Thai Journal of Mathematics
Volume 8 (2010) Number 3 : 535–543

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Characterization of Cayley Graphs of Rectangular Groups

S. Panma

Abstract : A digraph (V, E) is a *Cayley graph of semigroup(group)* if there exists a semigroup(group) S and $A \subseteq S$ such that (V, E) is isomorphic to the Cayley graph $Cay(S, A)$. In this paper, we characterize digraphs which are Cayley graphs of rectangular groups.

Keywords : Cayley graph; Rectangular group; Cayley graph of Rectangular group.
2000 Mathematics Subject Classification : 05C25; 20B25.

1 Introduction

One of the previously known investigations of algebraic structures on Cayley graphs can be found in Maschke's Theorem from 1896 about groups of genus zero. A group of genus zero is a group G which possess a generating system A such that the Cayley graph $Cay(G, A)$ is planar, see for example [16]. In [15] Cayley graphs which represent groups are described. It is natural to investigate Cayley graphs for semigroups which are unions of groups, so-called completely regular semigroups, see for example [14]. In [1,13] Cayley graphs which represent completely regular semigroups which are right (left) groups and Clifford semigroups are characterized. We now characterize digraphs which are Cayley graphs of rectangular groups.

2 Basic definitions and results

All sets in this paper are assumed to be finite. A *groupoid* is a non-empty set G together with a binary operation on G. A *semigroup* is a groupoid G which is associative. A *monoid* is a semigroup G which contains an (two-sided) identity element $e_G \in G$. A *group* is a monoid G such that for every $a \in G$ there exists a group inverse $a^{-1} \in G$ such that $a^{-1}a = aa^{-1} = e_G$.

A semigroup S is said to be a *right (left) zero semigroup* if $xy = y$ $(xy = x)$ for all $x, y \in S$.A semigroup S is called a *right(left) group* if it is isomorphic to the direct product of a group and a right (left) zero semigroup. A semigroup S is *rectangular band* if it is isomorphic to the direct product of a left zero semigroup and a right zero semigroup. A semigroup S is called a *rectangular group* if it is isomorphic to the direct product of a group and a rectangular band. It is clear that a right (left) zero semigroup, a right(left) group, and a rectangular band are rectangular groups.

Let (V_1, E_1) and (V_2, E_2) be digraphs. A mapping $\varphi : V_1 \to V_2$ is called a *(digraph) homomorphism* if $(u, v) \in E_1$ implies $(\varphi(u), \varphi(v)) \in E_2$, i.e. φ preserves arcs. We write $\varphi : (V_1, E_1) \to (V_2, E_2)$. A digraph homomorphism $\varphi : (V, E) \to (V, E)$ is called an *(digraph) endomorphism.* If $\varphi : (V_1, E_1) \to (V_2, E_2)$ is a bijective digraph homomorphism and φ^{-1} is also a digraph homomorphism, then φ is called an *(digraph) isomorphism*. A digraph isomorphism $\varphi : (V, E) \to (V, E)$ is called an *(digraph) automorphism*. All digraph automorphisms form a group, called the automorphism group of (V, E) and denoted by $Aut(V, E)$.

Let S be a semigroup(group) and $A \subseteq S$. We define the *Cayley graph* $Cay(S, A)$ as follows: S is the vertex set and (u, v), $u, v \in S$, is an arc in $Cay(S, A)$ if there exists an element $a \in A$ such that $v = ua$.

Theorem 2.1. ([2], [11], [15]) *A digraph (V, E) is a Cayley graph of a group G if and only if $Aut(V, E)$ contains a subgroup $\triangle$ which is isomorphic to G and for any two vertices $u, v \in V$ there exists $\sigma \in \triangle$ such that $\sigma(u) = v$.*

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs, $V_1 \cap V_2 = \emptyset$. The *disjoint union of G_1 and G_2* is defined as $G_1 \dot{\bigcup} G_2 := (V_1 \cup V_2, E_1 \cup E_2)$.

For terms in Graph Theory not defined here see for example [2].

3 Main results

A subdigraph (V', E') of a graph (V, E) is called a *strong subdigraph of* (V, E) if whenever $u, v \in V'$ and $(u, v) \in E$, then $(u, v) \in E'$. In the next theorem, we characterize digraphs which are Cayley graphs of rectangular groups.

Theorem 3.1. *A digraph (V, E) is a Cayley graph of a rectangular group if and only if then the following conditions hold:*

(1) (V, E) is the disjoint union of n isomorphic subdigraphs (V_1, E_1), (V_2, E_2),..., (V_n, E_n) for some $n \in \mathbb{N}$,

(2) there exists a group G and $m \in \mathbb{N}$ such that for each $i \in \{1, 2, .., n\}$, (V_i, E_i) contains m disjoint strong subdigraphs $(V_{i1}, E_{i1}), (V_{i2}, E_{i2}), ..., (V_{im}, E_{im})$ which are Cayley graphs of G, and $V_i = \bigcup_{\alpha=1}^{m} V_{i\alpha}$,

(3) for each $\alpha \in \{1, 2, .., m\}$, there exists a digraph isomorphism $\varphi_{i\alpha} : (V_{i\alpha}, E_{i\alpha}) \to Cay(G, A_{i\alpha})$ for some $A_{i\alpha} \subseteq G$, such that $A_{j\alpha} = A_{k\alpha}$ for all $j, k \in \{1, 2, ..., n\}$,

(4) for each $\alpha, \beta \in \{1, 2, ..., m\}$, and for each $u \in V_{i\alpha}, v \in V_{i\beta}$, $(u, v) \in E$ if and only if $\varphi_{i\beta}(v) = \varphi_{i\alpha}(u)a$ for some $a \in A_{i\beta}$.

Proof. ($\Rightarrow$) Let (V, E) be a Cayley graph of rectangular group. Then there exists a rectangular group $S = G \times L_n \times R_m$ where G is a group, $L_n = \{l_1, l_2, ..., l_n\}$ a left zero semigroup, and $R_m = \{r_1, r_2, ..., r_m\}$ a right zero semigroup, such that $(V, E) \cong Cay(S, A)$ for some $A \subseteq S$. Let f be an isomorphism from $Cay(S, A)$ onto (V, E).

(1) For each $i \in \{1, 2, ..., n\}$, set $V_i := f(G \times \{l_i\} \times R_m)$, and $E_i := E \cap (V_i \times V_i)$. Hence (V_i, E_i) is a strong subdigraph of (V, E). We will show that $(V_1, E_1), (V_2, E_2), ..., (V_n, E_n)$ are isomorphic subdigraphs. Let $p, q \in \{1, 2, .., n\}, p \neq q$, define a map ϕ from (V_p, E_p) to (V_q, E_q) by $\phi(f(g, l_p, r)) = f(g, l_q, r)$. Since f is an isomorphism and $|G \times \{l_p\} \times R_m| = |G \times \{l_q\} \times R_m|$, $|V_p| = |V_q|$. Therefore ϕ is a well defined bijection.

For $f(g, l_p, r), f(g', l_p, r') \in V_p$, take $(f(g, l_p, r), f(g', l_p, r')) \in E_p$. Since f is an isomorphism and $E_p \subseteq E$, $((g, l_p, r), (g', l_p, r'))$ is an arc in $Cay(S, A)$. Then there exists $(a, l, r'') \in A$ such that $(g', l_p, r') = (g, l_p, r)(a, l, r'') = (ga, l_p, r'')$. Hence, $g' = ga$, $r' = r''$, and thus $(g', l_q, r') = (ga, l_q, r'') = (g, l_q, r)(a, l, r'')$. Then $((g, l_q, r), (g', l_q, r'))$ is an arc in $Cay(S, A)$. Since f is an isomorphism, it follows that $(f(g, l_q, r), f(g', l_q, r')) \in E_q$. This shows that ϕ is a digraph homomorphism. Similarly, ϕ^{-1} is a digraph homomorphism. Hence ϕ is a digraph isomorphism.

Next, we will prove that $(V, E) = \dot{\bigcup}_{i=1}^{n}(V_i, E_i)$, i.e. $V = \dot{\bigcup}_{i=1}^{n} V_i$ and $E = \dot{\bigcup}_{i=1}^{n} E_i$. By the definition of V_i and f is a digraph isomorphism, we get $V_i \bigcap V_j = \emptyset$ for every $i \neq j$ in $\{1, 2, ..., n\}$. Hence $\dot{\bigcup}_{i=1}^{n} V_i := \dot{\bigcup}_{i=1}^{n} f(G \times \{l_i\} \times R_m) = f(\dot{\bigcup}_{i=1}^{n} G \times \{l_i\} \times R_m) = f(S) = V$. Suppose that $E \neq \dot{\bigcup}_{i=1}^{n} E_i$. By the definition of E_i, we get $\dot{\bigcup}_{i=1}^{n} E_i \subsetneq E$ Then there exists $(x, y) \in E$ such that $(x, y) \notin \dot{\bigcup}_{i=1}^{n} E_i$. Therefore $x = f(g, l_k, r) \in V_k$ and $y = f(g', l_t, r') \in V_t$ for some $k, t \in \{1, 2, ..., n\}$. . Hence $(f(g, l_k, r), f(g', l_t, r')) \in E$, and thus $((g, l_k, r), (g', l_t, r'))$ is an arc in $Cay(S, A)$, since f is an isomorphism. Then there exists $(a, l, r'') \in A$ such that $(g', l_q, r') = (g, l_p, r)(a, l, r'') = (ga, l_p, r'')$. Therefore $l_q = l_p$ and thus $q = p$. This is a contradiction, so $E = \dot{\bigcup}_{i=1}^{n} E_i$.

(2) For each $i \in \{1, 2, ..., n\}$, and $\alpha \in \{1, 2, ..., m\}$, set $V_{i\alpha} := f(G \times \{l_i\} \times \{r_\alpha\})$, $E_{i\alpha} := E \cap (V_{i\alpha} \times V_{i\alpha})$, and $B_{i\alpha} := \{(g, l_i, r_\alpha) | (g, l, r_\alpha) \in A\}$. Therefore $(V_{i1}, E_{i1}), (V_{i2}, E_{i2}), ..., (V_{im}, E_{im})$ are strong subdigraphs of (V_i, E_i). It is clear that $G \times \{l_i\} \times \{r_\alpha\}$ is a group, and $B_{i\alpha} \subseteq G \times \{l_i\} \times \{r_\alpha\}$. Define $\psi_{i\alpha} : (V_{i\alpha}, E_{i\alpha}) \to Cay(G \times \{l_i\} \times \{r_\alpha\}, B_{i\alpha})$ by

$$\psi_{i\alpha}(f(g, l_i, r_\alpha)) = (g, l_i, r_\alpha).$$

Since f is an isomorphism, $\psi_{i\alpha}$ is also an isomorphism. In particular, $\psi_{i\alpha} = f^{-1}|_{V_{i\alpha}}$, where $f^{-1}|_{V_{i\alpha}}$ is the restriction of f^{-1} to $V_{i\alpha}$. Hence $(V_{i\alpha}, E_{i\alpha})$ is a Cayley graph of group $G \times \{l_i\} \times \{r_\alpha\}$.

Let $\alpha, \beta \in R_m$ and $\alpha \neq \beta$. Since $(G \times \{l_i\} \times \{r_\alpha\}) \cap (G \times \{l_i\} \times \{r_\beta\}) = \emptyset$ and f is an isomorphism, we get $f(G \times \{l_i\} \times \{r_\alpha\}) \cap f(G \times \{l_i\} \times \{r_\beta\}) = \emptyset$, thus $V_{i\alpha} \cap V_{i\beta} = \emptyset$. By the definition of $E_{i\alpha}$ and $E_{i\beta}$, we have $E_{i\alpha} \cap E_{i\beta} = \emptyset$. Therefore $(V_{i\alpha}, E_{i\alpha})$ and $(V_{i\beta}, E_{i\beta})$ are disjoint subdigraphs of (V_i, E_i). Hence $\bigcup_{\alpha=1}^{m} Vi\alpha = \bigcup_{\alpha=1}^{m} f(G \times \{l_i\} \times \{r_\alpha\}) = f(\bigcup_{\alpha=1}^{m}(G \times \{l_i\} \times \{r_\alpha\})) = f(G \times \{l_i\} \times R_m) = V_i$

(3) From (2), we have $(V_{i\alpha}, E_{i\alpha}) \cong Cay(G \times \{l_i\} \times \{r_\alpha\}, B_{i\alpha})$. Let p_1 be the projection of $G \times \{l_i\} \times \{r_\alpha\}$ onto its first coordinate. Then p_1 is a group isomorphism from $G \times \{l_i\} \times \{r_\alpha\}$ onto G, and $p_1(G \times \{l_i\} \times \{r_\alpha\}) = G$. Hence $Cay(G \times \{l_i\} \times \{r_\alpha\}, B_{i\alpha}) \cong Cay(p_1(G \times \{l_i\} \times \{r_\alpha\}), p_1(B_{i\alpha})) = Cay(G, p_1(B_{i\alpha}))$. Let $A_{i\alpha} := p_1(B_{i\alpha})$. Therefore $(V_{i\alpha}, E_{i\alpha}) \cong Cay(G, A_{i\alpha})$, thus we have an isomorphism

$$\varphi_{i\alpha} = p_1 \circ \psi_{i\alpha} : (V_{i\alpha}, E_{i\alpha}) \to Cay(G, A_{i\alpha}).$$

Let $k, t \in \{1, 2, ..., n\}$. Take $g \in A_{k\alpha}$. Then we get $(g, l_k, r_\alpha) \in B_{k\alpha}$. By the definition of $B_{k\alpha}$, there exists $l \in L_n$ such that $(g, l, r_\alpha) \in A$. Therefore we have $(g, l_t, r_\alpha) \in B_{t\alpha}$, hence $g \in A_{t\alpha}$. This shows that $A_{k\alpha} \subseteq A_{t\alpha}$. Similarly, $A_{t\alpha} \subseteq A_{k\alpha}$. Thus $A_{i\alpha} = A_{j\alpha}$ for all $i, j \in \{1, 2, ..., n\}$.

(4) For each $i \in \{1, 2, ..., n\}$, and $\alpha, \beta \in \{1, 2, ..., m\}$, take $f(g, l_i, r_\alpha) \in V_{i\alpha}$, and $f(g', l_i, r_\beta) \in V_{i\beta}$. We will prove that $(f(g, l_i, r_\alpha), f(g', l_i, r_\beta)) \in E$ if and only if $\varphi_{i\alpha}(f(g', l_i, r_\beta)) = \varphi_{i\alpha}(f(g, l_i, r_\alpha))a$ for some $a \in A_{i\beta}$.
($\Rightarrow$) Let $(f(g, l_i, r_\alpha), f(g', l_i, r_\beta)) \in E$. Then $((g, l_i, \alpha), (g', l_i, \beta))$ is an arc in $Cay(S, A)$, since f is an isomorphism. Hence there exists $(a, l_j, r_\xi) \in A$ such that $(g', l_i, r_\beta) = (g, l_i, \alpha)(a, l_j, r_\xi) = (ga, l_i, r_\xi)$. Therefore $g' = ga$, $r_\beta = r_\xi$. Then we have $(a, l_j, r_\beta) = (a, l_j, r_\xi) \in A$. By the definition of $B_{i\beta}$, there exists $(a, l_i, r_\beta) \in B_{i\beta}$, and hence $a = p_1((a, l_i, r_\beta)) \in p_1(B_{i\beta}) = A_{i\beta}$. Since $\psi_{i\alpha} = f^{-1}|_{V_{i\alpha}}$, we get $\psi_{i\alpha}(f(g', l_i, r_\beta)) = (g', l_i, r_\beta)$ and $\psi_{i\alpha}(f(g, l_i, r_\xi)) = (g, l_i, r_\xi)$. Therefore $p_1 \circ \psi_{i\alpha}(f(g', l_i, r_\beta)) = g'$ and $p_1 \circ \psi_{i\alpha}(f(g, l_i, r_\xi)) = g$. Hence

$$p_1 \circ \psi_{i\alpha}(f(g', l_i, r_\beta)) = g' = ga = p_1 \circ \psi_{i\alpha}(f(g, l_i, r_\xi))a.$$

Since $p_1 \circ \psi_{i\alpha} = \varphi_{i\alpha}$, we have $\varphi_{i\alpha}(f(g', l_i, r_\beta)) = \varphi_{i\alpha}(f(g, l_i, r_\xi))a$.
($\Leftarrow$) Let $\varphi_{i\alpha}(f(g', l_i, r_\beta)) = \varphi_{i\alpha}(f(g, l_i, r_\alpha))a$ for some $a \in A_{i\beta}$. Then there exists $(a, l_i, r_\beta) \in B_{i\beta}$. Since $\psi_{i\alpha} = f^{-1}|_{V_{i\alpha}}$ and $\psi_{i\beta} = f^{-1}|_{V_{i\beta}}$, we get $\psi_{i\alpha}(f(g, l_i, r_\alpha)) = (g, l_i, r_\alpha)$ and $\psi_{i\beta}(f(g', l_i, r_\beta)) = (g', l_i, r_\beta)$, respectively. Therefore $\varphi_{i\alpha}(f(g, l_i, r_\alpha)) = p_1 \circ \psi_{i\alpha}(f(g, l_i, r_\alpha)) = g$ and $\varphi_{i\alpha}(f(g', l_i, r_\beta)) = p_1 \circ \psi_{i\beta}(f(g', l_i, r_\beta)) = g'$. Hence $g' = \varphi_{i\alpha}(f(g', l_i, r_\beta)) = \varphi_{i\alpha}(f(g, l_i, r_\alpha))a = ga$. By the definition of $B_{i\beta}$ and $(a, l_i, r_\beta) \in B_{i\beta}$, we have $(a, l, r_\beta) \in A$ for some $l \in L_m$. Therefore $(g', l_i, r_\beta) = (ga, l_i, r_\beta) = (g, l_i, r_\alpha)(a, l, r_\beta)$. Then $((g, l_i, r_\alpha), (g', l_i, r_\beta))$ is an arc in $Cay(S, A)$ and thus $(f(g, l_i, r_\alpha), f(g', l_i, r_\beta)) \in E$.

($\Leftarrow$) By (1) and (2), we get $V = \bigcup_{i=1}^{n}\bigcup_{\alpha=1}^{m} V_{i\alpha}$ is the disjoint union. Choose $k \in \{1, 2, ..., n\}$, and let $A := \bigcup_{\alpha=1}^{m}(A_{k\alpha} \times \{l_k\} \times \{r_\alpha\})$. We will show that $(V, E) \cong Cay((G \times L_n \times R_m), A)$. Define a map f from (V, E) to $Cay((G \times L_n \times R_m), A)$ by

$$f(v) = (\varphi_{i\alpha}(v), l_i, r_\alpha) \text{ for any } v \in V_{i\alpha},\ i\{1, 2, ..., n\}, \text{ and } \alpha \in \{1, 2, ..., m\}.$$

Let $u, v \in V$ and $u = v$. Then $u = v \in V_{j\beta}$ for some $j \in \{1, 2, ..., n\}$ and $\beta \in \{1, 2, ..., m\}$. Hence $\varphi_{j\beta}(u) = \varphi_{j\beta}(v)$ and $(\varphi_{j\beta}(u), l_j, r_\beta) = (\varphi_{j\beta}(v), l_j, r_\beta)$. Therefore f is well defined. Let $u, v \in V$ and $f(u) = f(v)$. Then $u \in V_{j\beta}$ and $v \in V_{t\delta}$ for some $j, t \in \{1, 2, ..., n\}$ and $\beta, \delta \in \{1, 2, ..., m\}$, thus

$$(\varphi_{j\beta}(u), l_j, r_\beta) = f(u) = f(v) = (\varphi_{t\delta}(v), l_t, r_\delta).$$

Hence $\varphi_{j\beta}(u) = \varphi_{t\delta}(v)$, $l_j = l_t$, and $r_\beta = r_\delta$. Therefore $j = t$ and $\beta = \delta$. Then $u, v \in V_{j\beta}$ and $\varphi_{j\beta}(u) = \varphi_{j\beta}(v)$. Since $\varphi_{j\beta}$ is an isomorphism, $u = v$. This shows that f is an injection.

By (2), we get $|G| = |V_{i\alpha}|$ for all $i \in \{1, 2, ..., n\}$ and $\alpha \in \{1, 2, ..., m\}$. Thus $|G \times L_n \times R_m| = |\dot{\bigcup}_{i=1}^{n}\dot{\bigcup}_{\alpha=1}^{m} V_{i\alpha}| = |V|$. Hence f is a surjection.

Let $u, v \in V$ and $(u, v) \in E$. By (1), we get $u, v \in V_j$ for some $j \in \{1, 2, ..., n\}$. Then there are $\beta, \delta \in \{1, 2, ..., m\}$ such that $u \in V_{j\beta}$ and $v \in V_{j\delta}$ by (2). From (4), we get $\varphi_{j\delta}(v) = \varphi_{j\beta}(u)a$ for some $a \in A_{j\delta}$. By (3), $a \in A_{k\delta}$. Hence $(a, l_k, r_\delta) \in (A_{k\delta} \times \{l_k\} \times \{r_\delta\}) \subseteq A$. Since $f(v) = (\varphi_{j\delta}(v), l_j, r_\delta) = (\varphi_{j\beta}(u)a, l_j, r_\delta) = (\varphi_{j\beta}(u), l_j, r_\beta)(a, l_k, r_\delta) = f(u)(a, l_k, r_\delta)$, we have $(f(u), f(v))$ is an arc in $Cay((G \times L_n \times R_m), A)$. This shows that f is a digraph homomorphism.

Let $g, g' \in G$, $j, t \in \{1, 2, ..., n\}$, $\beta, \delta \in \{1, 2, ..., m\}$, and let $((g, l_j, r_\beta), (g', l_t, r_\delta))$ be an arc in $Cay(G \times L_n \times R_m, A)$. Then there exists $(a, l_q, r_\xi) \in A$ such that $(g', l_t, r_\delta) = (g, l_j, r_\beta)(a, l_q, r_\xi) = (ga, l_j, r_\xi)$. Therefore $g' = ga$, $l_t = l_j$, and $r_\delta = r_\xi$. Thus $t = j$, and $\delta = \xi$. By (3) and $g, g' \in G$, there exists $u \in V_{j\beta}$ and $v \in V_{j\delta}$ such that $\varphi_{j\beta}(u) = g$ and $\varphi_{j\delta}(v) = g'$. Therefore $\varphi_{j\delta}(v) = g' = ga = \varphi_{j\beta}(u)a$. Since $A = \bigcup_{\alpha=1}^{m}(A_{k\alpha} \times \{l_k\} \times \{r_\alpha\})$ and $(a, l_q, r_\delta) \in A$, we get $q = k$ and $a \in A_{k\delta}$. By (3) again, $a \in A_{j\delta}$. From (4), we get $(f^{-1}(g, l_j, r_\beta), f^{-1}(g', l_t, r_\delta)) = (f^{-1}(\varphi_{j\beta}(u), l_j, r_\beta), f^{-1}(\varphi_{j\delta}(v), l_j, r_\delta)) = (u, v) \in E$. Thus f^{-1} is a digraph homomorphism. □

Example 3.5 will illustrate this result.

Consider a right group $S = G \times R_m$ where G is a group, and $R_m = \{r_1, r_2, ..., r_m\}$ an n-element right zero semigroup. It is clear that $G \times R_m \cong G \times L_1 \times R_m$ where L_1 is the 1-element left zero semigroup. Hence we get a Cayley graph of a right group is a Cayley graph of a rectangular group. Hence we have the following result.

Corollary 3.2. [1] *Let (V, E) is a digraph. Then (V, E) is a Cayley graph of right group if and only if the following conditions hold:*

(1) there exists a group G and $m \in \mathbb{N}$ such that (V, E) contains m disjoint strong subdigraph Cayley graphs of G $(V_1, E_1), (V_2, E_2), ..., (V_m, E_m)$, and $V_i = \bigcup_{\alpha=1}^{m} V_{i\alpha}$,

(2) *for each $\alpha \in \{1, 2, .., m\}$, there exists a digraph isomorphism $\varphi_\alpha : (V_\alpha, E_\alpha) \to Cay(G, A_\alpha)$, for some $A_\alpha \subseteq G$,*

(3) *for each $\alpha, \beta \in \{1, 2, ..., m\}$, and for each $u \in V_\alpha, v \in V_\beta$, $(u, v) \in E$ if and only if $\varphi_\beta(v) = \varphi_\alpha(u)a$ for some $a \in A_\beta$.*

Consider a rectangular band $S = L_n \times R_m$ where $L_n = \{l_1, l_2, ..., l_n\}$ is a left zero semigroup, and $R_m = \{r_1, r_2, ..., r_m\}$ a right zero semigroup. It is clear that $L_n \times R_m \cong G \times L_n \times R_m$ when $G = \{e\}$ is the trivial group. Hence we have the following result.

Corollary 3.3. [1] *Let (V, E) is a digraph. Then (V, E) is a Cayley graph of left group if and only if the following conditions hold:*

(1) *(V, E) is the disjoint union of n isomorphic subdigraphs (V_1, E_1), (V_2, E_2),..., (V_n, E_n) for some $n \in \mathbb{N}$,*

(2) *there exists a group G such that (V_i, E_i), $i \in \{1, 2, .., n\}$, are strong subdigraph Cayley graphs of G,*

(3) *there exists a digraph isomorphism $\varphi_i : (V_i, E_i) \to Cay(G, A_i)$, for some $A_i \subseteq G$, and $A_j = A_k$ for all $j, k \in \{1, 2, ..., n\}$,*

(4) *for each $\alpha, \beta \in \{1, 2, ..., m\}$, and $u, v \in V_i$, $(u, v) \in E$ if and only if $\varphi_i(v) = \varphi_i(u)a$ for some $a \in A_i$.*

Consider a rectangular band $S = L_n \times R_m$ where $L_n = \{l_1, l_2, ..., l_n\}$ is a left zero semigroup, and $R_m = \{r_1, r_2, ..., r_m\}$ a right zero semigroup. It is clear that $L_n \times R_m \cong G \times L_n \times R_m$ when $G = \{e\}$ is the trivial group. Hence we have the following result.

Corollary 3.4. *Let (V, E) is a digraph. Then (V, E) is a Cayley graph of rectangular band if and only if the following conditions hold:*

(1) *(V, E) is the disjoint union of n isomorphic subdigraphs (V_1, E_1), (V_2, E_2),..., (V_n, E_n) for some $n \in \mathbb{N}$,*

(2) *there exists $m \in \mathbb{N}$ such that (V_i, E_i), $i \in \{1, 2, .., n\}$, contains m disjoint strong subdigraphs $(\{v_{i1}\}, E_{i1}), (\{v_{i2}\}, E_{i2}), ..., (\{v_{im}\}, E_{im})$ and $V_i = \{v_{i1}, v_{i2}, ..., v_{im}\}$.*

(3) *for each $\alpha \in \{1, 2, .., m\}$, $|E_{i\alpha}| = |E_{j\alpha}|$ for all $i, j \in \{1, 2, ..., n\}$.*

(4) *for each $i \in \{1, 2, .., n\}$, $\alpha, \beta \in \{1, 2, ..., m\}$, and for each $u \in V_{i\alpha}, v \in V_{i\beta}$, $(u, v) \in E$ if and only if $(v, v) \in E_{i\beta}$.*

Example 3.6 will illustrate this result.

Example 3.5. *Consider the rectangular group $S = \mathbb{Z}_4 \times L_2 \times R_3$ where $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ denotes the 4-element cyclic group, $L_2 = \{l_1, l_2\}$ the 2-element left zero semigroup, and $R_3 = \{r_1, r_2, r_3\}$ the 3-element right zero semigroup. For any element $(g, l, r) \in S$, we may write $(g, l, r) = glr$. Let $A = \{(\bar{1}, l_1, r_1), (\bar{2}, l_2, r_2)\}$. Then we give the Cayley graph $Cay(S, A)$.*

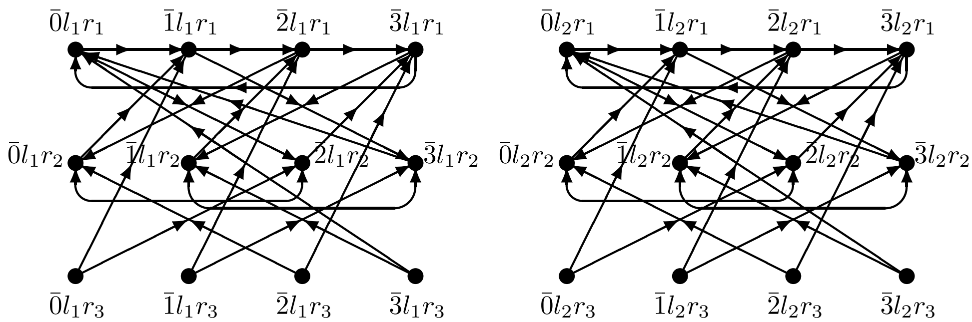

Fig. 1.

From the picture, we have

(1) $Cay(S, A)$ is the union of two isomorphic subdigraphs $((\mathbb{Z}_4 \times \{l_1\} \times R_3), E_1)$ and $((\mathbb{Z}_4 \times \{l_2\} \times R_3), E_2)$.

(2) For each $i \in \{1, 2\}$, $((\mathbb{Z}_4 \times \{l_i\} \times R_3), E_i)$ contains three strong subdigraph Cayley graphs of $\mathbb{Z}_4$
$((\mathbb{Z}_4 \times \{l_i\} \times \{r_1\}), E_{i1}) \cong Cay((\mathbb{Z}_4 \times \{l_i\} \times \{r_1\}), \{(1, l_i, r_1)\}) \cong Cay(\mathbb{Z}_4, \{\bar{1}\})$,
$((\mathbb{Z}_4 \times \{l_i\} \times \{r_2\}), E_{i2}) \cong Cay((\mathbb{Z}_4 \times \{l_i\} \times \{r_2\}), \{(2, l_i, r_2)\}) \cong Cay(\mathbb{Z}_4, \{\bar{2}\})$,
and $((\mathbb{Z}_4 \times \{l_i\} \times \{r_3\}), E_{i3}) \cong Cay((\mathbb{Z}_4 \times \{l_i\} \times \{r_3\}), \emptyset) \cong Cay(\mathbb{Z}_4, \emptyset)$.

(3) From (2), we have $A_{12} = A_{22} = \{\bar{2}\}$, $A_{13} = A_{23} = \emptyset$, $A_{11} = A_{21} = \{\bar{1}\}$, and $p_1 = \varphi_{i\alpha} : ((\mathbb{Z}_4 \times \{l_i\} \times \{r_\alpha\}), E_{i\alpha}) \to Cay(\mathbb{Z}_4, A_{i\alpha})$ is a digraph isomorphism for all $i \in \{1, 2\}$ and $\alpha \in \{1, 2, 3\}$.

(4) We see that $((g, l_i, r_\alpha), (g', l_j, r_\beta))$ is an arc in $Cay(S, A)$ if and only if $g' = ga$ for some $a \in A_{j\beta}$. For example, we have $((\bar{1}, l_1, r_3), (\bar{3}, l_1, r_2))$ is an arc in $Cay(S, A)$, $\bar{3} = \bar{1} + \bar{2}$, and $\bar{2} \in A_{12}$.

Example 3.6. *Consider the rectangular band $S = L_4 \times R_3$ where $L_4 = \{l_1, l_2, l_3, l_4\}$ the 4-element left zero semigroup, and $R_3 = \{r_1, r_2, r_3\}$ the 3-element right zero semigroup. Let $A = \{(l_1, r_1), (l_2, r_2)\}$. Then we give the Cayley graph $Cay(S, A)$.*

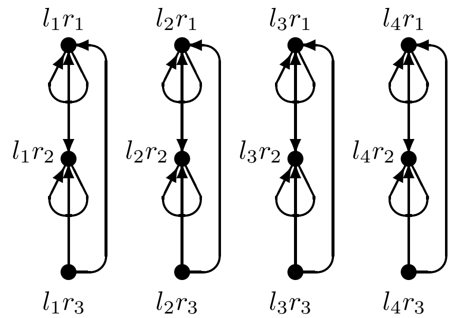

Fig. 2.

From the picture, we have

(1) $Cay(S, A)$ is the union of four isomorphic subdigraphs $((\{l_1\}\times R_3), E_1)$,$((\{l_2\}\times R_3), E_2)$, $((\{l_3\} \times R_3), E_3)$, and $((\{l_4\} \times R_3), E_4)$.

(2) For each $i \in \{1, 2, 3, 4\}$, $((\{l_i\} \times R_3), E_i)$ contains three strong subdigraphs $(\{l_ir_1\}, E_{i1})$, $(\{l_ir_2\}), E_{i2})$, $(\{l_ir_3\}), E_{i3})$, where $E_{i1} = \{(l_ir_1, l_ir_1)\}$, $E_{i2} = \{(l_ir_2, l_ir_2)\}$, and $E_{i3} = \emptyset$.

(3) From (2), we have $|E_{11}| = |E_{21}| = |E_{31}| = |E_{41}| = 1$, $|E_{12}| = |E_{22}| = |E_{32}| = |E_{42}| = 1$, $|E_{13}| = |E_{23}| = |E_{33}| = |E_{43}| = 0$.

(4) We see that $((l_i, r_\alpha), (l_j, r_\beta))$ is an arc in $Cay(S, A)$ if and only if $((l_j, r_\alpha), (l_j, r_\beta)) \in E_{j\beta}$. For example, we have $((l_1, r_3), (l_1, r_1))$ is an arc in $Cay(S, A)$, and $((l_1, r_1), (l_1, r_1)) \in E_{11}$.

Acknowledgements : This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

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(Received 3 August 2010)

S. Panma
Department of Mathematics,
Faculty of Science,
Chiang Mai University,
Chiang Mai 50200, Thailand, and
Centre of Excellence in Mathematics,
CHE, Si Ayutthaya Rd.,
Bangkok 10400, Thailand
e-mail : `panmayan@yahoo.com`