Algebra Problems

July 1998

1. If H_1 and H_2 are subgroups of the abelian group G such that $H_1 \subseteq H_2$, prove that $H_1 + H_2 = H_2$

2. Suppose that H_1 and H_2 are subgroups of the abelian group G such that $G = H_1 \oplus H_2$ If K is a subgroup of G such that $K \supseteq H_1$, prove that $K = H_1 \oplus (K \cap H_2)$.

3. Assume that $H_1, H_2, \ldots, H_n$ are subgroups of the abelian group G such that the sum $H_1 + H_2 + \ldots + H_n$ is direct. If K_i is a subgroup of H_i for n = 1, 2, ..., n, prove that $K_1 + K_2 + \ldots + K_n$ is a direct sum.

4. Prove that if each H_i is a subgroup of the abelian group G, then $H_1 + H_2 + \ldots + H_n$ is the smallest subgroup of G that contains all the subgroups H_i.

5. If $H_1, H_2, \ldots, H_n$ are subgroups of the abelian group G, prove that $G = H_1 + H_2 + \ldots + H_n$ if and only if G is generated by $\bigcup_{i=1}^{n} H_i$.

6. Let G be an abelian group of order mn where m and n are relatively prime. If $H_1 = \{x \in G \mid mx = 0\}$ and $H_2 = \{x \in G \mid nx = 0\}$, prove that $G = H_1 \oplus H_2$.

7. Let H_1 and H_2 be cyclic subgroups of the abelian group G where $H_1 \cap H_2 = \{0\}$. Prove that $H_1 \oplus H_2$ is cyclic if and only if $o(H_1)$ and $o(H_2)$ are relatively prime.

8. Assume that a is an element of order $r_1 r_2 \ldots r_n$ in an abelian group where r_i and r_j are relatively prime if $i \neq j$. Prove that a can be written in the form $a = b_1 + b_2 + \ldots + b_n$ where each b_i has order r_i.

9. Prove that if r and s and relatively prime positive integers, then any cyclic group of order rs is the direct sum of a cyclic group of order r and a cyclic group of order s.

10. Assume that G can be written as the direct sum $G = C_2 \oplus C_2 \oplus C_3 \oplus C_3$ where C_n is a cyclic group of order n.

    a. Prove that G has element of order 6, but no element of order greater than 6.

    b. Fine the number of distinct elements of G that have order 6.

11. For each of the following values of n, describe all the abelian groups of order n, up to isomorphism.

    a. n = 6, b. n = 12, c. n = 36.

12. Prove that a regular permutation can be expressed as the power of a cycle and that, conversely, if $\gamma = (1, 2, ..., m)$, then γ^s is a regular permutation consisting of d cycles of degree r , where d = g.c.d. (m, s) and r = m/d.

13. The **left regular representation** of a group G is defind as follows : corresponding to a fixed element u of G there is a permutation λ_u, acting on the elements of G in accordance with the rule $x\lambda_u = u^{-1}x (x \in G)$. Verify that

    (i) $\lambda_u \lambda_v = \lambda_{uv}$;

    (ii) $\lambda_u = \iota$ if and only if u = 1 ;

    (iii) $\lambda_u \rho_a = \rho_a \lambda_u$, where , for each $a \in G$, the function $\rho_a : G \rightarrow S_G$ from G into the symmetric group S_G of G sending x to $xa (x \in G)$;

    (iv) If θ is a permutation of the elements of G which commutes with all the λ_u, then $\theta = \rho_a$ for some a ; and if η commutes with all the ρ_a then $\eta = \lambda_u$ for some u.

14. Prove that if G is a simple group of order 168 and H is a proper subgroup of G, then $[G : H] \geq 7$.

15. Prove that when the non-trivial elements of a transitive group G of degree n are written as products of mutually exclusive cycles of degrees greater than one, then they involve between them $(n-1)\,|G|$ letters.

16. Let D_m be the dihedral group of order 2m where $m > 2$. Show that the centre of D_m has one or two elements according as n is odd or even.

17. Let x, y be two elements of order 2, which generate a group G, and suppose that xy has order $m \geq 3$. Show that G is isomorphic with D_m.

18. Show that A_4 has one Sylow subgroup of order 4 and four Sylow subgroups of order 3

19. Prove that there is no simple group of order 56.

20. Let G be a group of order p^2q where p and q are primes such that q is less than p and is not a factor of p^2-1. Prove that G is Abelian.

21. Let P be a Sylow subgroup of G which is normal in G. Prove that P is a characteristic subgroup of G.

22. Show that a normal p-subgroup is contained in every Sylow p-subgroup.

23. Let P be a Sylow p-subgroup of a finite group G and suppose that H is a normal subgroup of G. Prove that

(i) HP/H is a Sylow p-subgroup of G/H and

(ii) $H \cap P$ is Sylow p-subgroup of H.

24. Show that if the order of a finite Abelian group is not divisible by a square (>1), then the group is cyclic.

25. Prove that in a finite Abelian group

(i) the maximal order of an element is equal to the greatest invariant and

(ii) the order of any element divides the maximal order.

26. Show that the (multiplicative) group of residue classes coprime with 24 is elementary Abelian of order 8.

27. Find the elementary divisors and invariants of the following Abelian groups defined by generators and relations : (i) $15a = 4b = 0$, (ii) $20a = 6b = 5c = 0$.

28. The Abelian group A is generated by a, b, c with the defining relations $3a + 9b + 9c = 0$, $6a - 12b = 0$. Express A as a direct sum of cyclic groups.

29. Find the rank and invariants of the following Abelian groups :(i) with generators a, b and relation $2(a+b) = 0$; (ii) with generators a, b, c, d and relations $3a + 5b - 3c = 0$, $4a + 2b - 2d = 0$

30. The free Abelian group F is generated by u_1, u_2, u_3 and R is the subgroup generated by

$$r_1 = ku_1 + u_2 + u_3, \quad r_2 = u_1 + ku_2 + u_3, \quad r_3 = u_1 + u_2 + ku_3$$

where k is an integer greater than one. Find generator v_1, v_2, v_3 of F and s_1, s_2, s_3 or R such that $s_i = e_i v_i (i = 1,2,3)$ and e_1, e_2, e_3 are integers satisfying $e_1 | e_2 | e_3$.

31. Show that the derived group of a free group consists of those words in which the sum of the exponents for each generator is equal to zero (for example $x_1 x_2^{-1} x_1^{-2} x_2 x_1$).

32. Let F be the free group generated by $x_1, x_2, \ldots, x_r$. Show that each element of F/F' is of the form $\left(x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r}\right)F'$

Prove that F/F' is a free Abelian group of rank r.

33. Determine the structure of G/G', when G is given by

(i) $a^6 = b^2 = (ab)^2 = 1$; (ii) $a^6 = 1, b^2 = (ab)^2 = a^3$.

34. Let D_n be the dihedral group defined by $a^n = b^2 = (ab)^2 = 1$.

Find the structure of D_n / D_n' (i) when n is odd; (ii) when n is even.

35. Consider the free group F(X) andF(X′) in the case where X = {x,y} and X′ = {x,z} $(z = x^{-1}yx)$. Find the lengths of the elements y and $x^{-1}yx$ with respect to X and X′, respectively. Here F(X) is the free group on a generating set X.

36. Find a composition series (i) for the dihedral group of order 8 and (ii) for the quaternion group. Determine the composition factors in each case.

37. Prove that every subgroup and quotient group of soluble group is soluble.

38. Show that if G is nilpotent of class 2, then G' lies in the centre of G and deduce the identities.

$$[xy, z] = [x, z]\,[y, z], \; [x, yz] = [x,z]\,[x,y]$$

for such a group.

39. Prove that every subgroup and factor group of nilpotent group is nilpotent.

40. Let G be nilpotent of class 3. Show that, if $v \in G'$ and $x \in G$, then $x^v = cx$, where $c \in Z$, the centre of G. Deduce that G' is Abelian.

41. Prove that if M is a maximal subgroup of a nilpotent group G, then $M \triangleleft G$ and $|G/M| = p$, where p is a prime. (A maximal subgroup is a proper subgroup which is not contained in any other proper subgroup. Infinite groups need not possess maximal subgroups.)

42. Let $m = 2^n$ $(n \geq 2)$ and consider the dihedral group D_m of order 2m given by

$$a^m = b^2 = (ab)^2 = 1.$$

Prove that if Z is the centre of D_m, then $D_m/Z \simeq D_{m/2}$.

Deduce that D_m is nilpotent of class n.

43. Prove that $\mathbb{Z}_n$ is a field if and only if n is prime.

44. Prove that if D is an integral domain, then D is of characteristic o or p , a prime number.

45. In $\mathbb{Z}$, prove that I is a maximal ideal if and only if I generates by a prime number.

46. Show that every finite integral domain is a field.

47. Prove that every integral domain can be embedded in a field.

48. Let R, R' be rings and ϕ a homomorphism of R onto R' with kernel U. Prove that

 (1) $R' \cong R/U$

 (2) There is a one-to-one correspondence between the set of ideals of R' and the set of ideals of R which contain U.

49. If $e = e^2$ is an idempotent in a ring R , write $eRe = \{ere \mid r \in R\}$. Prove that

 (1) eRe is a ring with identity e and $eRe = \{a \in R \mid ea = a = ae\}$.

 (2) If $S \subseteq R$, then S is a ring using the operations of R if and only if S is a subring of eRe for some $e^2 = e \in R$.

50. Prove that if R is a division ring, then $M_n(R)$ is simple where $M_n(R)$ is the set of all $n \times n$ matrices with entries from R.

51. Prove that an ideal M of a ring R is maximal if and only if R/M is simple.

52. Let R be a commutative ring and suppose that A is an ideal of R.

Let $N(A) = \{x \in R \mid x^n \in A \;\exists\, n \in \mathbb{N}\}$. Prove

 (1) N(A) is an ideal of R which contains A

 (2) $N(N(A)) = N(A)$

53. If R is a ring, let $Z(R) = \{x \in R \mid xy = yx \text{ all } y \in R\}$.

Prove that

 (1) Z(R) is a subring of R;

 (2) If R is a division ring, then Z(R) is a field.

54. If L is a finite extension of K and if K is a finite extension of F, prove that L is a finite extension of F , and $[L : F] = [L : K]\,[K : F]$.

55. If L is an algebraic extension of K and if K is an algebraic extension of F, prove that L is an algebraic extension of F.

56. Prove that any two finite fields having the same number of elements are isomorphic.

57. Let F be a finite field. Prove that

(1) If F has q elements and $F \subset K$ where K is also a finite field, then K has q^n elements when $n = [K : F]$.

(2) F has p^m elements where the prime number p is the characteristic of F.

58. Prove that for every ring R there exists a ring R^* with unit such that R is isomorphic to a subring of R^*.

59. Prove that for a ring R, R is simple and $Z(R) \neq 0$ if and only if R has unit and 0 is a maximal ideal.

60. Let I, J be ideals in the ring R. Prove that if $I \subset J$, then J/I is an ideal in R/I and there is a ring isomorphism $(R/I)/(J/I) \cong R/J$.