forall a \in \mathcal{A}.$$

($1_\mathcal{A}$ is necessarily unique).
An element a in a unital algebra $\mathcal{A}$ is said to be **invertible** if there exists (a necessarily unique) element b such that:

$$ab = ba = 1_\mathcal{A}.$$

The set of invertible elements is denoted by $\mathrm{Inv}(\mathcal{A})$.

C^*-algebras 7.

If $\mathcal{A}$ is unital, the **spectrum** of an element $a \in \mathcal{A}$ is by definition the set $\mathrm{Sp}_{\mathcal{A}}(a)$ of complex numbers λ such that $a - \lambda 1_{\mathcal{A}}$ is not invertible:

$$\mathrm{Sp}_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} \mid a - \lambda 1_{\mathcal{A}} \notin \mathrm{Inv}(\mathcal{A})\}.$$

The complement of the spectrum of a is called the **resolvent set** of a and denoted by $\mathrm{Res}_{\mathcal{A}}(a) := \mathbb{C} - \mathrm{Sp}_{\mathcal{A}}(a)$.

Theorem

In a unital Banach algebra, $\mathrm{Sp}_{\mathcal{A}}(a)$ is always a nonempty compact set and the **spectral radius of** *a, is well defined by $r(a) := \sup\{|\lambda| \mid \lambda \in \mathrm{Sp}_{\mathcal{A}}\}$ and satisfies $r(A) \leq \|a\|$.*

The proof needs several lemmas.

C*-algebras 8.

Proposition (C. Neumann)

Let a be an alement of the unital Banach algebra $\mathcal{A}$.
If $\|a\| < 1$ then $(1_{\mathcal{A}} - a) \in \mathrm{Inv}(\mathcal{A})$ and $(1_{\mathcal{A}} - a)^{-1} = \sum_{n=0}^{\infty} a^n$.

Proof.

Since $\sum_{n=0}^{\infty} \|a\|^n$ is a convergent geometric serie, the sequence $N \mapsto \sum_{n=0}^{N} a^n$ is Cauchy, hence convergent in the Banach space $\mathcal{A}$. Now $(1_{\mathcal{A}} - a) \cdot (\sum_{n=0}^{N})a^n = 1_{\mathcal{A}} - a^{N+1} = (\sum_{n=0}^{N})a^n \cdot (1_{\mathcal{A}} - a)$ and passing to the limit, by continuity of product in $\mathcal{A}$ we get the result. ∎

If $|\lambda| > \|a\|$, $\|\lambda^{-1}a\| < 1$ hence $a - \lambda 1_{\mathcal{A}} = \lambda(\lambda^{-1}a - 1_{\mathcal{A}}) \in \mathrm{Inv}(\mathcal{A})$ so $\lambda \in \mathrm{Res}_{\mathcal{A}}(a)$ i.e. $\mathrm{Sp}_{\mathcal{A}}(a)$ is bounded and $|\lambda| \leq \|a\|$ for all $\lambda \in \mathrm{Sp}_{\mathcal{A}}(a)$. It follows that $r(a) \leq \|a\|$.

C*-algebras 9.

Proposition

In a unital Banach algebra $\mathcal{A}$, the group $\mathrm{Inv}(\mathcal{A})$ is open.
For all $a \in \mathcal{A}$, $\mathrm{Res}_{\mathcal{A}}(a)$ is open and $\mathrm{Sp}_{\mathcal{A}}(a)$ is closed.

Proof.

For every $x \in \mathrm{Inv}(\mathcal{A})$, the continous map $L_x : \mathrm{Inv}(\mathcal{A}) \to \mathrm{Inv}(\mathcal{A})$ defined by $L_x(a) := x \cdot a$ is an endo-homeomorphism of $\mathrm{Inv}(\mathcal{A})$ with inverse $L_{x^{-1}}$. Since $1_{\mathcal{A}}$ is an interior point of $\mathrm{Inv}(\mathcal{A})$ and $a = L_a(1_{\mathcal{A}})$, every point of $\mathrm{Inv}(\mathcal{A})$ is interior.
The function $f : \lambda \mapsto (\lambda 1_{\mathcal{A}} - a)$ is continuous and $\mathrm{Res}_{\mathcal{A}}(a) = f^{-1}(\mathrm{Inv}(\mathcal{A}))$ is open. ∎

By Heine-Borel theorem, it follows that $\mathrm{Sp}_{\mathcal{A}}(a)$ is always compact (closed and bounded in $\mathbb{C}$).

C*-algebras 10.

Proposition

In a unital Banach algebra $\mathcal{A}$, $\mathrm{Sp}_{\mathcal{A}}(a)$ is always nonempty.

Proof.

For every continuous linear functional $\phi \in \mathcal{A}^*$, the $\mathbb{C}$-valued function $h_\phi : \lambda \mapsto \phi((\lambda 1_{\mathcal{A}} - a)^{-1})$ is differentiable on the resolvent set $\mathrm{Res}_{\mathcal{A}}(a)$, hence analytic.
Suppose by contradiction that $\mathrm{Sp}_{\mathcal{A}}(a) = \varnothing$, then h_ϕ is an entire function and since $\lim_{\lambda \to \infty} h_\phi(\lambda) = 0$, h_ϕ is bounded and by Liouville theorem, h_ϕ is constant, hence $h_\phi(\lambda) = 0$ for all $\lambda \in \mathbb{C}$.
Since $\phi((\lambda 1_{\mathcal{A}} - a)^{-1}) = 0$ for all $\phi \in \mathcal{A}^*$, Hahn-Banach theorem gives $(\lambda 1_{\mathcal{A}} - a)^{-1} = 0_{\mathcal{A}}$ that implies $\|1_{\mathcal{A}}\| = 0$ (impossible by definition in a unital Banach algebra). ∎

C*-algebras 11.

Theorem (Gel'fand-Mazur)

A unital Banach algebra $\mathcal{A}$ that is a division algebra (i.e. $\mathrm{Inv}(\mathcal{A}) = \mathcal{A} - \{0_{\mathcal{A}}\}$) is canonically isomorphic to $\mathbb{C}$.

Proof.

The map $\lambda \mapsto \lambda \cdot 1_{\mathcal{A}}$ is a unital homomorphism from $\mathbb{C}$ to $\mathcal{A}$ that is injective because isometric: $\|\lambda 1_{\mathcal{A}}\| = |\lambda|$.
The surjectivity follows from the fact that for any $a \in \mathcal{A}$, $\mathrm{Sp}_{\mathcal{A}}(a) \neq \varnothing$ so there exists λ_o such that $a - \lambda_o 1_{\mathcal{A}}$ is not invertible, hence $a = \lambda_o 1_{\mathcal{A}}$. ∎

C*-algebras 12.

Proposition (polynomial functional calculus)

If $\mathcal{A}$ is a unital Banach algebra and $a \in \mathcal{A}$, there exists a unique unital homomorphism from the algebra $\mathbb{C}[z]$ of polynomials onto the unital Banach subalgebra generated by a such that $\iota \mapsto a$, where $\iota(z) := z$ is the identity function on $\mathbb{C}$.

Proof.

Any homomorphism with the given properties must coincide with the homomorphism $p(z) \mapsto p(a)$ where $p(z) := \sum_{n=0}^{N} \alpha_n z^n$ and $p(a) := \sum_{n=0}^{N} \alpha_n a^n$. This homomorphism is onto the unital algebra generated by a. ∎

C*-algebras 13.

Proposition (polynomial spectral mapping)

In a unital Banach algebra $\mathcal{A}$, for all $a \in \mathcal{A}$ and for all polymonials $p \in \mathbb{C}[z]$, we have $p(\mathrm{Sp}_{\mathcal{A}}(a)) = \mathrm{Sp}(p(a))$ i.e. the image of the spectrum of a under the polymonial p coincides with the spectrum of the element $p(a)$ obtained by polynomial functional calculus.

Proof.

Fix $\lambda \in \mathbb{C}$ and consider the polynomial $p(z) - \lambda$. By the foundamental theorem of algebra we have $p(z) - \lambda = \beta \prod_{j=1}^{N}(z - \alpha_j)$ where $\alpha_j \in \mathbb{C}$ are the roots (with multiplicity) and $\beta \in \mathbb{C}_0$. From polynomial functional calculus we get $p(a) - \lambda 1_{\mathcal{A}} = \beta \prod_{j=1}^{N}(a - \alpha_j 1_{\mathcal{A}})$.
Now $\lambda \notin \mathrm{Sp}_{\mathcal{A}}(p(a))$ if and only if $p(a) - \lambda 1_{\mathcal{A}}$ is invertible, if and only if $a - \alpha_j 1_{\mathcal{A}}$ are invertible for all $j = 1$, if and only if $\alpha_j \notin \mathrm{Sp}_{\mathcal{A}}(a)$ for all j.
On the other side, $\lambda \notin p(\mathrm{Sp}_{\mathcal{A}}(a))$ if and only if $p(z) - \lambda \neq 0$ for all $z \in \mathrm{Sp}_{\mathcal{A}}(a)$ if and only if $\alpha_j \notin \mathrm{Sp}_{\mathcal{A}}(a)$ for all j. ∎

C*-algebras 14.

Proposition (spectral radius formula)

In a unital Banach algebra $\mathcal{A}$ we have, for all $a \in \mathcal{A}$,
$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

From polymonial spectral mapping, $r(a)^n = r(a^n)$ and so $r(a) \leq \|a^n\|^{1/n}$, hence $r(a) \leq \liminf_n \|a^n\|^{1/n}$.
On the other side, given $\phi \in \mathcal{A}^*$, by Neumann serie, we have $f_\phi(\lambda) = -1/\lambda \cdot \sum_{n=0}^{\infty} \phi(a^n)/\lambda^n$ if $|\lambda| > \|x\|$ and since we know that the function $f_\phi : \lambda \mapsto \phi((a - \lambda 1_\mathcal{A})^{-1})$ is analytic in every point with $|\lambda| > r(a)$, the series converges for $|\lambda| > r(a)$.
It follows that, for $|\lambda| > r(a)$, $\lim_{n\to\infty} \phi(a^n)/\lambda^n = 0$, hence the sequence $\phi(a^n/\lambda^n)$ is bounded for all ϕ and by Banach-Steinhouse it is norm bounded i.e. $\|a^n\|/\lambda^n < k$ and so $\|a^n\|^{1/n} \leq k^{1/n}|\lambda|$ so $\limsup_n \|a^n\|^{1/n} \leq |\lambda|$ for $|\lambda| > r(a)$. Hence $\limsup_n \|a^n\|^{1/n} \leq r(a)$.

C*-algebras 15.

The following theorem of Gel'fand is a milestone for all the following and it provides the main philosophical motivation for the development of non-commutative geometry:

Theorem (Gel'fand)

If $\mathcal{A}$ is a commutative unital C-algebra, then $\mathcal{A}$ is isomorphic to an algebra of continuous complex valued functions on a compact Hausdorff topological space.*

The proof is based on the "spectral theory" of commutative Banach algebras that we are going to expose.

A **character** of a unital Banach algebra $\mathcal{A}$ is a unital homomorphism $\omega : \mathcal{A} \to \mathbb{C}$

The **spectrum** (also called the "structure space") Sp($\mathcal{A}$) of the unital Banach algebra $\mathcal{A}$ is the set of all characters on $\mathcal{A}$.

C*-algebras 16.

Proposition

Every character ω on a unital Banach algebra $\mathcal{A}$ is surjective continuous with $\|\omega\| = 1$ and for all $a \in \mathcal{A}$, for all $\omega \in \mathrm{Sp}(\mathcal{A})$, $\omega(a) \in \mathrm{Sp}_{\mathcal{A}}(a)$

Proof.

Since $\omega(1_{\mathcal{A}}) = 1$ the image of ω is all of $\mathbb{C}$.
For every $a \in \mathcal{A}$, set $\{\omega(a) \mid \omega \in \mathrm{Sp}(\mathcal{A})\}$ is contained in $\mathrm{Sp}_{\mathcal{A}}(a)$ because $\omega(a - \omega(a)1_{\mathcal{A}}) = 0$ so that $a - \omega(a)1_{\mathcal{A}}$ is not invertible.
It follows that $|\omega(a)| \leq \|a\|$ so that ω is continuous with norm $\|\omega\| \leq 1$. Since $\omega(1_{\mathcal{A}}) = 1$ we have $\|\omega\| = 1$. ∎

C*-algebras 17.

A (bilateral) **ideal** $\mathcal{I}$ in an algebra $\mathcal{A}$ is a vector subspace of $\mathcal{A}$ that is "stable" under multiplication by arbitrary elements of $\mathcal{A}$ i.e.:

$$a \cdot x, x \cdot a \in \mathcal{I}, \quad \forall i \in \mathcal{I}, \ \forall a \in \mathcal{A}.$$

An ideal $\mathcal{I} \subset \mathcal{A}$ is **maximal** if $\mathcal{I} \neq \mathcal{A}$ and every ideal $\mathcal{J}$ in $\mathcal{A}$ containing $\mathcal{I}$ coincides either with $\mathcal{I}$ or with $\mathcal{A}$.

In a unital Banach algebra, maximal ideals are closed.

The quotient of a unital algebra by an ideal has a natural structure of unital algebra.

The quotient of a commutative unital algebra by an ideal is a field if and only if the ideal is maximal.

The quotient of a unital Banach algebra $\mathcal{A}$, by a closed ideal $\mathcal{I}$ is a Banach algebra with the "quotient norm" defined by $\|a + \mathcal{I}\| := \inf\{\|a + i\| \mid i \in \mathcal{I}\}$.

C*-algebras 18.

Proposition

In a unital Banach algebra $\mathcal{A}$ there is a bijective correspondence between characters and maximal ideals.

Proof.

To every character $\omega \in \mathrm{Sp}(\mathcal{A})$ we associate $\mathrm{Ker}(\omega)$ that is a maximal ideal. The map $\omega \mapsto \mathrm{Ker}(\omega)$ is injective.
If $\mathcal{I}$ is a maximal ideal in $\mathcal{A}$, the quotient $\mathcal{A}/\mathcal{I}$ is a field and a unital Banach algebra with the quotient norm, hence by Gel'fand-Mazur theorem, there is an isomorphism $\phi : \mathcal{A}/\mathcal{I} \to \mathbb{C}$ and so the composition $\phi \circ \pi : \mathcal{A} \to \mathbb{C}$ of ϕ with the quotient homomorphism $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is a character with $\mathrm{Ker}(\pi \circ \phi) = \mathcal{I}$. ∎

C*-algebras 19.

Recall that on the dual $\mathcal{A}^*$ of a Banach space $\mathcal{A}$ the **weak*-topology** is the weakest topology making continuous all the evaluation maps $\phi \mapsto \phi(a)$, for all $a \in \mathcal{A}$.

Proposition

If $\mathcal{A}$ is a unital Banach algebra, $\mathrm{Sp}(\mathcal{A})$ *is a compact Hausdorff space of $\mathcal{A}^*$ with the weak*-topology.*

Proof.

The set $\mathrm{Sp}(\mathcal{A})$ is closed in $\mathcal{A}$ for the weak*-topology because $\mathrm{Sp}(\mathcal{A})$ is the intersection of the set $\{\omega \in \mathcal{A}^* \mid \omega(1_\mathcal{A}) = 1\}$ with all the sets $X_{ab} := \{\omega \in \mathcal{A}^* \mid \omega(ab) = \omega(a)\omega(b)\}$, for $a, b \in \mathcal{A}$, that are weak*-closed because evaluations are continuous.
Since $\|\omega\| = 1$ for all characters $\omega \in \mathrm{Sp}(\mathcal{A})$, we have that $\mathrm{Sp}(\mathcal{A})$ is a norm-bounded subset of the dual $\mathcal{A}^*$. By Banach-Alaouglu theorem, a norm-bounded closed set in $\mathcal{A}^*$ is compact. Since $\mathcal{A}^*$ is Hausdorff in the weak*-topology so is $\mathrm{Sp}(\mathcal{A})$.

C*-algebras 20.

If $\mathcal{A}$ is a unital Banach algebra, to every $a \in \mathcal{A}$ there an associated "evaluation" function $\hat{a} : \mathrm{Sp}(\mathcal{A}) \to \mathbb{C}$ on the set of characters, called the **Gel'fand transform of** a, defined by:

$$\hat{a} : \mathrm{Sp}(\mathcal{A}) \to \mathbb{C}, \quad \hat{a}(\omega) := \omega(a).$$

From the definition of weak*-topology, we see that for all $a \in \mathcal{A}$, its Gel'fand transform $\hat{a}$ is a continuous map on $\mathrm{Sp}(\mathcal{A})$.
The **Gel'fand transform** of the unital Banach algebra $\mathcal{A}$ is the map that to every element $a \in \mathcal{A}$ associate its Gel'fand transform $\hat{a} \in C(\mathrm{Sp}(\mathcal{A}); \mathbb{C})$:

$$\mathfrak{G}_{\mathcal{A}} : \mathcal{A} \to C(\mathrm{Sp}(\mathcal{A}); \mathbb{C}), \quad a \mapsto \mathfrak{G}_{\mathcal{A}}(a) := \hat{a}.$$

C^*-algebras 21.

Proposition

The spectrum of the Gel'fand transform of a in the unital Banach algebra $C(\mathrm{Sp}(\mathcal{A});\mathbb{C})$ always coincide with the spectrum of a in $\mathcal{A}$:

$$\mathrm{Sp}_{C(\mathrm{Sp}(\mathcal{A});\mathbb{C})}(a) = \mathrm{Sp}_{\mathcal{A}}(a), \quad \forall a \in \mathcal{A}.$$

Proof.

The spectrum of $\hat{a}$ in $C(\mathrm{Sp}(\mathcal{A});\mathbb{C})$ is the image of the function $\hat{a}$. Since we already know that $\hat{a}(\omega) = \omega(a) \in \mathrm{Sp}_{\mathcal{A}}(a)$ for all $\omega \in \mathrm{Sp}(\mathcal{A})$, we have $\mathrm{Sp}_{C(\mathrm{Sp}(\mathcal{A});\mathbb{C})}(\hat{a}) \subset \mathrm{Sp}_{\mathcal{A}}(a)$.
If $\lambda \in \mathrm{Sp}_{\mathcal{A}}(a)$, $(a - \lambda 1_{\mathcal{A}})$ is not invertible, hence it generates an ideal $(a - \lambda 1_{\mathcal{A}}) \cdot \mathcal{A}$ that is proper (i.e. different form $\mathcal{A}$).
By Zorn's lemma, every proper ideal in a unital algebra is contained in a maximal ideal. It follows that there exists a character ω such that $(a - \lambda 1_{\mathcal{A}}) \cdot \mathcal{A} \subset \mathrm{Ker}(\omega)$ and so $\omega(a - \lambda 1_{\mathcal{A}}) = 0$, hence $\lambda = \omega(a) \in \mathrm{Sp}_{C(\mathrm{Sp}(\mathcal{A}),\mathbb{C})}$.

C^*-algebras 22.

Proposition

The Gel'fand transform $\mathfrak{G}_\mathcal{A}$ of a unital Banach algebra $\mathcal{A}$ is a unital homomorphism of unital Banach algebras. For all $a \in \mathcal{A}$, $\|\mathfrak{G}_\mathcal{A}(a)\|_\infty \leq \|a\|$.

Proof.

The algebraic properties are immediate from calculation on a character $\omega \in \mathrm{Sp}(\mathcal{A})$.
Since $\hat{a}(\omega) = \omega(a) \in \mathrm{Sp}_\mathcal{A}(a)$ we know that $|\hat{a}(\omega)| \leq \|a\|$ hence $\|\hat{a}\|_\infty := \sup_{\omega \in \mathrm{Sp}_\mathcal{A}} |\hat{a}(\omega)| \leq \|a\|$. ■

It follows that the image of the Gel'fand transform $\mathfrak{G}_\mathcal{A}(\mathcal{A}) \subset C(\mathrm{Sp}(\mathcal{A}); \mathbb{C})$ is a unital subalgebra of $C(\mathrm{Sp}(\mathcal{A}); \mathbb{C})$.
If $\omega_1 \neq \omega_2$ there exists at least one $a \in \mathcal{A}$ such that $\omega_1(a) \neq \omega_2(a)$, hence the subalgebra $\mathfrak{G}_\mathcal{A}(\mathcal{A})$ separates the points of $\mathrm{Sp}(\mathcal{A})$.

C*-algebras 23.

An element a in a C*-algebra $\mathcal{A}$ is said to be:

- **selfadjoint** if: $a = a^*$,
- **normal** if: $aa^* = a^*a$,
- **unitary** if: $aa^* = a^*a = 1_\mathcal{A}$ (in this case $\mathcal{A}$ must be unital),
- **isometry** if: $a^*a = 1_\mathcal{A}$ (again $\mathcal{A}$ must be unital),
- **coisometry** if: $aa^* = 1_\mathcal{A}$ (for $\mathcal{A}$ unital as well),
- (selfadjoint) **projection** if: $aa = a$ (and $a = a^*$),
- **partial isometry** if: a^*a is a projection,

C*-algebras 24.

Proposition

In a $*$-algebra $\mathcal{A}$, every element a has a unique decomposition as $a = x + iy$ with x, y Hermitian.

Proof.

If $a = x + iy$, taking the adjoints, we have $a^* = x - iy$ hence $a = (a + a^*)/2 + i(a - a^*)/(2i)$. ∎

The unique elements described above are called the **real part** and the **imaginary part** of a and are denoted by $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$.

C^*-algebras 25.

Proposition

Let h be a Hermitian in a (commutative) unital C^*-algebra. Then $\mathrm{Sp}_{\mathcal{A}}(h) \subset \mathbb{R}$.

Proof.

If we have an Hermitian element h_1 such that $\alpha + i\beta \in \mathrm{Sp}_{\mathcal{A}}(h_1)$ with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$, i.e. we have a character ω such that $\omega(h_1) = \alpha + i\beta$, then we have an Hermitian element $h := h_1 - \alpha 1_{\mathcal{A}}$ with $i\beta \in \mathrm{Sp}_{\mathcal{A}}(h)$ and $\omega(h) = i\beta$. Now $|\omega(h + it1_{\mathcal{A}})|^2 \leq \|h + it1_{\mathcal{A}}\|^2 = \|h^2 + t^2 1_A\| \leq \|h\|^2 + t^2$. Since $|\omega(h + it1_{\mathcal{A}})|^2 = \beta^2 + 2\beta t + t^2$, we get for all $t \in \mathbb{R}$, $\beta^2 + \beta t \leq \|h\|^2$ that is impossible if $\beta > 0$ (for $\beta < 0$ take $-h$). ∎

C*-algebras 26.

Proposition

The Gel'fand transform of a unital commutative C^*-algebra is a $*$-homomorphism.

Proof.

If $a = \mathrm{Re}(a) + i\,\mathrm{Im}(a)$, we have $a^* = \mathrm{Re}(a) - i\,\mathrm{Im}(a)$ and for all characters $\omega \in \mathrm{Sp}(\mathcal{A})$, since $\omega(\mathrm{Re}(a)), \omega(\mathrm{Im}(a)) \in \mathbb{R}$, we get $\widehat{a^*}(\omega) = \omega(\mathrm{Re}(a)) - i\omega(\mathrm{Im}(a)) = \overline{\hat{a}(\omega)}$. ∎

C*-algebras 27.

Proposition

The Gel'fand transform of a unital commutative C-algebra $\mathcal{A}$ is isometric, hence injective.*

Proof.

If $h \in \mathcal{A}$ is Hermitian, $\|h^2\| = \|h\|^2$ and by induction $\|h^{2^n}\| \leq \|h\|^{2^n}$ for $n \in \mathbb{N}_0$. Since, by spectral radius formula, $r(h) = \lim_{n\to\infty} \|h^n\|^{1/n}$, we have $r(h) = \|h\| = \lim_{n\to\infty} \|h^{2^n}\|^{1/2^n}$. Since we know that $r(h) = \|\hat{h}\|_{C(\mathrm{Sp}(\mathcal{A});\mathbb{C})}$ we see that the Gel'fand transform is isomtric on Hermitian elements.

For general elements

$$\|a\|^2 = \|a^*a\| = \|\widehat{a^*a}\|_{C(\mathrm{Sp}(\mathcal{A};\mathbb{C}))} = \|\bar{\hat{a}}\hat{a}\| = \|\hat{a}\|^2_{C(\mathrm{Sp}(\mathcal{A});\mathbb{C})}.$$ ∎

C*-algebras 28.

Proposition

The Gel'fand transform of a unital commutative C-algebra is an isomorphism.*

Proof.

The image of the Gel'fand transform $\mathfrak{G}_{\mathcal{A}}(\mathcal{A})$, is an ivolutive unital subalgebra of the C*-algebra $C(Sp(\mathcal{A}); \mathbb{C})$ that separates the points of $\mathrm{Sp}(\mathcal{A})$. Since the Gel'and transform is isometric, $\mathfrak{G}_{\mathcal{A}}(\mathcal{A})$ is also a closed subalgebra.

The surjectivity follows from Stone-Weierstrass theorem. ∎

C*-algebras 29.

We thus competed the proof of Gel'fand theorem for commutative unital C*-algebras.
The most important lesson that we learn from the proof of this theorem is that, when we are given an abstract unital commutative C*-algebra $\mathcal{A}$, we can construct out of it a compact Hausdorff space denoted $\mathrm{Sp}(\mathcal{A})$ (called the spectrum of the C*-algebra $\mathcal{A}$) in such a way that the the original algebra is isomorphic (hence indistinguishable from) $C(\mathrm{Sp}(\mathcal{A}); \mathbb{C})$.

C*-algebras 30.

To understand how this "magic" thing is possible, we have to rethink (in a completely algebraic way) what is the meaning of a point $x \in X$ in terms of the algebra $C(X,\mathbb{C})$: if we are given a point $x \in X$, we can associate to it a function $\mu_x : C(X,\mathbb{C}) \to \mathbb{C}$, defined by: $\mu_x : f \mapsto f(x), \quad \forall f \in C(X,\mathbb{C})$.
For every $x \in X$, μ_x is simply the evaluation of the function f over the point x. Well, it is an easy game to see that the function μ_x is a character over the C*-algebra $C(X,\mathbb{C})$ It is possible to see that every character over $C(X,\mathbb{C})$ is obtained as an evaluation in a point of x i.e. the map $x \mapsto \mu_x$ that associates to a point its character is a bijective one in such a way that, if we want, we can say that a point of the space X "is" a character on the algebra $C(X;\mathbb{C})$. Once this identification has been done, we associate to every element $f \in C(X;\mathbb{C})$ its Gel'fand transform $\hat{f}(\mu) := \mu(f)$.

C*-algebras 31.

Theorem (Gel'fand theorem - second part)

Let X be a compact Hausdorff space. X is naturally homeomorphic to the compact Hausdorff space $\mathrm{Sp}(C(X;\mathbb{C}))$.

Proof (1).

Consider the map $\mathfrak{E}_X : X \to \mathrm{Sp}(C(X;\mathbb{C}))$ that to every point $x \in X$ associates the character $\mu_x \in \mathrm{Sp}(C(X;\mathbb{C}))$ given by the evaluation in x.

$\mathfrak{E}_X$ is injective, because if $x \neq y$, by Urysohn lemma, there exists $f \in C(X;\mathbb{C})$ with $\mu_x(f) = f(x) \neq f(y) = \mu_y(f)$. ∎

C*-algebras 32.

Proof (2).

$\mathfrak{E}_X$ is surjective because if the maximal ideal $\mathrm{Ker}(\mu)$ of the character $\mu \in \mathrm{Sp}(C(X;\mathbb{C}))$ is not of the form $\mathrm{Ker}(\mu_x)$ for a certain $x \in X$, there exists $f_x \in \mathrm{Ker}(\mu)$ with $f_x(x) \neq 0$ and by compactness of X we get a finite number of points $x_1, \dots, x_n$ and functions $f_{x_1}, \dots, f_{x_n}$ such that the invertible function $\sum_{j=1}^n f_{x_j}^2 \in \mathrm{Ker}(\mu)$.
$\mathfrak{E}_X$ is continuous if and only if all its compositions $\hat{f} \circ \mathfrak{E}_X$ with the $\hat{f}$ are continous, for all $f \in C(X;\mathbb{C})$ and this is true, since $\hat{f} \circ \mathfrak{E}_X = f \in C(X;\mathbb{C})$.
Since X is Hausdorff and $\mathrm{Sp}(C(X;\mathbb{C}))$ is compact, it follows that Eg_X is a homeomorphism. ∎

C*-algebras 33.

Gel'fand theorems tells us that to study compact Hausdorff topological spaces is the same thing as to study unital commutative C*-algebras.

To every compact Hausdorff topological space there is an associated unital commutative C*-algebra of continuous complex functions and, thanks to the theorem, we know that every unital commutative C*-algebra arises in this way.

Similarly every commutative unital C*-algebra gives a compact Hausdorff space of characters and every compact Hausdorff space arises in this way.

If commutative unital C*-algebras are nothing else but compact Hausdorff topological spaces, we can start to think that maybe, when we study a non-commutative (unital) C*-algebra, we are actually studying non-commutative topology: the study of noncommutative C*-algebras "is" the study of "non-commutative spaces".

C*-algebras 34.

This is the starting point of non-commutative geometry:

- We look at the standard "commutative" spaces and we construct algebras of complex functions over them.
- We try to express the geometric properties of the base space making use only of its algebra of functions.
- Finally we try to see if the geometrical information codified in the commutative algebra of functions still make sense if we take a non-commutative algebra (that of course cannot arise as an algebra of functions on a space).
- We call the non-commutative algebras "non-commutative spaces".

C*-algebras 35.

The basic idea of non-commutative geometry is to study spaces through their algebras of functions and to define more "abstract" spaces using only their non-commutative algebras.

This approach is not only useful to identify the "geometrical" behaviour hidden in non-commutative algebras, but can be used to study in a more efficient way classical (hence commutative) spaces that are too complicate for ordinary geometry.

Non-commutative geometry has already been used to study the topology of "bad" spaces like Penrose tilings, quotient spaces, foliations on a manifold, and it is also expected that non-commutative geometry will cast some new light on an intrinsic definition of fractal spaces.

Of course Gel'fand theorem is not only the "starting of an ideology", it is also the starting point for the development of "technical tools" in the study of C*-algebras.

C*-algebras 36.

Proposition

If $\mathcal{B}$ is a unital Banach subalgebra of the unital Banach algebra $\mathcal{A}$ and $b \in \mathcal{B}$, then $\mathrm{Sp}_{\mathcal{A}}(b) \subset \mathrm{Sp}_{\mathcal{B}}(b)$ and the border points of the spectra satisfy $\partial(\mathrm{Sp}_{\mathcal{B}}(b)) \subset \partial(\mathrm{Sp}_{\mathcal{A}}(b))$.

Proof.

Of course $\mathrm{Res}_{\mathcal{B}}(b) \subset \mathrm{Res}_{\mathcal{A}}(b)$.
If λ is a border point of $\mathrm{Sp}_{\mathcal{B}}(b)$ then $\lambda \in \mathrm{Sp}_{\mathcal{B}}(b)$ and there exists a sequence $\lambda_n \to \lambda$ with $\lambda_n \in \mathrm{Res}_{\mathcal{B}}(b) \subset \mathrm{Res}_{\mathcal{A}}(b)$ Hence it is enough to show that $\lambda \in \mathrm{Sp}_{\mathcal{A}}(b)$.
If by contradiction $\lambda \notin \mathrm{Sp}_{\mathcal{A}}(b)$, then definitely $\lambda_n \in \mathrm{Res}_{\mathcal{A}}(b)$ hence $(b - \lambda_n 1) \in \mathrm{Inv}(\mathcal{A})$ and $(b - \lambda_n 1)^{-1}$ converges to $(b - \lambda 1)^{-1} \in \mathrm{Inv}(\mathcal{A})$ and since $\mathcal{B}$ is closed we have $(b - \lambda 1)^{-1} \in \mathcal{B}$ so $\lambda \notin \mathrm{Sp}_{\mathcal{B}}(b)$ that is a contradiction. ∎

C^*-algebras 37.

Proposition

If $\mathcal{B}$ is a unital C^*-subalgebra of the C^*-algebra $\mathcal{A}$, and $b \in \mathcal{B}$ then $\mathrm{Sp}_{\mathcal{B}}(b) = \mathrm{Sp}_{\mathcal{A}}(b)$.

Proof.

If b is Hermitian in $\mathcal{B}$, consider the unital C^*-algebra $C^*(b)$ generated by b. We have $\mathrm{Sp}_{\mathcal{A}}(b) \subset \mathrm{Sp}_{\mathcal{B}}(b) \subset \mathrm{Sp}_{C^*(b)}(b)$. Since $\mathrm{Sp}_{C^*(b)}(b) \subset \mathbb{R} \subset \mathbb{C}$, we have $\mathrm{Sp}_{C^*(b)(b)} = \partial(\mathrm{Sp}_{C^*(b)}(b)) \subset \partial(\mathrm{Sp}_{\mathcal{B}}(b)) \subset \partial(\mathrm{Sp}_{\mathcal{A}}(b))$. Hence $\mathrm{Sp}_{\mathcal{B}}(b) = \mathrm{Sp}_{\mathcal{A}}(b)$.

If $b - \lambda \in \mathcal{B} \cap \mathrm{Inv}(\mathcal{A})$, $(b-\lambda)^*(b-\lambda)$ is Hermitian in $\mathcal{B}$ and invertible in $\mathcal{A}$ hence (from above) $(b-\lambda)^*(b-\lambda) \in \mathrm{Inv}(\mathcal{B})$ so $((b-\lambda)^*(b-\lambda))^{-1}(b-\lambda)^* \in \mathcal{B}$ and it is the inverse of $(b-\lambda)$ in $\mathcal{A}$ hence $(b-\lambda)^{-1} \in \mathcal{B}$.

C*-algebras 38.

Theorem (continuous functional calculus)

If a is a normal element in a unital C^*-algebra $\mathcal{A}$, there is a unique isometric isomorphism of C^*-algebras $\Phi : C(\mathrm{Sp}_{\mathcal{A}}(a); \mathbb{C}) \to C^*(a)$, onto the unital C^*-algebra generated by a, mapping the identity function $\iota : z \mapsto z$ to a.

Proof.

The unital C*-algebra $C^*(a)$ generated by a is commutative. Furthermore $C^*(a)$ is the closure in $\mathcal{A}$ of the involutive subalgebra of polynomials in a, a^*. The unicity follows from Stone-Weierstrass
The Gel'fand transform $\hat{a} : \mathrm{Sp}(C^*(a)) \to \mathrm{Sp}_{\mathcal{A}}(a)$ of a is a surjective continous map from a compact to a Hausdorff space. $\hat{a}$ is injective because if $\hat{a}(\omega_1) = \hat{a}(\omega_2)$ then ω_1 and ω_2 coincide on a and hence on all of $C^*(a)$. Hence $\hat{a}$ is a homeomorphism. Since the "pull-back" by $\hat{a}$ is an isometric isomorphism from $C(\mathrm{Sp}_{\mathcal{A}}(a)), \mathbb{C}$ to $C^*(a)$, the required isometric isomorphism is $\Phi : f \mapsto f(a) := \mathfrak{G}^{-1}_{C^*(a)}(f \circ \hat{a})$.

C*-algebras 39.

Theorem (continuous spectral mapping theorem)

If $a \in \mathcal{A}$ is a normal element if a unital C^*-algebra, and $f \in C(\mathrm{Sp}(a); \mathbb{C})$ is a continuous function on $\mathrm{Sp}(a)$, we have $\mathrm{Sp}(f(a)) = f(\mathrm{Sp}(a))$.

Proof.

$f(a) - \lambda 1_{\mathcal{A}}$ is invertible if and only if $f - \lambda$ is invertible in $C(\mathrm{Sp}(a); \mathbb{C})$. This is true if and only if $\lambda \notin f(\mathrm{Sp}(a))$. ∎

C*-algebras 40.

Theorem

In a unital C-algebra $\mathcal{A}$, an element a is Hermitian if and only if it is normal and $\mathrm{Sp}_{\mathcal{A}}(a) \subset \mathbb{R}$.*

Proof.

If a is normal, from functional calculus we have an isometric isomorphism of C*-algebras $C(\mathrm{Sp}(a); \mathbb{C})$ and $C^*(a)$ mapping $\iota : z \mapsto z$ to a.
a is Hermitian if and only if ι is Hermitian if and only if $\mathrm{Sp}(a) = \mathrm{Sp}(\iota) \subset \mathbb{R}$. ∎

C*-algebras 41.

Theorem

In a unital C-algebra $\mathcal{A}$, an element a is unitary if and only if it is normal and $\mathrm{Sp}_{\mathcal{A}}(a) \subset \mathbb{T}$.*
An element is a projection if and only if it is normal and $\mathrm{Sp}(a) = \mathrm{Sp}(\iota) \subset \{0, 1\}$.

Proof.

If a is normal, from functional calculus we have an isometric isomorphism of C*-algebras $C(\mathrm{Sp}(a); \mathbb{C})$ and $C^*(a)$ mapping the identity function $\iota : z \mapsto z$ to a.
a is unitary (projection) if and only if ι is unitary (projection) if and only if if and only if $\mathrm{Sp}(a) = \mathrm{Sp}(\iota) \subset \mathbb{T}$ $(\mathrm{Sp}(a) = \mathrm{Sp}(\iota) \subset \{0, 1\})$. ∎

C*-algebras 42.

Theorem

In a unital C^*-algebra $\mathcal{A}$, an element a is equal to h^2, with h Hermitian, if and only if it is normal and $\mathrm{Sp}_{\mathcal{A}}(a) \subset [0, +\infty[$.

Proof.

If a is normal with $\mathrm{Sp}(a) \subset \mathbb{R}_+$, take $f(z) := \sqrt{z} \in C(\mathrm{Sp}(a); \mathbb{C})$ and by continuous functional calculus get $f(a)$. Since f is Hermitian in $C(\mathrm{Sp}(a); \mathbb{C})$ also $f(a)$ is and since $f^2 = \iota$, $f(a)^2 = a$. If h is Hermitian with $h^2 = a$, by continuous functional calculus for h, $\iota^2(h) = h^2 = a$ and $\mathrm{Sp}_{\mathcal{A}}(a) = \mathrm{Sp}_{C^*(h)}(a) = \mathrm{Sp}(\iota^2) \subset \mathbb{R}_+$. ∎

C*-algebras 43.

In a C*-algebra $\mathcal{A}$, we define the set $\mathcal{A}_+$ of **positive elements** as:

$$\mathcal{A}_+ := \{a \in \mathcal{A} \mid [a, a^*] = 0_\mathcal{A},\ \mathrm{Sp}(a) \subset \mathbb{R}_+\}$$

or equivalently as the set of "squares" of Hermitian elements.

Theorem

The set $\mathcal{A}_h$ of Hermitian elements in a unital C-algebra is a real vector space and the set $\mathcal{A}_+ \subset \mathcal{A}_h$ is a positive convex sharp cone:*

$$a \in \mathcal{A}_+,\ t \in \mathbb{R}_+ \Rightarrow ta \in \mathcal{A}_+,$$
$$a, b \in \mathcal{A}_+ \Rightarrow a + b \in \mathcal{A}_+,$$
$$a \in \mathcal{A}_+, -a \in \mathcal{A}_+ \Rightarrow a = 0_\mathcal{A}.$$

Hence there exists a natural linear partial order relation in $\mathcal{A}_h$ given by $a \leq b \Leftrightarrow b - a \in \mathcal{A}_+$.

C*-algebras 44.

Proof.

If $a \in (\mathcal{A}_+) \cap (-\mathcal{A}_+)$, $\mathrm{Sp}(a) \subset \mathbb{R}_+ \cap (-\mathbb{R}_+) = \{0\}$ hence $r(a) = 0$ so that $\|a\| = 0$ and $a = 0_\mathcal{A}$. This proves that $\mathcal{A}_+$ is sharp.
Note that $h \in \mathcal{A}_+$ if and only if there exists a positive real number $t \in \mathbb{R}_+$ such that $\|t1_\mathcal{A} - h\| \leq t$ because by Gel'fand $\|t - \iota\|_\infty \leq t$ is equivalent to say that for all $\lambda \in \mathrm{Sp}(h)$, $\lambda \in [0, 2t] \subset \mathbb{R}_+$.
Now if $a, b \in \mathcal{A}_+$ we have $\|t - a\| \leq t$ and $\|s - b\| \leq s$ for $t, s \in \mathbb{R}_+$, hence $\|t + s - (a + b)\| \leq \|t - a\| + \|s - b\| \leq t + s$ and so $a + b \in \mathcal{A}_+$. ∎

C*-algebras 45.

Theorem

In a unital C^*-algebra $\mathcal{A}$, we have $\mathcal{A}_+ = \{a^*a \mid a \in \mathcal{A}\}$.

The proof uses several lemmata.

Proposition

In a unital Banach algebra $\mathcal{A}$, $\mathrm{Sp}_{\mathcal{A}}(ab) \cup \{0\} = \mathrm{Sp}_{\mathcal{A}}(ba) \cup \{0\}$ for all $a, b \in \mathcal{A}$.

Proof.

Need to see that for $\lambda \neq 0$, $(\lambda - ab) \in \mathrm{Inv}(\mathcal{A})$ if and only if $(\lambda - ba) \in \mathrm{Inv}(\mathcal{A})$ and it is enough to see $(1_{\mathcal{A}} - ab) \in \mathrm{Inv}(\mathcal{A})$ if and only if $(1_{\mathcal{A}} - ba) \in \mathrm{Inv}\,(\mathcal{A})(\mathcal{A})$. If $x := (1_{\mathcal{A}} - ab)^{-1}$ we take $y := 1_{\mathcal{A}} + bxa$ and we see by direct computation that $y = (1_{\mathcal{A}} - ba)^{-1}$. ∎

C*-algebras 46.

Proposition

In a unital C^*-algebra $\mathcal{A}$, if $-a^*a \in \mathcal{A}_+$ then $a = 0_{\mathcal{A}}$.

Proof.

If $-a^*a \in \mathcal{A}_+$ also $-aa^* \in \mathcal{A}_+$. Hence $-a^*a - aa^* \in \mathcal{A}_+$. Now $-(a^*a + aa^*) = -2(\mathrm{Re}(a)^2 + \mathrm{Im}(a)^2)$ hence $(a^*a + aa^*) \in \mathcal{A}_+$ so that $(a^*a + aa^*) = 0_{\mathcal{A}}$. Hence $\mathrm{Re}(a)^2 = -\mathrm{Im}(a)^2$ hence $\mathrm{Re}(a)^2 = 0_{\mathcal{A}}$ so $\mathrm{Re}(a) = 0$ and $\mathrm{Im}(a) = 0$ and finally $a = 0_{\mathcal{A}}$. ∎

C*-algebras 47.

Proposition

In a unital C-algebra $\mathcal{A}$, every Hermitian element $a \in \mathcal{A}$ can be written as $a = a_+ - a_-$ with $a_+, a_- \in \mathcal{A}_+$ and such that $a_+ a_- = 0_{\mathcal{A}}$. Such decomposition is unique.*

Proof.

Consider $f_+(z) := \max(0, z)$ and $f_-(z) := \max(0, -z)$. We have $f_+ - f_- = \iota$ and $f_+ f_- = 0$. By continuous functional calculus for a, we get $a_+ := f_+(a)$, $a_- := f_-(a)$ and this satisfy the requirements. If x_+, x_- is another decomposition, the unital C*-algebra generated by x_+, x_- is commutative and contains a hence also a_+, a_-. The unicity follows by Gel'fand theorem, since we can see unicity for the pairs of functions on the spectrum of $C^*(x_+, x_-)$. ∎

C*-algebras 48.

Proposition

In a unital C-algebra $\mathcal{A}$, if $a = x^*x$, $a \in \mathcal{A}_+$.*

Proof.

$a = a_+ - a_-$ and $a_- a a_- = -(a_-)^3$ that by functional calculus is negative.

Now $a_- a a_- = a_- x^* x a_- = (xa_-)^*(xa_-)$ is negative hence zero and so $a_-^3 = 0$ and $a_- = 0$. Hence $a = a_+ \in \mathcal{A}_+$. ∎

If $a \in \mathcal{A}_+$, taking $x := \sqrt{a} \in \mathcal{A}_+$, we see that $a = x^*x$ and this completes the proof of the theorem.

C*-algebras 49.

A **state** on a unital C*-algebra $\mathcal{A}$ is a function $\omega : \mathcal{A} \to \mathbb{C}$ such that:

$$\omega(a+b) = \omega(a) + \omega(b), \quad \forall a, b \in \mathcal{A},$$
$$\omega(\lambda \cdot a) = \lambda \cdot \omega(a), \quad \forall \lambda \in \mathbb{C}, \ \forall a \in \mathcal{A},$$
$$\omega(a^* a) \geq 0 \quad \forall a \in \mathcal{A},$$
$$\omega(1_{\mathcal{A}}) = 1_{\mathbb{C}}.$$

It is possible to see that ω is always continuous and $\|\omega\| = 1$

C*-algebras 50.

Proposition

If $\omega : \mathcal{A} \to \mathbb{C}$ is a linear functional on a unital C-algebra $\mathcal{A}$, ω is a state if and only if ω is continuous with $\|\omega\| = 1 = \omega(1_{\mathcal{A}})$.*

Proof.

If ω is a state, the sesquilinear form $(x, y) \mapsto \omega(x^* y)$ is positive and by Schwarz inequality $|\omega(x^* y)|^2 \leq \omega(x^* x)\omega(y^* y)$.
Since $\|y\|^2 \cdot 1_{\mathcal{A}} - y^* y$ is positive, $\omega(y^* y) \leq \|y\|^2 \omega(1_{\mathcal{A}})$ hence $|\omega(y)|^2 \leq \omega(1_{\mathcal{A}})\|y\|^2$ hence $\|\omega\| \leq 1$ and since $\omega(1_{\mathcal{A}}) = 1$ we have $\|\omega\| = 1$. ∎

C*-algebras 51.

A **representation** of a unital C*-Algebra on a Hilbert space $\mathcal{H}$ is simply a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ from the unital involutive algebra $\mathcal{A}$ with values in the unital involutive algebra $\mathcal{B}(\mathcal{H})$.

Two representations $\pi_1 : \mathcal{A} \to \mathcal{B}(\mathcal{H}_1)$ and $\pi_2 : \mathcal{A} \to \mathcal{B}(\mathcal{H}_2)$ of an involutive algebra $\mathcal{A}$ are said to be **equivalent** if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that:

$$U\pi_1(a)U^* = \pi_2(a), \quad \forall a \in \mathcal{A}.$$

A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is called **cyclic** if there exists a **cyclic vector for** π i.e. a vector $\xi \in \mathcal{H}$ such that $\{\pi(a)\xi \mid a \in \mathcal{A}\}$ is dense in $\mathcal{H}$.

C*-algebras 52.

A representation of a C*-algebra is always continuous because it is possible to prove that a unital $*$-homomorphism between unital C*-algebras is always continuous.

Proposition

If $\phi : \mathcal{A} \to \mathcal{B}$ is a unital $*$-homomorphism of unital C-algebras, $\|\phi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$.*

C*-algebras 53.

Proof.

$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ maps $\mathrm{Inv}\,\mathcal{A}$ into $\mathrm{Inv}\,\mathcal{B}(\mathcal{H})$ hence $\mathrm{Sp}_{\mathcal{B}(\mathcal{H})}(\pi(a)) \subset \mathrm{Sp}_{\mathcal{A}}(a)$ hence $\rho(\pi(a)) \leq \rho(a)$.
Since π maps Hermitian elements h into Hermitian elements $\pi(h)$, and the norm of Hermitian elements coincides with the spectral radius, we get $\|\pi(h)\| = \rho(\pi(h)) \leq \rho(h) = \|h\|$.
Finally $\|\pi(a)\|^2 = \|\pi(a)^*\pi(a)\| = \|\pi(a^*a)\| \leq \|a^*a\| = \|a\|^2$. ∎

C*-algebras 54.

There is a very important theorem of Gel'fand and Naĭmark that says that all the C*-algebras are (isomorphic to) some (norm) closed subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. In this way, the study of C*-algebras is essentially the study of the closed $*$-subalgebras of the algebras $\mathcal{B}(\mathcal{H})$.
The proof of this result is based on the following fundamental construction, called Gel'fand-Naĭmark-Segal representation, that allows to associate to every state ω of a C*-algebra a representation of the algebra as an algebra of operators on a Hilbert space.

C*-algebras 55.

Theorem

Given a state ω on a C^*-algebra $\mathcal{A}$, it is possible to construct a representation π_ω of the algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$, such that $\omega(A) = (\Omega_\omega | \pi_\omega(A)\Omega_\omega)$, for all $A \in \mathcal{A}$, where Ω_ω is a norm one cyclic vector in $\mathcal{H}_\omega$. Any two such representations are equivalent via a unique unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U(\xi_1) = \xi_2$.

Proof.

Define a sesquilinear form on $\mathcal{A}$ by $B_\omega(a, b) := \omega(a^* b)$. Note (by Schwarz inequality for B_ω) that the set $N_\omega := \{a \in \mathcal{A} \mid \omega(a^* a) = 0\}$ is a left ideal in $\mathcal{A}$. Consider $\mathcal{A}/N_\omega$, with the "quotient inner product", and define a representation of $\mathcal{A}$ on it by $L_a([b]) := [ab]$, for $x \in \mathcal{A}$ that are continuous linear maps.

Completing $\mathcal{A}/N_\omega$ gives our Hilbert space and continous linear extension provides the representation. The cyclic vector is $[1_\mathcal{A}]$.

C*-algebras 56.

Proposition

If $h \in \mathcal{A}$ is a Hermitian element in the unital C^*-algebra $\mathcal{A}$, there exists a cyclic representation $\pi_h : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\|\pi_h(h)\| = \|h\|$.

Proof.

Consider $C^*(h)$ the unital C*-algebra generated by h. Since $C^*(h)$ is commutative, via Gel'fand transform, it is isomorphic to $C(\mathrm{Sp}_{\mathcal{A}}(h))$. The modulus of the Gel'fand transform of h attains the value $\|h\| = \|\hat{h}\|_\infty$ in at least a point $\omega_o \in \mathrm{Sp}\, C^*(h)$ that is a state on $C^*(h)$ hence a linear continuous map with $\omega_o(1_{\mathcal{A}}) = 1 = \|\omega_o\|$. By Hanh-Banach there exists a linear continuous extension $\omega : \mathcal{A} \to \mathbb{C}$ with $\|\omega\| = \|\omega_o\|$. Hence ω is a state with $|\omega(h)| = |\omega_o(h)| = \|h\|$. The GNS-representation π_ω satisfies.

$$\|h\| = |\omega(h)| = |(\Omega_\omega \mid \pi_\omega(h)\Omega_\omega)| \leq \|\pi_\omega(h)\|$$

and form the contractivity of π_ω it follows $\|\pi_\omega(h)\| = \|h\|$. ∎

C^*-algebras 57.

Theorem (Gel'fand- Naĭmark)

If $\mathcal{A}$ is a unital C^*-algebra, there exists a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ that is isometric.
Hence $\mathcal{A}$ is isomorphic $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$.

Proof.

For any element $a \in \mathcal{A}$, pick a cyclic representation $\pi_a : \mathcal{A} \to \mathcal{B}(\mathcal{H}_a)$ such that $\|\pi_a(a^*a)\| = \|a^*a\|$ and note that $\|\pi_a(a)\| = \|a\|$ and construct the direct sum $\oplus_{a\in\mathcal{A}}\pi_a : \mathcal{A} \to \mathcal{B}(\oplus_{a\in\mathcal{A}}\mathcal{H}_a)$. Note that for all $a \in \mathcal{A}$ $\| \oplus_{a\in\mathcal{A}} \pi_a(a)\| = \|a\|$ hence $\oplus_{a\in\mathcal{A}}\pi_a$ is isometric. ∎

C^*-algebras 58.

On the algebra $\mathcal{B}(\mathcal{H})$ it is possible to introduce several different useful topologies for example:

- **Norm Topology** In this topology a sequence of elements $A_n \in \mathcal{B}(\mathcal{H})$ converges to an element $A \in \mathcal{B}(\mathcal{H})$ if and only if $A_n \xrightarrow{n\to\infty} A$ i.e. if and only if $\|A_n - A\| \xrightarrow{n\to\infty} 0$.
- **Strong Operator Topology** In this topology a sequence of operators $A_n \in \mathcal{B}(\mathcal{H})$ converges to $A \in \mathcal{B}(\mathcal{H})$ if and only if for all the vectors $\xi \in \mathcal{H}$, $A_n\xi \xrightarrow{n\to\infty} A\xi$.
- **Weak Operator Topology** In this topology a sequence of operators $A_n \in \mathcal{B}(\mathcal{H})$ converges to $A \in \mathcal{B}(\mathcal{H})$ if and only if for all the pairs of vectors $\xi, \eta \in \mathcal{H}$, $(\eta|A_n\xi) \xrightarrow{n\to\infty} (\eta|A\xi)$.

C*-algebras 59.

If $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$, we can define the **commutant** $\mathcal{S}'$ of $\mathcal{S}$ as:

$$\mathcal{S}' := \{L \in \mathcal{B}(\mathcal{H}) \mid L \cdot T = T \cdot L, \ \forall T \in \mathcal{S}\}.$$

$\mathcal{S}'$ is the set of linear continuous functions on $\mathcal{H}$ that commute with all the functions in $\mathcal{S}$. We can also consider $\mathcal{S}''$, the **bicommutant of** $\mathcal{S}$ and so on $\mathcal{S}'''$, $\dots$, but since $\mathcal{S} \subset \mathcal{S}''$ and $\mathcal{S}' = \mathcal{S}'''$, we have actually that $\mathcal{S}'$ and $\mathcal{S}''$ are the only two "commutant sets".

It is easy to see that $\mathcal{S}'$ is always a unital subalgebra. Furthermore $\mathcal{S}'$ is weakly closed. If $\mathcal{S} = \mathcal{S}^*$ then $\mathcal{S}'$ is a $*$-algebra (hence a C*-subalgebra of $\mathcal{B}(\mathcal{H})$) because $(\mathcal{S}^*)' = (\mathcal{S}')^*$.

A **Von Neumann algebra** $\mathcal{R}$ is by definition a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{R} = \mathcal{R}''$. A Von Neumann algebra is always a unital C*-algebra.

C*-algebras 60.

Proposition

Let $\mathcal{K}$ be a closed subspace of $\mathcal{H}$ and $P_{\mathcal{K}}$ the ortogonal projection onto $\mathcal{K}$. $\mathcal{K}$ is $\mathcal{S}$-invariant i.e. $T(\mathcal{K}) \subset \mathcal{K}, \quad \forall T \in \mathcal{S}$ if and only if $TP_{\mathcal{K}} = P_{\mathcal{K}} T p_{\mathcal{K}}$ for all $T \in \mathcal{S}$ and,in the case $\mathcal{S} = \mathcal{S}*$, if only if $P_{\mathcal{K}} \in \mathcal{S}'$.

C*-algebras 61.

Theorem (Von Neumann Density Theorem)

If $\mathcal{R}$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, then $\mathcal{R}''$ is the the closure of $\mathcal{R}$ in the strong operator topology.

Proof.

Since $\mathcal{R}''$ is weakly hence strongly closed, we need to see that $\mathcal{R}$ is strongly dense in $\mathcal{R}''$ i.e. $\forall y \in \mathcal{R}''$, $\forall \epsilon > 0$, $\forall n \in \mathbb{N}_0$, $\forall \xi_1, \dots, \xi_n \in \mathcal{H}$ there exists $x \in \mathcal{R}$ such that $\|y\xi_j - x\xi_j\| \leq \epsilon$ for all $j = 1, \dots, n$.
The case $n = 1$. Consider the closure of the subspace $\mathcal{R}\xi$, clearly invariant under $\mathcal{R}$. The ortogonal projection p onto $\mathcal{R}\xi$ is in $\mathcal{R}'$ hence if $y \in \mathcal{R}''$, $yp = py$ so $y\xi = y(1_\mathcal{R}\xi) = yp\xi = p(y\xi) \in (\mathcal{R}\xi)^-$ and so for all $\epsilon > 0$, there exists $x \in \mathfrak{R}$ such that $\|y\xi - x\xi\| \leq \epsilon$.
The general case is reduced to $n = 1$ considering the "n-amplification" $\mathcal{H}^{(n)} := \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ of $\mathcal{H}$ and the "n-amplification" $\mathcal{R}^{(n)} := \mathcal{R} \oplus \cdots \oplus \mathcal{R}$ of $\mathcal{R}$ and noting that $(\mathcal{R}^{(n)})'' = (\mathcal{R}'')^{(n)}$, because $(\mathcal{R}^{(n)})' = \{[a_{ij}] \mid a_{ij} \in \mathcal{R}', \ \forall i, j\}$.

C*-algebras 62.

A very famous theorem of Von Neumann states that a unital involutive subalgebra of $\mathcal{B}(\mathcal{H})$ is a Von Neumann algebra if and only if it is closed in the weak (or in the strong) topology.

Theorem (Von Neumann Double Commutant Theorem)
If $\mathcal{R}$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, the following are equivalent:

- *$\mathcal{R}$ is closed in the weak operator topology,*
- *$\mathcal{R}$ is closed in the strong operator topology,*
- *$\mathcal{R} = \mathcal{R}''$, i.e. $\mathcal{R}$ is a Von Neumann algebra.*

Since a weakly closed set is always strongly closed and a strongly closed set is always a norm closed one, we see that a Von Neumann algebra is a special case of C*-algebra that is not only norm closed, but also weakly (or equivalently strongly) closed.

Von Neumann Algebras 1.

If C*-algebras are the study of topology (locally compact and Hausdorff) what is the algebraic analog of measure theory and probability?
Von Neumann algebras are the setting for the study of "non-commutative measure theory" and "non-commutative probability" as can be seen from the following theorem of Von Neumann:

Theorem

A commutative Von Neumann algebra is always isomorphic to the algebra $L^\infty(\Omega, \mu)$ of complex valued essentially bounded measurable functions on a compact Hausdorff space Ω with a positive measure μ on it.

Von Neumann Algebras 2.

Of course in the non-commutative setting some "unexpected" completely new phenomena can appear: as an example, in the contest of non-commutative measure theory, we have the following astonishing theorem of Tomita-Takesaky:

Theorem

To every normal faithful state on a Von Neumann algebra $\mathcal{R}$ is associated a one parameter group of automorphism $t \mapsto \sigma_t^\omega$ of the algebra $\mathcal{R}$, called the modular automorphism group of ω.

Quantum Vector Bundles = Hilbert-C*-modules.

Bundles 1.

A **bundle** is by definition a surjective map $\pi : E \to X$ from a set E called the **total space** and a set X called the **base space**. For $x \in X$, the set $E_x := \pi^{-1}(x)$ is called the **fiber** at the point x.
A **section** of a bundle is a map $\sigma : X \to E$ such that $\pi \circ \sigma = \mathrm{Id}_X$. In other words, a section associates to every point of the base set a point of E taken from the fiber over x. The set of section of the bundle is denoted by $\Gamma(X; E)$. It also possible to say that sections are a generalization of functions on X taking the value, at the point x, in the fiber over x.

Bundles 2.

A **bundle morphism** between to bundles E_1 and E_2 is a pair of functions $(f, \mathcal{F})$, where $f : X_1 \to X_2$ is a function between the bases sets and $\mathcal{F} : E_1 \to E_2$ is a function between the total sets, such that the following diagram is commutative

$$\begin{tikzcd} E_1 \arrow[r, "\mathcal{F}"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "f"'] & X_2, \end{tikzcd}$$

i.e. $\pi_2 \circ \mathcal{F} = f \circ \pi_1$. In other words, a bundle morphism sends the fiber over x into the fiber over $f(x)$.
Bundles morphisms constitute a category.

Bundles 3.

In a topological bundle (E, π, X), it is supposed that both E and X are topological spaces and that π is a continuous function.
A (topological) **vector bundle** is a bundle (E, π, X) such that all the fibers are (topological) vector spaces (topo-)isomorphic to a given (topological) (complex) vector space F called the **typical fiber** of the vector bundle.
A morphism $(E_1, \pi_1, X_1) \xrightarrow{(f,\mathcal{F})} (E_2, \pi_2, X_2)$ between (topological) vector bundles must be given, by definition, by a (continuous) map $f : X_1 \to X_2$ and a (continuous) map $\mathcal{F} : E_1 \to E_2$ that is linear (continuous) when "restricted to the fibers". In particular, for every $x \in X$, the function $\mathcal{F}_x : E_x \to E_{f(x)}$, defined by $\mathcal{F}_x(e) := \mathcal{F}(e)$, $e \in E_x$, is (continuous) linear.

Bundles 4.

A topological vector bundle is said to be **trivial** if it is isomorphic to the topological vector bundle $(X \times F, \pi, X)$, where $\pi : X \times F \to X$ is the projection onto the fist component of the Cartesian product of topological spaces.
A topological vector bundle is said **locally trivial** if for every point $x \in X$ of the base space, it is possible to find a neighborhood U of x such that $\pi^{-1}(U)$ is isomorphic to the vector bundle $U \times F$.
This requirement is equivalent to the existence for every point $x \in X$ of a continuous section $\sigma \in \Gamma(X; E)$ such that $\sigma(x) \neq 0_{E_x}$.

Hilbert-C*-modules 1.

By definition a **right $\mathcal{A}$-module $\mathcal{E}$ over a ring $\mathcal{A}$** is an Abelian group $(\mathcal{E}, +)$ equipped with an operation:

$$\cdot : \mathcal{E} \times \mathcal{A} \to \mathcal{E}, \quad \cdot : (x, a) \mapsto xa, \quad \text{such that:}$$
$$x \cdot (a + b) = (x \cdot a) + (x \cdot b), \quad \forall x \in \mathcal{E}, \ \forall a, b \in \mathcal{A},$$
$$(x + y) \cdot a = (x \cdot a) + (y \cdot a), \quad \forall x, y \in \mathcal{E}, \ \forall a \in \mathcal{A},$$
$$x \cdot (ab) = (x \cdot a) \cdot b, \quad \forall x \in \mathcal{E}, \ \forall a, b \in \mathcal{A}.$$

If $\mathcal{A}$ is a unital ring and

$$x \cdot 1_{\mathcal{A}} = x, \quad \forall x \in \mathcal{E},$$

we will say that $\mathcal{E}$ is a **unital right $\mathcal{A}$-module**.

Hilbert-C*-modules 2.

The definition of left $\mathcal{A}$-module is given in the same way using an operation $\cdot : \mathcal{A} \times \mathcal{E} \to \mathcal{E}$.
An $\mathcal{A}$-$\mathcal{B}$-**bimodule** is at the same time a right $\mathcal{B}$-module and left $\mathcal{A}$-module over the rings $\mathcal{A}$ and $\mathcal{B}$ with the supplementary property:

$$(a \cdot x) \cdot b = a \cdot (x \cdot b), \quad \forall a \in \mathcal{A}, \ \forall b \in \mathcal{B}, \ \forall x \in \mathcal{E}.$$

In practice, a module is a generalization of a vector space: it is no more required that $\mathcal{A}$ is a field, but only that $\mathcal{A}$ is a ring, in particular an algebra.[2]

[2] If $\mathcal{E}$ is a unital module over a unital $\mathbb{K}$-algebra $\mathcal{A}$, it is naturally a vector space over $\mathbb{K}$ defining $x \cdot \lambda := x \cdot (\lambda 1_{\mathcal{A}})$ for all $\lambda \in \mathbb{K}$. If $\mathcal{E}$ is not a unital module over the $\mathbb{K}$-algebra $\mathcal{A}$, we assume that it is also a vector space over $\mathbb{K}$ such that $(x \cdot \lambda) \cdot a = (x \cdot a) \cdot \lambda$, for $\lambda \in \mathbb{K}$, $a \in \mathcal{A}$ and $x \in \mathcal{E}$.

Hilbert-C*-modules 3.

As an example of (left) module consider the set $\mathcal{E} := \Gamma^0(X, E)$ of continuous sections of a given vector bundle and the ring (algebra) $\mathcal{A} := C^0(X, \mathbb{C})$ of complex valued continuous functions over X.
The set of sections $\mathcal{E}$ becomes an Abelian group defining the sum of two section σ_1, σ_2 to be: $(\sigma_1 + \sigma_2)(x) := \sigma_1(x) + \sigma_2(x)$ (the sum on the right is the sum in fiber at the point x). Furthermore we can define the multiplication of a section $\sigma \in \mathcal{E}$ by a complex function $\alpha \in \mathcal{A}$ to be: $(\alpha \cdot \sigma)(x) := \alpha(x) \cdot (\sigma(x))$.
With this definitions the set of sections $\mathcal{E}$ of a vector bundle becomes a left module over the algebra $\mathcal{A}$ of continuous functions over the base space X.
Actually in this example $\mathcal{E}$ is a bimodule on $\mathcal{A}$ by defining $\sigma \cdot \alpha := \alpha \cdot \sigma$.

Hilbert-C*-modules 4.

A **morphism** (also called a $\mathcal{A}$-linear function) between two modules over the same ring $\mathcal{A}$ is a function $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ such that:

$$\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2);$$
$$\phi(av) = a\phi(v) \quad \forall a \in \mathcal{A}.$$

An isomorphism is a bijective morphism.

Hilbert-C*-modules 5.

A module $\mathcal{P}$ over a ring is said to be **projective** if, given a surjective morphism $\pi : \mathcal{M} \to \mathcal{N}$ between to $\mathcal{A}$-modules and any morphism $\phi : \mathcal{P} \to \mathcal{N}$, there exists a morphism $\psi : \mathcal{P} \to \mathcal{M}$ such that $\pi \circ \psi = \phi$. This means that any morphism from $\mathcal{P}$ with values in $\mathcal{N}$ can be "lifted" to a morphism from $\mathcal{P}$ to $\mathcal{M}$.
It is possible to show that the module $\Gamma^0(X, E)$ of continuous sections of a locally trivial vector bundle E over a compact Hausdorff topological space X is always a projective module over the algebra $C^0(X, \mathbb{C})$.

Hilbert-C*-modules 6.

Now, exactly as happened in the study of topological spaces, where every locally compact Hausdorff topological space was uniquely associated to a commutative C*-algebra, here we will associate to every locally trivial vector bundle over a Hausdorff compact topological space a projective module over the previous commutative C*-algebra of continuous complex functions over X. Furthermore every projective module $\mathcal{E}$ over a commutative unital C*-algebra $\mathcal{A}$ arises in this way i.e. can be uniquely identified with a modulus of sections of a locally trivial vector bundle over the spectrum of the C*-algebra $\mathcal{A}$.

Hilbert-C*-modules 7.

This is done thanks to a theorem of Swan-Serre:

Theorem

A projective module on the algebra $C^0(X, \mathbb{C})$ of continuous complex functions on a compact Hausdorff topological space X is always isomorphic to the module of continuous sections of a locally trivial vector bundle E over X. $\square$

In this spirit, instead of studying (locally trivial) vector bundles over X, we can study (projective) modules over the algebra $C^0(X, \mathbb{C})$. When we substitute the commutative algebra $C^0(X, \mathbb{C})$ with a non-commutative one $\mathcal{A}$, we cannot define any more a vector bundle, but we can still talk about projective modules $\mathcal{E}$ over the non-commutative algebra $\mathcal{A}$ and so we will define a (locally trivial) vector bundle over a non-commutative algebra to be a projective module over $\mathcal{A}$.

Hilbert-C^*-modules 8.

Vector bundles' theory is the theory of vector spaces parametrized by the points of a topological space: there is a vector space (the fiber at x) attached to every point of the base space X.
Vector bundles are especially useful in mathematics to study the topology of the base space X: **topological K-Theory** is the branch of algebraic topology that extract information on the topology of the base space X studying the algebraic properties of the vector bundles that can be constructed over X.

Hilbert-C*-modules 9.

Now, in the same way as topology has an algebraic analog in the study of commutative C*-algebras, topological K-Theory, has an analog in the **algebraic K-theory** of C*-Algebras. This is the study of some of the "topological" properties of the "non-commutative" topological space described by the non-commutative algebra $\mathcal{A}$ through the consideration of the "non-commutative" vector bundles, that in our language are the modules over the algebra $\mathcal{A}$. In the non-commutative case, some complications obviously arise: since the algebra $\mathcal{A}$ is no more commutative, in general left and right modules over $\mathcal{A}$ are not the same (they are not bimodules) and we must distinguish between the two cases.

Quantum Differential Forms = Hochschild Homology.

After the study of topology and vector bundles, the next step in the structural exam of the properties of a geometrical space is the study of the "smoothness" or "differentiability" properties of the space. Let us start again to review the basic notions in the case of "ordinary" "commutative" spaces.
[Here vector spaces are supposed to be over the real numbers $\mathbb{R}$].

Differentials 1.

Given a vector space W a function $G : W \to W$ is called a **translation** if there exists a vector $w \in W$ such that:

$$G(h) = w + h, \quad \forall h \in W.$$

A function $G : V \to W$ is said to be an **affine map** if it is a composition of a linear map $L : V \to W$ with a translation of W i.e. if there exists a vector $w \in W$ such that:

$$F(h) = w + L(h).$$

Given two normed spaces V and W, a function $\omega : S \to W$ from a subset $S \subset V$ to W is said to be an **infinitesimal** in the point $x = 0_V$ if ω is continuous in 0_V and $\omega(0_V) = 0_W$ (of course we suppose $0_V \in S$). In practice, an infinitesimal in one point is a function that goes to zero in that point actually reaching the zero value in that point.

Differentials 2.

Given two functions $F : S \to W$ and $G : S \to W'$ (maybe W and W' different normed spaces) we say that F is **a small "o" of** G if:

$$F(h) = \|G(h)\| \cdot \omega(h)$$

where $\omega : S \to W$ is an infinitesimal. In practice F is a small "o" of G whenever measuring the length of the vector $F(h)$ in the unit of length given by $\|G(h)\|$ gives a "measure" that is an infinitesimal. This means that F is "going to zero" at a faster rate than G. Two functions $F, G : S \to W$ are said to be **tangent in** x_0 (where S is supposed to be a neighborhood of x_0) if, the function

$$h \mapsto F(x_0 + h) - G(x_0 + h),$$

is a small "o" of the function $h \mapsto \|h\|$. This means that the difference between the two functions is going to zero faster that the identity function $h \mapsto h$.

Differentials 3.

A function $F : S \to W$ is said to be **differentiable in** x_0 (S is supposed to be a neighborhood of x_0) if the "graph" of the function F "admits a tangent plane" at the point x_0 i.e. if the function F is tangent to at least a continuous affine map $G : V \to W$. This means that there exists a linear continuous map $L : V \to W$ and a vector $w \in W$ such that:

$$F(x_0 + h) = w + L(h) + \|h\|\omega(h).$$

It is possible to show that if F is differentiable in x_0 then necessarily $w = F(x_0)$ and the linear continuous function $L : V \to W$ is unique. This unique linear continuous function is called the **differential of** F **in** x_0 and is denoted DF_{x_0}.

Differentials 4.

Hence, F is differentiable in x_0 if and only if there is a continuous linear function $DF_{x_0} : V \to W$ and an infinitesimal ω such that for all $h \in S - x_0$:

$$F(x_0 + h) = F(x_0) + DF_{x_0}(h) + \|h\|\omega(h).$$

A function $F : S \to W$ is said to be C^1 in one point x_0 if is differentiable on a neghborhood U of x_0 and the "differential" $DF : U \to \mathcal{B}(V, W)$ is continuous in x_0 (remember that the space $\mathcal{B}(V, W)$ of linear continuous maps from V to W is a normed space too).

If the "differential function" $DF : U \to \mathcal{B}(V, W)$ defined by $x \mapsto DF_x$ is differentiable in x_0, we can go on to define the second differential $D^2F_{x_0}$ that is the differential of the differential function and so on . . .

Differentials 5.

In this way it is possible to define "smooth" functions, called C^∞ **functions**, in one point as those funtions that admit arbitrary "high" differentials on a (maybe shrinking) family of neighborhoods of the point.

The set of functions $C_o^\infty(V, \mathbb{R})$ of real valued "smooth" functions (going to zero at infinity) on the finite dimensional normed space V is a subalgebra of the algebra $C_o^0(V, \mathbb{R})$ of the C*-algebra of continuous functions (going to zero at infinity), but it is not a C*-algebra because it is not closed in norm. It is anyway possible to see that it is a dense subalgebra of $C_o^0(V, \mathbb{R})$.

Hence we can associate to a finite dimensional normed space a so called **differential algebra** i.e. a pair $(\mathcal{A}, \mathcal{A}^\infty)$ where $\mathcal{A}$ is a real C*-algebra and $\mathcal{A}^\infty$ is a dense subalgebra of $\mathcal{A}$ taking the role of the algebra of C_o^∞ functions.

Manifolds 1.

Let M be a set. A **chart** on M is a bijective function $\phi : U_\phi \to V$ from a subset $U_\phi \subset M$ with values in an open set $\phi(U_\phi)$ in a fixed normed space V. A chart is simply a vay to attach coordinates to the points of the set U_ϕ.

Given two charts $\phi_1 : U_{\phi_1} \to V$ and $\phi_2 : U_{\phi_2} \to V$, we can consider the "transition" function

$\phi_{1,2} : \phi_1(U_{\phi_1} \cap U_{\phi_2}) \to \phi_2(U_{\phi_1} \cap U_{\phi_2})$ defined by:

$$\phi_{1,2}(x) := \phi_2 \circ \phi_1^{-1}(x), \quad \forall x \in U_{\phi_1} \cap U_{\phi_2}$$

and in the same way the function:

$\phi_{2,1} : \phi_2(U_{\phi_1} \cap U_{\phi_2}) \to \phi_1(U_{\phi_1} \cap U_{\phi_2})$ defined by:

$$\phi_{2,1}(x) := \phi_1 \circ \phi_2^{-1}(x), \quad \forall x \in U_{\phi_1} \cap U_{\phi_2}$$

(actually $\phi_{2,1} = \phi_{1,2}^{-1}$). These functions are essentially "changes of coordinates" for the same points in $U_{\phi_1} \cap U_{\phi_2}$.

Manifolds 2.

Two carts $\phi_1 : U_{\phi_1} \to V$ and $\phi_2 : U_{\phi_2} \to V$ are said to be C^∞**-compatible** if the "change of coordinates" $\phi_{1,2}$ and $\phi_{2,1}$ are C^∞ functions.

A **differentiable atlas** on a set M is a family Φ of bijective maps $\phi : U_\phi \to V$ called **charts of** M defined on subsets U_ϕ of M with values in the normed space V, and satisfying the following conditions:

- $\bigcup_{\phi \in \Phi} U_\phi = M$ (i.e. the domains of the charts are a covering of M);
- Any two charts in the family Φ are compatible;

A **differential structure** on a set M is a differentiable atlas that is "maximal" in the sense that if a chart ϕ is compatible with all the charts in Φ, then $\phi \in \Phi$.

Manifolds 3.

Once we have been given a differential structure on a set, we can start to develop a differential calculus on this set as if it were a normed space, the only difference is that we have to work locally.
A **differential manifold** (M, Φ) is a set equipped with a differential structure.
A function $f : M \to \mathbb{R}$ is said to be a C^∞ **function** if for every chart $\phi \in \Phi$ we have: $f \circ \phi^{-1} : \phi(U_\phi) \to \mathbb{R}$ is a C^∞ function from the open set $\phi(U_\phi) \in V$ to the normed space V. [Notice that to define the meaning of "smooth" functions on a differential manifold we have to rely on the already developed notion of differentiability in normed vector spaces!].

Manifolds 4.

It is possible to show that a differentiable manifold (modelled on a finite dimensional normed space) is always a locally compact topological space. In fact there is one and only one topology on a manifold that is the smallest topology making all the domains of the charts open sets in M and with this topology M is a locally compact space.

Hence, we can define the C*-algebra $C^0_o(M, \mathbb{R})$ of continuous functions (vanishing at infinity) on M and if we furthermore consider its dense subalgebra $C^\infty_o(M, \mathbb{R})$, we get a structure of differential algebra associated to every (finite dimensional) manifold.

Manifolds 5.

All the (finite dimensional) manifolds can be associated to a differential algebra $(C^0_o(M,\mathbb{R}), C^\infty_o(M,\mathbb{R}))$. Is it true that any commutative differential algebra $(\mathcal{A},\mathcal{A}^\infty)$ is a differential algebra arising from a differential manifold? Up till now I do not know of the existence of a definitive answer to the problem of finding the conditions under which this statement is valid. Anyway the notion of "smooth" differential algebra has already been used many times in the literature as a possible convenient generalization of the notions of differentiable structure also for the noncommutative case.

Vector Fields 1.

If we are given a manifold and we fix a point $m \in M$ of the manifold, it is possible to construct in a completely intrinsic way a vector space $T_m(M)$ of "tangent" vectors at the point m. To do so, we first give an algebraic characterization of vectors in a finite dimensional vector space, and then we use this algebraic formulation to define tangent vectors to a point of a manifold.

Let $x_0 \in V$ be a fixed point in a finite dimensional normed space V. Given a vector $v \in V$, we can consider the function $v_{x_0} : C^\infty(V, \mathbb{R}) \to \mathbb{R}$, defined by:

$$f \mapsto Df_{x_0}(v), \quad \forall f \in C^\infty(V, \mathbb{R}).$$

This function, takes a smooth function f on V and associates to it a real number $v_{x_0}(f)$ that is the differential of f in x_0 calculated along the vector v i.e. it is the "directional derivative" of f in the direction of v.

Vector Fields 2.

It is possible to show that the function v_{x_0} has the following properties:

$$v_{x_0}(f+g) = v_{x_0}(f) + v_{x_0}(g);$$
$$v_{x_0}(\lambda f) = \lambda v_{x_0}(f);$$
$$v_{x_0}(f \cdot g) = v_{x_0}(f) \cdot g(x_0) + f(x_0) \cdot v_{x_0}(g).$$

Hence v_{x_0} is a linear map from the algebra of smooth functions on V to the real numbers, that satisfies the Leibnitz rule (it is a directional derivative!).
Well, it is possible to show that, if V is finite dimensional, any linear function with the Leibnitz property is the directional derivative associated to a certain vector $v \in V$.

Vector Fields 3.

This extremely important algebraic characterization of vectors (in the commutative and finite dimensional case) open the way to a very elegant and intrinsic definition of tangent vectors to a point of a manifold. We know that for a vector space, to talk about vectors is the same thing as to talk about linear funtionals with the Leibnitz property. On a manifold we have no idea of what are the tangent vectors at a point $m \in M$ (actually our scope is to define these!) but we know exactly what are the smooth functions on a manifold and so we can immediately define the set of linear functionals with the Leibnitz property in m.

Vector Fields 4.

Hence, we will define the set of tangent vectors to the manifold M in the point m to be the set $T_m(M)$ of all the functions $v_m : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ such that:

$$v_m(f+g) = v_m(f) + v_m(g);$$
$$v_m(\lambda f) = \lambda v_m(f);$$
$$v_m(f \cdot g) = v_m(f) \cdot g(m) + f(m) \cdot v_m(g).$$

It is possible to see immediately that $T_m(M)$ is a real vector space with the operations defined by:

$$(v_m + w_m)(f) := v_m(f) + w_m(f);$$
$$(\lambda v_m)(f) := \lambda(v_m(f)).$$

Vector Fields 5.

In this way, to every point m of a manifold we have associated the vector space $T_m(M)$ of the "tangent vectors" in m to the manifold. If we consider now the set

$$T(M) := \bigcup_{m \in M} T_m(M),$$

and the "projection" function $\pi : T(M) \to M$ defined by:

$$\pi(v_m) := m, \quad \forall v_m \in T(M),$$

we see immediately that we have actually defined a vector bundle on the manifold M called the **tangent bundle** of M.

Vector Fields 6.

Vector fields on a manifold are simply functions v that associate to every point m a vector v_m tangent to that point. This means that a vector field is a section of the tangent bundle i.e. a function

$$v : M \to T(M) \quad \text{such that} \quad \pi \circ v = \mathrm{Id}_M.$$

A vector field v is said to be C^∞ or "smooth" (resp. C^r) if, given an arbitrary C^∞ function $f \in C^\infty(M)$ over the manifold M, the new function defined by: $[v(f)](m) := v_m(f)$ is again a "smooth" (resp. C^r) function on M.

The set of smooth (or C^r) vector fields over M is denoted by $\mathcal{X}^\infty(M)$ (respectively by $\mathcal{X}^r(M)$) and is a bimodule over the algebra $C^\infty(M)$ (resp. $C^r(M)$) of smooth (resp. C^r) functions over the manifold if we define:

$$(f \cdot v)(m) := f(m) \cdot v_m \in T_m(M).$$

Vector Fields 7.

Given an algebra $\mathcal{A}$, a **derivation** of the algebra $\mathcal{A}$ is, by definition, a function $v : \mathcal{A} \to \mathcal{A}$ that is linear and that satisfy:

$$v(f \cdot g) = v(f) \cdot g + f \cdot v(g)$$

(where the product is the multiplication in the algebra $\mathcal{A}$).
It is immediate to see that on a manifold, every smooth vector field gives rise to a derivation of the algebra of smooth functions on the manifold M.
It is possible to show that, on a finite dimensional manifold, every smooth vector field arises in this way as a derivation of the algebra of smooth functions on the manifold.

Vector Fields 8.

This means that, if we have at desposition the algebra of "smooth" functions on M, we can define "smooth" vector fields immediately, in a global way as derivations of the algebra $C^\infty(M)$. This elegant method (in several variants) has been actually used in many places as a possible definition of the bimodule of smooth vector fields over a non-commutative algebra: given a smooth algebra $(\mathcal{A}, \mathcal{A}^\infty)$, the non-commutative vector fields should be the elements of the bimodule (over the center $Z(\mathcal{A}^\infty)$ of the algebra $\mathcal{A}^\infty$) $\mathrm{Der}(\mathcal{A}^\infty)$ of derivations of $\mathcal{A}^\infty$. In some cases, this approach has been working, but actually there is not a complete agreement on this definition as the "good" one!

Tensor Fields 1.

There are several ways to define what is a tensor and a tensor product, we will choose the most intuitive one.
Given a real vector space V, its **dual** space V^* is defined to be the set of all the linear functionals on V. This means that V^* is the family of all the linear functions $\omega : V \to \mathbb{R}$.
V^* is of course a real vector space with the operations defined by;

$$(\omega_1 + \omega_2)(v) := \omega_1(v) + \omega_2(v);$$
$$(\lambda \cdot \omega)(v) := \lambda(\omega(v)).$$

The element of V^* are called **covectors**.
In V is a finite dimensional vector space, a well known relult tells us that V is isomorphic to its double dual V^{**} so that we can always limit ourselves to consider only V and V^*.

Tensor Fields 2.

A **tensor** (contravariant of order p and covariant of order q) on the vector space V is simply a multilinear function

$$t : \underbrace{V^* \times \cdots \times V^*}_{p \text{ times}} \times \underbrace{V \times \cdots \times V}_{q \text{ times}} \to \mathbb{R}$$

that "eats" a certain number (q) of vectors and a certain number (p) of covectors to produce a real number. The set of p contravariant and q covariant tensors on V is denoted by $\mathcal{T}^p_q(V)$. As an example, if $t \in \mathcal{T}^1_2(V)$ then given a covector $\omega \in V^*$ and two vectors $v_1, v_2 \in V$, $t(\omega, v_1, v_2)$ is a real number that depends linearly on each of the entries i.e. for $w \in V$, $\mu \in V^*$, $\lambda \in \mathbb{R}$:

$$t(\omega + \lambda\mu, v_1, v_2) = t(\omega, v_1, v_2) + \lambda t(\mu, v_1, v_2),$$
$$t(\omega, v_1 + \lambda w, v_2) = t(\omega, v_1, v_2) + \lambda t(\omega, w, v_2),$$
$$t(\omega, v_1, v_2 + \lambda w) = t(\omega, v_1, v_2) + \lambda t(\omega, v_1, w),$$

Tensor Fields 3.

It is easy to see that $\mathcal{T}^p_q(V)$ is a vector space with the operation defined by:

$$(t+s)(\omega_1,\dots,\omega_p;v_1,\dots,v_q) := t(\omega_1,\dots,\omega_p;v_1,\dots,v_q)+ \\ +s(\omega_1,\dots,\omega_p;v_1,\dots,v_q); \\ (\lambda\cdot t)(\omega_1,\dots,\omega_p;v_1,\dots,v_q) := \lambda(t(\omega_1,\dots,\omega_p;v_1,\dots,v_q)).$$

Given two tensors $t\in\mathcal{T}^{p_1}_{q_1}$ and $s\in\mathcal{T}^{p_2}_{q_2}$, we can define the tensor product $t\otimes s$ to be the new tensor $t\otimes s\in T^{p_1+p_2}_{q_1+q_2}$ defined by:

$$t\otimes s(\omega_1,\dots,\omega_{p_1+p_2};v_1,\dots,v_{q_1+q_2}) := t(\omega_1,\dots,\omega_{p_1};v_1,\dots,v_{q_1})\cdot \\ \cdot s(\omega_{p_1+1},\dots,\omega_{p_1+p_2};v_{q_1+1},\dots,v_{q_1+q_2}).$$

Tensor Fields 4.

The set of all tensors of all the possible orders on V is denoted by:

$$\mathcal{T}(V) := \bigcup_{p,q\in\mathbb{N}} \mathcal{T}^p_q(V)$$

and becomes an algebra called the **tensor algebra** of V with the operation of tensor product.
Now, if we have a manifold M, at each point m of M we can consider the "tangent" vector space $T_m(M)$ and over it start to construct all the tensor spaces: $\mathcal{T}^p_q(T_m(M))$.
The vector space $\mathcal{T}^0_1(T_m(M))$ is called the **cotangent** space and is indicated by $T^*_m(M)$.

Tensor Fields 5.

We can also construct the tensor bundles:

$$\mathcal{T}_q^p(M) := \bigcup_{m \in M} \mathcal{T}_q^p(T_m(M))$$

Note that here $\mathcal{T}_0^1(M)$ is the tangent bundle already denoted by $T(M)$ and that in an analogous way $\mathcal{T}_1^0(M)$ is called the cotangent bundle and is for short denoted by $T^*(M)$.
In this bundles, every fiber $\mathcal{T}_q^p(M)_m$ is the tensor space $\mathcal{T}_q^p(T_m(M))$ over the tangent space in m.

Tensor Fields 6.

Furthermore we can consider the **tensor algebra** $\mathcal{T}(M)$ of the manifold M to be:

$$\mathcal{T}(M) := \bigcup_{p,q\in\mathbb{N}} \mathcal{T}^p_q(M).$$

In this case, each fiber of the bundle is the tensor algebra $\mathcal{T}(T_m(M))$ over the tangent space $T_m(M)$.
A **tensor field** on a manifold M will be a function $t : M \to \mathcal{T}(M)$ that to every point $m \in M$ associates a tensor $t_m \in \mathcal{T}(T_m(M))$ over the tangent space in m. Hence a tensor field is a section of the tensor algebra bundle of the manifold.
Covector fields (i.e. covariant tensor fields of order 1) are denoted by $\mathcal{X}^*(M)$.

Tensor Fields 7.

A covector field $\omega \in \mathcal{X}^*(M)$ is said to be "smooth" if for all "smooth" vector fields $v \in \mathcal{X}^\infty(M)$ the function given by:

$$[\omega(v)](m) := \omega_m(v_m),$$

is a smooth function on M.

A tensor field t of order p, q is said to be "smooth" if given arbitrary p smooth vector fields and arbitrary q smooth covector fields, the function defined by:

$$t(\omega_1, \dots, \omega_p; v_1, \dots, v_q)(m) := t_m(\omega_{1\,m}, \dots, \omega_{p\,m}; v_{1\,m}, \dots, v_{q\,m}),$$

is a smooth function on M.

Tensor Fields 8.

The set of continuous (resp. smooth) tensor fields or order p, q over the manifold M is denoted by $\Gamma^0(M, \mathcal{T}^p_q(M))$ (resp. $\Gamma^\infty(M, \mathcal{T}^p_q(M))$) and the set of all the continuous (resp. smooth) tensor fields by $\Gamma^0(M, \mathcal{T}(M))$ (resp. $\Gamma^\infty(M, \mathcal{T}(M))$).
From the previous definitions we have $\mathcal{X}(M) = \Gamma(M, \mathcal{T}^1_0(M))$ and $\mathcal{X}^*(M) = \Gamma(M, \mathcal{T}^0_1(M))$ as the sets of all vector fields and covector fields.
Exactly as in the case of the module of vector fields, it is possible to see that $\Gamma^0(M, \mathcal{T}^p_q(M))$ and $\Gamma^0(M, \mathcal{T}(M))$ are symmetric bimodules over the algebra $C^0(M)$. In the same way $\Gamma^\infty(M, \mathcal{T}^p_q(M))$ and $\Gamma^\infty(M, \mathcal{T}(M))$ are symmetric bimodules over the algebra of smooth functions $C^\infty(M)$.

Tensor Fields 9.

Let us denote with $\mathcal{A}$ the algebra $C^0(M)$ (resp. $C^\infty(M)$). It is possible to see that (in the case of finite dimensional manifolds) continuous covector fields (resp. smooth covector fields) are simply given by the module of $\mathcal{A}$-linear functions on the $\mathcal{A}$-module $\mathcal{X}^0(M)$ (resp. $\mathcal{X}^\infty(M)$) with values in $\mathcal{A}$. In the same way the $\mathcal{A}$-module of continuous tensor fields $\Gamma^0(M, \mathcal{T}^p_q(M))$ (resp. smooth tensor fields) is given by the $\mathcal{A}$-module of multi $\mathcal{A}$-linear functions from $\mathcal{X}^*(M) \times \cdots \times \mathcal{X}^*(M) \times \mathcal{X}(M) \times \cdots \times \mathcal{X}(M)$ with values in $\mathcal{A}$.

Tensor Fields 10.

This suggest that, as long as it is possible to individuate a suitable $\mathcal{A}$-module that can work as a module of continuous (smooth) vector fields over the manifold M, the construction of all the other tensor fields amounts at a simply algebraic game: they are the $\mathcal{A}$-modules of multi $\mathcal{A}$-linear funtionals with values in $\mathcal{A}$. The difficult task, at least in the non-commutative case is to find a good $\mathcal{A}$-module to be used as a substitute for $\mathcal{X}^0(M)$.

Anyway, in the struggle to generalize to the non-commutative world the basic notions related to the differential structure of a space, people have been using with some success other algebraic constructs coming from the notion of exterior algebra associated to a given differential manifold.

Differential Forms 1.

If $t \in \mathcal{T}^0_q(V)$ is a covariant tensor of order q on the real vector space V and σ is a permutation of the set of natural numbers $\{1, \dots, q\}$, we define a new tensor σt by:

$$\sigma t(v_1, \dots, v_q) := t(v_{\sigma(1)}, \dots, v_{\sigma(q)}).$$

A q **form** on a vector space V is a covariant tensor of order q, $t \in \mathcal{T}^0_q(V)$ that is antisymmetric i.e.:

$$t(\dots, v, \dots, w, \dots) = -t(\dots, w, \dots, v, \dots)$$

for all the possible exchanges of two vectors.
The space of q forms is denoted by $\Lambda_q(V)$ and the set of all the forms by $\Lambda(V)$.

Differential Forms 2.

If $t \in \mathcal{T}_q^0(V)$, we can "antisymmetrize" it to get a q-form by:

$$A(t) := \frac{1}{q!} \sum_{\sigma} \sigma t,$$

where the sum is over all the possible permutations of $\{1, \dots, q\}$. $\Lambda(V)$ is a vector subspace of $\mathcal{T}(V)$ but it is not an algebra with the tensor product (the tensor product in general is not an antisymmetric tensor). Anyway it is possible to define a new useful product on $\Lambda(V)$ that makes it into an associative algebra called the **exterior algebra** of the vector space V.

Differential Forms 3.

The new product called the **wedge** product is very natural: if $\omega \in \Lambda_p(V)$ and $\eta \in \Lambda_q(V)$, we consider the tensor product $\omega \otimes \eta$ and since this is not an antisymmetric tensor, we "antisymmetrize" it defining:

$$\omega \wedge \eta := \frac{(p+q)!}{p!q!} A(\omega \otimes \eta).$$

If we have a manifold M. we can proceed exactly as we did in the case of tensors: we can define the vector bundle of q-forms on M denoted by $\Lambda_q(M)$ whose fiber in the point m is simply the vector space $\Lambda_q(T_m(M))$. We denote by $\Lambda(M)$ the vector bundle of exterior forms on M.

A **differential form** on M is a section of the bundle $\Lambda(M)$ i.e. it is a tensor field that at each point m of the manifold associates an antisymmetric tensor field on the tangent space $T_m(M)$.

Differential Forms 4.

The set of q-differential forms is denoted by $\Omega_q(M)$ i.e.:

$$\Omega_q(M) = \Gamma(M, \Lambda_q(M)).$$

The set of differential forms is denoted by $\Omega(M)$.
$\Omega(M)$ is an algebra (called the **exterior algebra of the manifold** M) with the operations defined point by point in the usual way:

$$(\omega + \eta)_m := \omega_m + \eta_m,$$
$$(\lambda \cdot \omega)_m := \lambda \cdot (\omega_m),$$
$$(\omega \wedge \eta)_m := \omega_m \wedge \eta_m.$$

A differential q-form ω is said to be smooth if it is smooth as a tensor field i.e. if for arbitrary smooth vector fields $v_1, \dots, v_q$, the function $\omega(v_1, \dots, v_q)$ is a smooth function on M. We denote by $\Omega_q^\infty(M)$ the family of smooth differential q-forms on M.

Differential Forms 5.

The modules $\Omega_q(M)$ and $\Omega_q^\infty(M)$ can be uniquely identified (in the finite dimensional case) as the symmetric bimodules of antisymmetric $\mathcal{A}$-valued (resp. $\mathcal{A}^\infty$-valued) multilinear maps on $\mathcal{X}(M)$ (resp. $\mathcal{X}^\infty(M)$).

The set $\Omega_q(M)$ (resp. $\Omega_q^\infty(M)$) is a symmetric bimodule over the algebra $\mathcal{C}^0(M)$ (resp. $\mathcal{C}^\infty(M)$). The same is true for $\Omega(M)$ (resp. $\Omega^\infty(M)$). We have furthermore the following important properties:

a- $\Omega^\infty(M)$ is a graded algebra [this means that under the wedge product we have $\Omega_p^\infty(M) \wedge \Omega_q^\infty(M) \subset \Omega_{p+q}^\infty(M)$];

b- $C^\infty \subset \Omega_0(M)$ [actually in this case $C^\infty = \Omega_0(M)$].

Exterior Differential 1.

It is possible to show that all the previously constructed tensor bundles $\mathcal{T}^p_q(M)$, $\Lambda_q(M)$ are actually manifolds in a natural way since they can be equipped with a differential structure.
A function $F : M \to N$ between manifolds is said to be smooth if for any possible choice of a smooth function $f : N \to \mathbb{R}$, the new function $F^*(f) : M \to \mathbb{R}$ defined by:

$$[F^*(f)](m) := f(F(m))$$

is a smooth function on M. [Note that in this definition, we do not make direct use of the differential structure on M whose only fundamental use has been to select a family of smooth real valued functions on M].

Exterior Differential 2.

If F is a smooth function between the manifolds M and N, we can define the **differential** (denoted by DF_m) of F in the point $m \in M$. The differential will be a linear continuous function from the tangent space $T_m(M)$ in m to the manifold M, to the tangent space $T_{F(m)}(N)$ in the point $F(m)$ to the manifold N. The actual definition is the following: taken a vector $v_m \in T_m(M)$, the vector $DF_m(v_m)$ is the vector over N (hence it is a linear functional with the Leibnitz property over the algebra of smooth functions over N) defined by:

$$[DF_m(v_m)](g) := v_m(g \circ F)$$

where g is an arbitrary smooth function over the manifold N.

Exterior Differential 3.

In the special case of a smooth real valued function $f \in C^\infty(M)$, the function $m \mapsto Df_m$ is actually a covector field: in fact at every point $m \in M$ is associated a linear map $Df_m \in T^*_m(M)$ hence Df is a smooth differential 1-form on M.
Since $\Omega^\infty_0(M)$ is by definition the set of smooth functions $C^\infty(M)$ on M, the differential D is a map that associate to a 0-form a 1-form. The map is linear: i.e. for all $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$, $D(f+g) = D(f) + D(g)$, and $D(\lambda f) = \lambda D(f)$.

Exterior Differential 4.

It is a well known result that the map $D : \Omega_0^\infty(M) \to \Omega_1^\infty(M)$ can be extended to a map $d : \Omega^\infty(M) \to \Omega^\infty(M)$ called the **exterior differential** and having the following properties:

1- For all $q \in \mathbb{N}$, $d(\Omega_q^\infty(M)) \subset \Omega_{q+1}^\infty(M)$,

2- d is $\mathbb{R}$-linear,

3- $d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-)^p \omega \wedge d(\eta)$, where $\omega \in \Omega_p^\infty(M)$ and $\eta \in \Omega_q^\infty(M)$,

4- $d \circ d = 0$,

5- $d(f) = D(f)$ for all $f \in \Omega_0^\infty(M)$.

Properties $1, 2, 3$ above amount to say that the map d is a **graded derivation** on $\Omega^\infty(M)$.

Exterior Differential 5.

In this way, starting with the algebra $\mathcal{C}^\infty(M)$ of smooth functions on a manifold, we have constructed a bigger algebra $\Omega(M)$ equipped with a differential $d : \Omega \to \Omega$ satisfying the properties $1, 2, 3, 4$.

This amounts to say that we have associated what is called a **differential algebra** $\Omega^\infty(M)$ to the algebra $C^\infty(M)$ i.e. a graded algebra with a graded derivation such that $d^2 = 0$.

To generalize the notion of differential algebra to the non-commutative setting is really very simple: given and algebra $\mathcal{A}$ (that plays the analogous role of the algebra $C^\infty(M)$ above) we look for pairs $(\Omega(\mathcal{A}), d)$ such that $\Omega(\mathcal{A})$ is a graded algebra on $\mathcal{A}$ (i.e. $\mathcal{A}$ is identified with a subalgebra of $\Omega_0(\mathcal{A})$) and d is a graded derivation with $d^2 = 0$.

Homology and Cohomology 1.

If $\mathcal{R}$ is a ring, let denote by E the sequence $E^0, E^1, \dots, E^n, \dots$ of $\mathcal{R}$-modules and let denote by d a sequence $d^n : E^n \longrightarrow E^{n+1}$ of $\mathcal{R}$-linear maps between the previous modules such that $d^n \circ d^{n-1} = 0$.

Such an algebraic structure is called a **cohomological complex**.

The maps d^n are called the (cohomological) differentials of the complex.

Homology and Cohomology 2.

The **cohomology sequence** $H(E,d)$ of the previous complex, is by definition the sequence of $\mathcal{R}$-modules given by the following quotients of $\mathcal{R}$-modules:

$$H^i(E,d) := \frac{Z^i(E,d)}{B^i(E,d)},$$

Where $Z^i(E,d)$ called the set of **cycles** of the complex is by definition:

$$Z^i(E,d) := \ker(d^i),$$

and $B^i(E,d)$, called the set of **boundaries** of the complex, is given by:

$$B^i(E,d) := \operatorname{im}(d^{i-1}), \quad B^0(E,d) := \{0\}.$$

Homology and Cohomology 3.

It is possible to define in an analogous way homology complexes E_i (where the differentials are acting in the reverse direction: $d_i : E_i \to E_{i-1}$) and their homology classes $H_i(E, d)$. Given a cohomology complex, we can always obtain a homology complex considering the dual modules and the "traposed" of the differentials.

Homology and Cohomology 4.

In the case of a differential manifold, we already have in our hands a cohomological complex obtained taking:

$$\mathcal{R} := \mathbb{R} \quad \text{the field of real numbers,}$$
$$E^i := \Omega_i^\infty(M) \quad \text{the set of smooth } i\text{-differential forms,}$$
$$d^i := d|_{E^i} \quad \text{the differential of } i\text{-forms with values in } (i+1)\text{-forms.}$$

This very famous cohomological complex is called **De Rham complex** and the cohomology sequence $H(E,d)$ obtained from it is called the the **De Rham cohomology** and we will denote if by $H_{DR}(M)$. The family of vector spaces H^i_{DR} gives information on the topology of the manifold M (even if for their construction we made use of the differential structure of M).

Cyclic Cohomology 1.

A lot of people have been trying to find a non-commutative version of the De Rham cohomology, but as far as I know, the only successful approach has been given by A. Connes.
If we try to generalize immediately the De Rham complex to the non-commutative case, we would take an algebra $\mathcal{A}$ (as an algebra of smooth functions over a manifold), and we would consider $\Omega(\mathcal{A})$ the universal differential algebra that is our best candidate for the non-commutative analogue of differential forms and we would consider the cohomology complex associated:

$$E^i := \Omega_i(\mathcal{A});$$

$$d^i := d \quad \text{the differential restricted to } E^i \text{ with values in } E^{i+i}.$$

Unfortunately this honest complex has a trivial cohomology i.e. $H^i(E, d) = 0$ if $i > 0$.

Cyclic Cohomology 2.

Instead of this, A. Connes considers a different cohomological complex known as the **Hochschild complex** and shows that the cohomology classes $HH(\mathcal{A})$ of the Hochschild complex can be identified (in the commutative case) with the dual of the vector space of the differential forms on a manifold (the De Rham currents). Then A. Connes introduces a new cohomological complex (called **cyclic complex**) that is simply a subcomplex of the previous Hochschild complex[3] and shows that the information contained in the cohomology $HC(\mathcal{A})$ of this new complex (called **cyclic cohomology**), in the commutative case, allows to reconstruct the De Rham homology spaces (and so, taking the duals, the De Rham cohomology spaces).

[3] In the sense that all the vector spaces of the cyclic complex are subspaces of the Hochschild complex and the differential is obtained by restriction.

Cyclic Cohomology 3.

This is not an isomorphism: every Connes' cyclic homology space is a direct sum a finite number of De Rahm homology spaces, but from the complete knowledge of the cyclic cohomology it is possible to find all the de Rham homology classes.
In the limited space available for this lecture it is not possible to give more details on this topic: for the easiest introduction to cyclic cohomology we suggest to consult M. Khalkhali notes, the two papers by R. Coquereaux and the book by J-L. Loday mentioned in the bibliography.

Quantum Spin Manifolds = Connes' Spectral Triples.

Riemannian Manifolds 1.

A **semi-Riemannian manifold** M is a manifold equipped with a smooth symmetric covariant tensor of order two g, called the metric tensor. The metric tensor g_x at the point x is simply a symmetric bilinear form on the tangent space $T_x(M)$ to the manifold M in the point x. In this way we see that a semi-Riemannian manifold is a manifold having a metric defined on every tangent space, with the only additional condition that the metric is changing in a smooth way from point to point.
We say that we have a **Riemannian manifold** if the metric tensor g is positive definite over every tangent space, this means that, for all points $x \in M$, the scalar product g_x is positive definite i.e.:

$$
\begin{aligned}
g_x(v,v) &\geq 0, \quad \forall v \in T_x(M), \\
g_x(v,v) &= 0 \quad \Rightarrow v = 0.
\end{aligned}
$$

Riemannian Manifolds 2.

Using the metric tensor, we can define the length $g_x(v, v)$ of a tangent vector $v \in T_x(M)$ and the angle between two tangent vectors.

Furthermore, the set $\mathcal{X}^\infty(M)$ of (smooth) vector fields over M becomes immediately equipped with a $\mathcal{A}$-bilinear map with values in the algebra $\mathcal{A}$ of continuous (smooth) functions on M that associates to a pair of vector fields $v, w \in \mathcal{X}^\infty(M)$ a continuous (smooth) function γ on M defined by $\gamma(x) := g_x(v(x), w(x))$.

This kind of algebraic structure is an example of a **pre-Hilbert module** over the algebra $\mathcal{A}$ i.e. a module over the algebra $\mathcal{A}$ equipped with a $\mathcal{A}$-bilinear map with values in $\mathcal{A}$.

Riemannian Manifolds 3.

One possible way to try to generalize the notion of Riemannian manifold to the non-commutative setting is to define a non-commutative Riemann manifold to be a smooth algebra equipped with a suitable pre-Hilbert module (or bimodule) structure. The difficult task is usually to find the appropriate module on the noncommutative algebra.
The approach of Alain Connes to this problem is a bit more subtle.

Clifford Algebras 1.

The first step to understand how to achieve the definition of noncommutative Riemann manifold Connes' way, is the study of Clifford algebras. Let us suppose that we are given a real vector space V equipped with a bilinear symmetric form $g : V \times V \to \mathbb{R}$. From the intuitive point of view, the **Clifford Algebra** $\mathrm{Cl}(V,g)$ of (V,g) is a unital associative algebra over the real numbers that contains the vector space V, is generated by the elements of V (i.e. every element of $\mathrm{Cl}(V,g)$ is a linear combination of products of vectors from V) and in which the product of vectors reproduce the inner product in V i.e.:

$$v \cdot v = -g(v,v) \cdot 1_{\mathcal{A}}.$$

In this way we see that the Clifford algebra of (V,g) contains all the information about the vector space V and its metric g.

Clifford Algebras 2.

A **Clifford module** is a (left) module $\mathcal{S}$ over a Clifford algebra i.e. it is an Abelian group (under an operation of addition) endowed with an operation of multiplication that associates to every element $c \in \mathrm{Cl}(V, g)$ and every element $s \in \mathcal{S}$ a new element of $\mathcal{S}$ denoted by $c \cdot s$.
If we have a semi-Riemannian manifold (M, g), then in any point $x \in M$ of the manifold, we can consider the Clifford algebra $\mathrm{Cl}(T_x(M), g_x)$ of the inner product space $(T_x(M), g_x)$. In this way we obtain a bundle whose fiber at the point $x \in M$ is the Clifford algebra of the tangent space $T_x(M)$ equipped with the inner product g_x. This special bundle is called the **Clifford bundle** of the semi-Riemannian manifold (M, g) and is denoted by $\mathrm{Cl}(M)$.

Clifford Algebras 3.

The set of (smooth) sections $\Gamma^\infty(M, \mathrm{Cl}(M))$ of the Clifford bundle is a unital associative algebra with the pointwise multiplication in the Clifford algebras of tangent spaces. Actually this algebra can be considered as the "Clifford Algebra" of the pre-Hilbert module $(\mathcal{X}^\infty(M), \gamma)$ and we denote it by $\mathrm{Cl}(\mathcal{X}^\infty(M), \gamma)$.

Covariant Exterior Derivatives 1.

Let us suppose that we are given a differentiable manifold M. Let f be a differentiable function (i.e. a tensor of order 0) defined on M, and let $v \in \mathcal{X}^\infty(M)$ be a smooth vector field on M. We already know that we can define the "directional derivative" $\nabla_v(f)$ of f along v in an intrinsic way calculating the differential df on the vector v i.e. $(\nabla_v(f))_m := (df)_m(v_m)$. In the same way, we would like to define the "directional derivative" $\nabla_v(w)$ of a vector field $w \in \mathcal{X}^\infty(M)$ along a vector field $v \in \mathcal{X}^\infty(M)$. As in the previous case, ∇ should be a one form in v for each fixed w and it should satisfy the Leibnitz property typical of derivatives.

Covariant Exterior Derivatives 2.

This means that we require $\forall v, w, z \in \mathcal{X}^\infty(M)$ and $\forall \alpha, \beta \in C^\infty(M)$:

$$\nabla_{\alpha v+\beta w}(z) = \alpha \cdot \nabla_v(z) + \beta \cdot \nabla_w(z);$$
$$\nabla_v(w+z) = \nabla_v(w) + \nabla_v(z);$$
$$\nabla_v(\alpha w) = d\alpha(v) \cdot w + \alpha \cdot \nabla_v(w).$$

Such a function ∇ is called a **Koszul connection** for the tangent bundle $T(M)$ of the manifold. The name connection comes from the fact that, to give a directional derivative of vector fields is equivalent to give rules to "connect" two different tangent spaces i.e. to move vectors along a path (in a "parallel way") between different tangent spaces of the manifold in order to compare them.

Covariant Exterior Derivatives 3.

If we are given a Koszul connection on vector field (i.e. a directional derivative) it is possible to extend it in a unique way to a directional derivative of arbitrary tensor fields provided that ∇_v "commutes" with contractions of tensor fields and satisfies the Leibnitz property. In this way we get a Koszul connection for any tensor bundle over M.

Covariant Exterior Derivatives 3.

Now, given an arbitrary vector bundle E (any tensor bundle for example) over the manifold M, it is always possible to consider the module of "differential forms with values in E". Such a module is denoted by $\Omega^\infty(M,E)$. An **exterior covariant derivative** d^E on E is, by definition, a real linear function with the following properties:

$$d^E(\Omega^\infty_q(M,E) \subset \Omega^\infty_{q+1}(M,E);$$
$$d^E(t_1 \wedge t_2) = (d^E t_1) \wedge t_2 + (-)^p t_1 \wedge (d^E t_2),$$

for all $t_1 \in \Omega^\infty_p(M,E)$ and $t_2 \in \Omega^\infty(M,E)$.

Covariant Exterior Derivatives 4.

A Koszul connection on E is, by definition, a function $\nabla : \mathcal{X}^\infty(M) \times E \to E$ such that for all $v \in \mathcal{X}^\infty(M)$, $t, t_1, t_2 \in \Omega_0^\infty(M, E) = \Gamma^\infty(M, E)$ and $\alpha, \beta \in C^\infty(M)$:

$$\nabla_{\alpha v + \beta w}(t) = \alpha \cdot \nabla_v(t) + \beta \cdot \nabla_w(t);$$
$$\nabla_v(t_1 + t_2) = \nabla_v(t_1) + \nabla_v(t_2);$$
$$\nabla_v(\alpha \cdot t) = d\alpha(v) \cdot t + \alpha \cdot \nabla_v(t).$$

It is a known result that any Koszul connection ∇ on E uniquely determines an exterior covariant derivative d^E such that:

$$(d^E t)(v) = \nabla_v(t), \quad \forall t \in \Omega_0^\infty(M, E), v \in \mathcal{X}^\infty(M).$$

Covariant Exterior Derivatives 5.

We take now: $E := \mathcal{T}^\infty(M)$. An exterior covariant derivative on a semi-Riemannian manifold is called **metric covariant derivative** if it satisfies the following additional property:

$$d^E g = 0.$$

The **torsion** τ of the connection is by definition the two form with values in the tensor fields over M obtained taking the exterior derivative of the following tensor valued one-form $\zeta : \mathcal{X}^\infty(M) \to \Gamma^\infty(M, \mathcal{T}(M))$ defined as: $\zeta(t) := t$ (ζ acts as the identity function, associating to a vector field the same vector field as an element of the tensor field algebra over M). In formulas: $\tau := d^E\zeta$.

Covariant Exterior Derivatives 6.

A fundamental theorem in differential geometry states that, on the $C^\infty(M)$-module $\Gamma^\infty(M, \mathcal{T}(M))$ of a semi-Riemannian manifold, there exists one and only one metric covariant exterior derivative without torsion. This covariant exterior derivative d_{LC} is called the **Levi Civita covariant exterior derivative**.

Furthermore, another theorem states that the Levi Civita covariant derivative can be uniquely extended to the module $\Gamma^\infty(M, \mathrm{Cl}(M))$ of smooth sections of the Clifford bundle of M in such a way that Leibnitz rule is satisfied i.e. there exists only one $d^{\mathrm{Cl}(M)} : \Omega^\infty(M, \mathrm{Cl}(M)) \to \Omega^\infty(M, \mathrm{Cl}(M))$ such that:

$$\forall c_1, c_2 \in \Omega^\infty(M, \mathrm{Cl}(M)),$$

$$d^{\mathrm{Cl}(M)}(c_1 \cdot c_2) = d^{\mathrm{Cl}(M)}(c_1) \cdot c_2 + c_1 \cdot d^{\mathrm{Cl}(M)}(c_2), \quad \text{and}$$

$$d^{\mathrm{Cl}(M)}(c) = d_{LC}(c), \quad \forall c \in \Omega^\infty(M, T(M)) \subset \Omega^\infty(M, \mathrm{Cl}(M)).$$

Covariant Exterior Derivatives 7.

In the same way, another important theorem states that there exists an extension d^S of the exterior covariant derivative, on any Clifford bundle $S(M)$ over M, with the property: $d^S(c \cdot s) = d^S(c) \cdot s + c \cdot d^S(s)$, where $s \in S(M)$ and $c \in \mathrm{Cl}(M)$.
On a semi-Riemannian manifold M, if $S(M)$ is any bundle of Clifford modules, we can define the **Dirac operator** associated to this bundle as the operator $D : \Gamma^\infty(M, S(M)) \to \Gamma^\infty(M, S(M))$ defined (locally in a chart) by:

$$Ds := \sum_k d^{S(M)}_{e_k}(s) \cdot e^k.$$

This amount to say that the Dirac operator is obtained by contracting the covariant derivative via the "Clifford multiplication".

Spectral Triples 1.

We are now ready to pass to the non-commutative side! The fundamental theorem of A. Connes states that if we are given a spin Riemannian manifold, then we can reconstruct all of the geometry of the manifold from the following set of algebraic data:

1. The Hilbert space $L^2(M, S(M))$ of square integrable sections of the spinor bundle;
2. The Dirac operator $D : L^2(M, S(M)) \to L^2(M, S(M))$;
3. The algebra $C^\infty(M)$ of smooth functions on the manifold M.

Spectral Triples 2.

The last step in non-commutative differential geometry, following A. Connes, is the definition of a non-commutative analogue of a spin$_c$, compact, finite dimensional manifold. This should be given by the so called **Connes' spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra of operators on the Hilbert space $\mathcal{H}$ ($\mathcal{A}$ plays the role of the algebra of smooth functions on the manifold, and $\mathcal{H}$ the role of the square integrable sections of the spinor bundle) and a self adjoint operator D on $\mathcal{H}$ (that mimic the Dirac operator) satisfying the following axioms:

- D has compact resolvent.
- $[D, a] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{A}$.
- There exists an antilinear involution $J : \mathcal{H} \to \mathcal{H}$ such that:
  - $[a, JbJ] = 0$ for all $a, b \in \mathcal{A}$.
  - $[[D, a], Jb^* J] = 0$ for all $a, b \in \mathcal{A}$.

Spectral Triples 3.

If we are given the full structure contained in a Connes' spectral triple, many of the difficult construct of non-commutative geometry simplify. For example, Connes has proved that we have an explicit formula to derive the metric space structure of the manifold M using only the Dirac operator:

$$d(x, y) := \sup\{|a(x) - a(y)| \mid a \in \mathcal{A}, \ \|[D, a]\| \leq 1\}.$$

This mean that the distance between two points of the manifold can be extracted from the knowlege of the spectral triple.

Spectral Triples 4.

The differential da of a "function" from $\mathcal{A}$ can be represented on the Hilbert space $\mathcal{H}$ by the commutator with the Dirac operator:

$$da = i[D, a]$$

and it turns out that in the same way all the differential forms (i.e. the elements of the universal differential algebra $\Omega(\mathcal{A})$) can be directly expressed (but only as a $\mathcal{A}$-module not as a differential algebra!) with operators in the Hilbert space $\mathcal{H}$ in the following natural way:

$$a_0 \delta a_1 \cdots \delta a_n \mapsto (-)^n a_0 [D, a_1] \cdots [D, a_n].$$

Spectral Triples 5.

Finally it is possible to introduce a non-commutative theory of integration of differential forms on the manifold M again making use of the Dirac operator alone. In this case the original formulation of A. Connes makes use of the very technical theory of Dixmier traces over algebras of operators, anyway as proved by A. Jaffe and collaborators there exist a very simple physical appealing formula for such an integral given by:

$$\int \omega := \lim_{h \to 0^+} \frac{\mathrm{Tr}(\omega e^{-hD^2})}{\mathrm{Tr}(e^{-hD^2})},$$

where ω is a differential form and "Tr" is the ordinary trace of operators on the Hilbert space $\mathcal{H}$.

Spectral Triples 6.

Many other important things must be said about non-commutative geometry, but the purpose of this notes is to give **only the most elementary ideas** as they are addressed mainly to help those who already have problems with the "commutative" side of the matter!! Hence we leave any further excursion in the non-commutative land to the study of some of the references reported in the guide to the bibliography.

Operator Algebras in Physics.

Since the very beginning of the subject, operator algebras have always been strongly linked with physics. In particular we can say that classical physics is essentially described by commutative algebras and that quantum physics is described by non-commutative algebras, so that the passage from classical to quantum physics is the passage from the study of commutative to more general noncommutative algebras. Let us justify a bit (in a very intuitive way!) the previous statement.

Classical Physics = Commutative Algebras 1.

In physics, we make experiments to get information from a system. We can imagine a physical system as a part of the real world about which we do not know anything at the beginning (if you want, think the system enclosed in a black box). Each experimental apparatus, will give us (when acting on the system) a result that is a real number that we call the measure of the physical quantity operationally defined by the instrument.
The physical system in the box can exist, usually, in many different ways (that we will call pure states of the systems). When I measure a physical quantity (called an observable) I will get different real numbers corresponding to different pure states of the system.

Classical Physics = Commutative Algebras 2.

From a mathematical point of view, an observable is a function from the space of pure states of the system to the real numbers. If $\mathcal{S}$ is the space of states, then the observables are elements of the $*$-algebra $C^0(\mathcal{S};\mathbb{C})$: they are the elements $f \in C^0(\mathcal{S};\mathbb{C})$ such that $f = \overline{f}$ having real values (self adjoint elements). In classical physics a state can also be seen as a function that associate to each observable f a real number $f(x)$. The possible values of an observable are the possible values of f. [α is a possible value for f if and only if $\alpha - f$ is not an invertible function].
The set $\{\alpha \in \mathbb{C} \mid \alpha - f$ is not invertible$\}$ is called the spectrum of f. So the possible values of an observable f are the elements of the spectrum of f in the algebra of observables.

Classical Physics = Commutative Algebras 3.

If I know that the system is in a pure state, I know everything about the system, because I know all the exact values of the observables: $f(x), g(x), \dots$
If the system is not in a pure state (a state in which I have the maximal obtainable information about the system) it is only because of our ignorance, we can make further measurements and get more information on the system until we identify its pure state.
In practice, all the observables have a definite value at each istant of time. If I measure f and then g, the information that I get from the first measurement of f is not lost: I can measure again f and I get again the same value. In fact (if the experiments are perfomed "with care") the system is always in the same state that is "approximately" not affected by the measurements so that $f \cdot g = g \cdot f$. Classical physics is commutative!

Quantum Mechanics = Non-commutative Algebras 1.

In quantum physics, when I measure a physical quantity, I disturb the system in a direct way, so that the state of the system will not be the same. If I measure f and then g, the measure of g modify the state of the system and if I measure f again, maybe I get another value. Information can be lost by making further observations!

Furthermore, the system can be in a pure state (a state of maximum information) without having a definite value of an observable so that if I measure this observable I will get different values: in this sense quantum mechanics is intrinsecally probabilistic.

Quantum Mechanics = Non-commutative Algebras 2.

The observables in quantum mechanics are elements of a non commutative $*$-algebra because in general $f \cdot g \neq g \cdot f$. As in classical mechanics, observables are self-adjoint elements of the algebra: $f = f^*$ (because self adjoint elements have real spectrum) and the possible values of an observable f are again the elements of the spectrum of f: $\{\alpha \mid \alpha - f \quad \text{is not invertible}\}$.
In the same way as before, the states are linear functionals ω on the $*$-algebra of observables. From the technical point of view, "bounded observables" are self adjoint elements of a C*-algebra $\mathcal{A}$ and states of the system are positive, normalized, linear functionals on $\mathcal{A}$ i.e. functions $\omega : \mathcal{A} \to \mathbb{C}$ such that:

$$\omega(\alpha f + \beta g) = \alpha\omega(f) + \beta\omega(g), \quad \omega(f^* f) \geq 0, \quad \omega(1_{\mathcal{A}}) = 1.$$

Quantum Mechanics = Non-commutative Algebras 3.

A famous theorem of Gel'fand-Naĭmark-Segal associates to a state of a C*-algebra a concrete C*-algebra of linear operators acting on a Hilbert space. This is why we can say that quantum mechanics is essentially the theory of selfadjoint infinite dimensional matrices in Hilbert space.

Another theorem of Von Neumann says that this representation does not depend on the state ω that we use for the Gel'fand-Naĭmark-Segal construction if we start with the C*-algebra of quantum mechanics of a system with a finite number of degree of freedom.

Quantum Mechanics = Non-commutative Algebras 4.

Big problems are still open in the foundations of quantum mechanics of systems with infinite degrees of freedom i.e. in the fields of quantum statistical mechanics and quantum field theory. It is here that the formalism of C^*-algebras is really very useful (because here the von Neumann theorem is no more true and so we cannot work on a fixed Hilbert space).

Special Relativity 1.

In the study of (relativistic) quantum field theory there are also other complications coming from the fact that the physical theory is actually trying to unify quantum mechanics with special relativity. Let us examine some of the features of A. Einstein's "special theory of relativity".

The special theory of relativity is based on the fact that in our world there exists a maximum speed with which it is possible to exchange signals (actually the speed of light signals) so that it is not possible to transmit information in an instantaneous way between points in space. This principle together with the principle of relativity (all the inertial observers performing the same experiment will get the same result) tells us that it is impossible to give an absolute meaning to the simultaneity of two events.

Special Relativity 2.

In this way it is not possible to give to the events an absolute time coordinate: this coordinate depend on the observer in the same way as the other space coordinates (it does not have any meaning to say that two events took place with the same space coordinates because the coordinates depend on the observer!). Under this point of view, the time coordinate of a physical event is not a preferred one, but it can be treated exactly in the same way as all the other three space coordinates of the event: this is why in special relativity we speak about **space-time**.

Special Relativity 3.

In special relativity, we assume that (fixed a base point) space-time can be described by a four dimensional vector space $\mathbb{M}$ (the number of dimensions being equal to the number of coordinates that are necessary to locate an event: three space coordinates x, y, z and one time coordinate t).
Furthermore, since the speed of light (the limit speed) should be the same for all the observers, we get that the quantity $t^2 - x^2 - y^2 - z^2$ does not depend on the observer and can be used to define on the vector space $\mathbb{M}$ an inner product $g : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ with signature $(+1, -1, -1, -1)$:

$$g(v_1, v_2) := t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2, \quad \forall v_1, v_2 \in \mathbb{M},$$

where $v_1 := (t_1, x_1, y_1, z_1)$ and $v_2 := (t_2, x_2, y_2, z_2)$.

Special Relativity 4.

The vector space $\mathbb{M}$ equipped with this semi-definite inner product is called Minkowsky space-time. Every inertial observer will assign to the points v in this space coordinates (t, x, y, z) (choosing an orthonormal basis for the Minkowsky space). The coordinates of an event depend on the observer, but the inner product not and can actually be used to define the absolute future, past and present of a given event. So, with respect to the origin $0_{\mathbb{M}}$ of the Minkowsky space:

- the future points (those that can be reached by physical signals travelling with speed inferior to the speed of light) are those v such that $g(v, v) > 0$ (timelike separated) with positive time coordinate;

Special Relativity 5.

- the past points are those v such that $g(v, v) > 0$ (timelike separated) with negative time coordinate;
- the point in the "present" are those spatially separated in the sense that $g(v, v) < 0$;
- finally the points of the "light-cone" are those v such that $g(v, v) = 0$ (lightlike separated).

Under the hypothesis that no signal can be sent with a speed exceeding the light speed, points that are spatially separated cannot influence each other (this is called the principle of locality).

Algebraic Quantum Field Theory 1.

In “algebraic (relativistic) quantum field theory”, we assume that we can perform experiments on the physical system and that the instruments performing the experiment will interact with the system only in a localized region of space-time. Given a certain region in space-time $\mathcal{O}$, we can associate to it a C*-algebra $\mathcal{A}(\mathcal{O})$ of all the possible observables that we can measure performing experiments “inside” $\mathcal{O}$. In technical terms we are given a net of C*-algebras on the Minkowsky space.

Algebraic Quantum Field Theory 2.

Some properties of this net of algebras are axiomatically imposed in order to select the nets that are physically relevant:

- **Isotony**: $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$. This property (called isotony) simply says that the observables that we can measure in a bigger region are more.
- **Locality**: if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spatially separated, the observables of $\mathcal{A}(\mathcal{O}_1)$ commute with the observables in $\mathcal{A}(\mathcal{O}_2)$. This property (called locality) encodes the fact that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spatially separated, then no signals can be sent between $\mathcal{O}_1$ and $\mathcal{O}_2$ so that if we disturb the system in $\mathcal{O}_1$ performing an experiment there, nothing will change for those experiments that are going on in $\mathcal{O}_2$ so that $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ are compatible observables.

Algebraic Quantum Field Theory 3.

- **Covariance**: there exists a representation α of the Poincaré group $\mathcal{P}$ as a group of automorphism of the algebra of observables such that: $\alpha_g(\mathcal{A}(\mathcal{O})) = \mathcal{A}(g\mathcal{O})$.
  The inertial observers that look at the system in Minkowsky space, will label the events with different coordinates. A change of coordinates (or a change of observer) correspond to a so called Poincaré trasformation of the Minkowsky space (translations, rotations and boosts). Each observer will have his experimental apparata to measure physical quantities so if $g \in \mathcal{P}$ is a transformation that relates two different observers, we can associate with it a transformation of the set of observables α_g that will tell us how the observables of the second observer are related to the observables of the first.

Algebraic Quantum Field Theory 4.

- **Existence of the vacuum**: there exists a state $\omega : \mathcal{A} \to \mathbb{C}$ that is invariant under Poincaré transformations: $\omega(\alpha_g(a)) = \omega(a)$, for all observables a, and such that the energy of the system is positive (spectral condition).
  From the technical point of view the energy is defined to be the generator of the one-parameter group of transformations given by the time translations in the GNS representation associated to the vacuum state ω.
  From the physical point of view we impose the existence of a state of the system (the "empty" state). In this state we cannot distinguish any preferred directions, positions or uniform motions (the state is invariant under the Poincaré group) and in this state the system has the lowest energy, that we impose to be anyway non negative.

Who Does NCG and Where 1.

[The following list is necessarily an extremely partial one and its only aim is to give to the newcomer in the field geographical coordinates and some well known names in order to perform a search of relevant papers on internet archives (http://xxx.lanl.gov) or in libraries].

- The absolute leader in the research on non-commutative geometry is still its genial creator: Alain Connes in College de France and IHES - Paris - France (and now also Vanderbildt in USA).
  A strong group of A. Connes' colleagues (among them G. Skandalis, J.-L. Sauvageot, M. Karoubi) are in Paris VI and Paris VII universities.

Who Does NCG and Where 2.

- Another great center especially for the application of non-commutative geometry to physics is CPT in Marseille (France) where, under the enthusiastic leadership of Daniel Kastler, generations of mathematical physicists have been initiated to the mysteries of algebraic quantum field theory and now to non-commutative geometry. The most active researchers in Marseille are now, R. Coquereaux, T. Schüker (non-commutative geometry and standard model), C. Rovelli (non-commutative geometry and quantum gravity), B. Iochum.

Who Does NCG and Where 3.

Other people in France include, among the many others:

- J. Renault (in Orleans) that has been studying the application of non-commutative geometry to groupoids;
- J.L. Loday (in Strasbourg) that is the specialist about cyclic homology;
- in Orsay the group of mathematical physicists directed by M. Dubois-Violette and J. Madore has developed a "derivation based" version of non-commutative geometry in some way different from the original one of A. Connes.
- T. Fack and M.-T. Benameur (in Lyon) work on "Von Neumann variants" of Connes' spectral triples.

Who Does NCG and Where 4.

In Germany there are two important centers for non-commutative geometry:

- The Max Planck Institute in Bonn is one of the most interesting places where to learn non-commutative geometry. M. Marcolli (now the main collaborator of A. Connes) and Y. Manin are the main researchers there.
- In Münster, J. Cuntz, one of the pioneers of non-commutative geometry, leads a strong research group.

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Always in Germany there are two big groups of people mainly involved in algebraic quantum field theory: one located in Hamburg, where the school of R. Haag is now under the direction of K. Fredhenhagen, and one in Göttingen where the school of H.J. Borchers is now under the supervision of D. Buchholtz.

Who Does NCG and Where 5.

In Italy, the main place for operator algebras and non-commutative geometry and especially algebraic quantum field theory is Rome where the main professors involved are S. Doplicher, J.E. Roberts, R. Longo, L. Zsido, C. D'Antoni, D. Guido, T. Isola, F. Fidaleo, C. Pinzari. Other italian people involved in non-commutative geometry are: G. Landi (Trieste), F. Lizzi (Napoli).
In Poland S.L. Woronowicz (in Warsaw) is one of the pioneers in the application of operator algebras to quantum groups; A. Sitartz (now at Jagiellonian - Krakow) works on the applications of non-commutative geometry to particle physics; W. Heller (now in Vatican City) is using non-commutative geometry in general relativity.
In Japan, under the leadership of H. Araki, there are now many mathematicians involved in operator algebras and their applications in most of the universities (especially in RIMS). Some of the most

Who Does NCG and Where 6.

In United States there are many people scattered in several centers:

- in Vanderbilt: D. Bisch and A. Connes have developed the main center in non-commutative geometry [www.math.vanderbilt.edu/ñcgoa] where every year there are introductory courses for beginners as well as state of the art workshops. Members of the group include D. Bisch, A. Connes, G. Kasparov, G. Yu.
- in Philadelphia: R. Kadison, M. Pimsner (operator algebras);
- in Penn State: P. Baum, N. Higson, V. Nistor, A. Ocneanu, J. Roe (ncg);
- in Atlanta (Georgia Insitute of Technology): J. Bellissard (ncg)
- in Ohio - Columbus: H. Moscovici (ngc, cyclic homology);

Who Does NCG and Where 7.

- in Berkeley: W. Arveson, V. Jones, D. Voiculescu (operator algebras),
  M.A. Rieffel, M. Wodzicki (non-commutative geometry);
- in California - Davis: A. Schwarz (ngc in physics);
- in Harvard: A. Jaffe (ncg in quantum field theory);
- in Los Angeles: M. Takesaki, S. Popa, D. Shlyakhtenko (operator algebras);
- in Reno: B. Blackadar (K-theory), A. Kumjian;
- in Riverside: M. Lapidus (ncg of fractal sets);
- in Evanston: B. Tsygan (ncg)
- in Gainesville (Florida): S.J. Summers (algebraic quantum field theory).

Who Does NCG and Where 8.

Operator algebras are now studied in many places at very high level. Just to mention a few names:

- In Denmark (Copenhagen - Odense) U. Haagerup, M. Rørdam, R. Nest, G. Elliott, A. Rennie (ncg).
- In Norway (Oslo - Trondheim) O. Bratteli, M. Landstadt, C. Skau.
- In Ireland (Cork) G. Murphy.
- In Great Britain (Cardiff) D. Evans; (Manchester) R. Plymen.
- In Romania (Bucharest) S. Strătilă.
- In Holland (Amsterdam) N. Landsman, M. Müger.

Who Does NCG and Where 9.

In Australia the main center for operator algebra is Newcastle (I. Raeburn, W. Szymanzig) and there are strong research groups in Adelaide (V. Mathai, F. Sukochev) and most of all in Canberra - ANU (A. Carey, B. Wang).

In Canada several people in operator algebra and non-commutative geometry work in Victoria (J. Phillips, M. Laca) in Western Ontario (M. Khalkhali) in Waterloo (K. Davidson)

People with research interests in non-commutative geometry are working also in Costa Rica (J.C. Varilly), in Lebanon (A. Chamseddine in Beirut), Brazil (R. Exel), Iran (M. Khalkhali now in Western Ontario - Canada), Vietnam (Do Ngoc Diep), . . .

Here in Thailand we have a small group of researchers already: P. Chaisuriya (Mahidol), S. Utudee (Chiang Mai) and our group in Bangkok.

A Guided Tour to the Bibliography 1.

In the following we list only some references that we think can be important as learning sources of the basic techniques and as general introductions to the subject treated in this workshop. For books we refer to last printed edition.
[Some of the references contained in the following are also available for free on the internet archives in Los Alamos:
`http://xxx.lanl.gov`
and in this case the relative reference number is given in square brackets].

A Guided Tour of the Bibliography 2.

The "Bible" of Noncommutative Geometry is represented by the famous Alain Connes' book (now available on-line):

- A. Connes,
  Noncommutative Geometry,
  Academic Press (1994).

This is not a paedagogical book, it is mainly a masterpiece of written mathematics that can be used much more as font of inspiration for future work as most of the proofs are not contained in the book!
A new book by A. Connes and M. Marcolli is (now November 2006) in preparation.

A Guided Tour of the Bibliography 3.

There are now several more or less introductory expositions devoted to non-commutative geometry: the most comprehensive graduate level textbook available is:

- J.M. Gracia-Bondia, H. Figueroa, J.C. Varilly,
  Methods of Noncommutative Geometry,
  Birkhäuser (2001).

The most elementary reference is still the book by G. Landi:

- G. Landi,
  An Introduction to Noncommutative Spaces and Their Geometry, [hep-th/97801078]
  Springer Verlag (1997).

A Guided Tour of the Bibliography 4.

An alternative good on-line introduction to some aspects of non-commutative geometry is

- M. Khalkhali
  Very Basic Noncommutative Geometry
  [math.KT/0408416] (2004).

A huge list of examples of spectral triples is provided in

- A. Connes, M. Marcolli
  A Walk in the Noncomutative Garden [math.QA/0601054] (2006).

Excellent dowloadable material can be found at the Warsaw University "Noncommutative Geometry and Quantum Groups" page: **[http://toknotes.mimuw.edu.pl]**.

A Guided Tour of the Bibliography 5.

Other very good introductions are given by (or contained in):

- J. Varilly, J. Gracia-Bondia,
  Connes' Noncommutative Differential Geometry and the Standard Model, J. Geom. Phys. **12** 223-301 (1993).
- J. Varilly,
  An Introduction to Noncommutative Geometry,
  Summer School "Noncommutative Geometry and Applications" Lisbon [physics/9709045] (1997).
- R. Coquereaux,
  Noncommutative Geometry and Theoretical Physics,
  J. Geom. Phys. **6** 425-490 (1989).
- R. Coquereaux,
  Noncommutative Geometry: a Physicist's Brief Survey,
  J. Geom. Phys. **11** 307-324 (1993).

A Guided Tour of the Bibliography 6.

- J. Fröhlich, O. Grandjean, A. Recknagel, **Supersymmetric Quantum Theory and Differential Geometry**, [hep-th/9612205] Commun. Math. Phys. **193** 527-594 (1998).
- J. Fröhlich, O. Grandjean, A. Recknagel, **Supersymmetric Quantum Theory and Noncommutative Geometry**, [math-ph/9807006], Commun. Math. Phys. **203** 119-184 (1999).

Another introduction to non-commutative geometric methods from a point of view different from that of A. Connes:

- J. Madore, **An Introduction to Noncommutative Differential Geometry and Its Physical Applications**, CUP (1999).

A Guided Tour of the Bibliography 7.

For Cyclic Cohomology and K-Theory:

- J.L. Loday,
  Cyclic Homology,
  Springer (1992).
- J. Brodzki,
  An Introduction to K-Theory and Cyclic Cohomology,
  [funct-an/9606001].
- N.E. Wegge Olsen,
  K-Theory and C*-Algebras a Friendly Approach,
  Oxford University Press (1993).

A Guided Tour of the Bibliography 8.

Easy and fast introductions to functional analysis as it is used for operator algebras are:

- V.S. Sunder,
  An Invitation to Von Neumann Algebras,
  Springer (1987),
- G.K. Pedersen,
  Analysis Now,
  Springer (1995).

A Guided Tour of the Bibliography 9.

For the subject of operator algebras several books can be recommended. Good textbooks are:

- R.V. Kadison, J.R. Ringrose,
  Fundamentals of the Theory of Operator Algebras,
  Vol. I - II, American Mathematical Society (1997).
- G.J. Murphy,
  C^*-Algebras and Operator Theory,
  Academic Press (1990).
- O. Bratteli, D.W. Robinson,
  Operator Algebras and Quantum Statistical Mechanics,
  Vol. I-II, Springer (1987-1997).
- S. Strătilă, L. Zsido,
  Lectures on von Neumann Algebras,
  Abacus Press (1979).

A Guided Tour of the Bibliography 10.

Complete reference books are:

- M. Takesaki,
  The Theory of Operator Algebras,
  Vol. I-II-III, Springer (2001-2002).
- B. Blackadar,
  Operator Algebras,
  Springer (2006).

For Tomita-Takesaki theory, beside Takesaki above, the best reference available is:

- S. Strătilă,
  Modular Theory in Operator Algebras,
  Abacus Press (1981).

A Guided Tour of the Bibliography 11.

For the subject of algebraic relativistic quantum field theory the basic reference book is:

- R. Haag,
  Local Quantum Physics,
  Springer (1996).

An easier introduction is:

- H. Araki,
  Mathematical Theory of Quantum Fields,
  Oxford University Press (2000).

For a more mathematical oriented presentation:

- H. Baumgärtel, M. Wöllemberg,
  Causal Nets of Operator Algebras,
  Akademie Verlag (1992).

A Guided Tour of the Bibliography 12.

For the background material on differential geometry and topology:

- M. Nakahara,
  Geometry, Topology and Physics,
  Institute of Physics Publishing (1990).

For Clifford algebras and Dirac operators:

- H.B. Lawson, M.L. Michelsohn,
  Spin Geometry,
  Princeton University Press (1989).
- N. Berline, E. Getzler, M. Vergne,
  Heat Kernels and Dirac Operators,
  Springer Verlag (1992).

Conclusions 1.

Finally, I would like to say a final word of motivation about the goals of this researches (addressed mainly to the young students and to those who may wonder if it can be valuable to undertake the effort to learn something on this new subject): why are people doing this kind of mathematics?

For sure not for the possibility of easy technological applications . . .

Not because it is an easy field for publications: actually operator algebras require at least some years of study only to grasp the fundamental techniques and, in this field, people usually publish at a lower rate compared to other branches of mathematics (but publications tend to have a stronger impact on the mathematical community).

Conclusions 2.

Maybe most of the people (especially physicists or "platonic" mathematicians) are interested in this questions because of a deep quest for the "true" about nature (as you know, in the west, philosophy has always been very much concerned with the absolute true under the surface of phenomena!), but actually this is only one part of the story (as for me, I don't even think about the possible existence of a final "true" physical theory that we can discover).
So why to study non-commutative geometry and its applications to fundamental physics?
Well ... because of "beauty"! These ideas are really very beautiful and elegant! Thailand is well known to be a place where people are great estimators of the beauty and elegance everywhere and I am sure that somehow (with some effort of course!) you will be able to percive the subtle fascination of these ideas ...

Conclusions 3.

Actually the final goal is to enjoy ourselves and our friends with original, new wild ideas (space-time, mind, logic, physics, geometry and all that): there is no better way to do so than to travel and play in a world of amazing things ...
For those who continue to ask for the "true" or for the easy practical or empyrical application, I can only recall the wonderful words of Dirac:

"If an idea is beautiful it will very likely be also true" ...

Thank You for Your Kind Attention!

Used Software.

The following file has been realized in $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX 2_ε using the **beamer** LaTeX-macro.