Numerical Optimization

A Workshop
At
Department of Mathematics
Chiang Mai University
August 4-15, 2009

Instructor: **Vira Chankong**

Electrical Engineering and Computer Science
Case Western Reserve University

Phone: 216 368 4054, Fax: 216 368 3123
E-mail: **vira@case.edu**

Session:

Optimization of Dynamic Systems

Vira Chankong
Case Western Reserve University
Electrical Engineering and Computer Science

Module Objective

To develop functional knowledge and skills in

- modeling dynamic systems and/or optimal control problems
- determining a state trajectory and/or a control of dynamic systems that optimizes a performance functional subject to constraints on control and/or states.

Module Highlights

Two pronged approach:

1. Focus on the two traditional approaches for dealing with dynamic optimization/optimal control problems
   - the variational approach based on calculus of variations leading to the maximum principle
   - the dynamic programming approach and its corresponding Hamilton-Jacobi-Bellman (HJB) equation. When applied to linear optimal control problem derivation of similar results based on the concept of Lyapunov stability will also be demonstrated.
2. Focus on numerical methods for solving large real-world dynamic optimization and optimal control problems with complex constraints. The two numerical approaches are the indirect approach and the direct approach. **MATLAB** will be used to implement methods discussed.

   Applications to engineering and economic problems will be illustrated throughout

References

Text:

Optimal Control Theory, D.E. **Kirk**, Dover Publications, ISBN: 0486434842, 2004

References:

1. *Practical Methods for Optimal Control Using Nonlinear Programming*, J.T. **Betts**, SIAM, ISBN: 0898714885, 2001

2. *Applied Dynamic Programming for Optimization of Dynamical Systems*, R.D. **R**obinett III, D.G. **W**ilson, G. Richard **E**isler and J. E. **H**urtado, SIAM, ISBN: 089715865, 2006

Dynamic Optimization

$$\min_{x:[t_0,t_f]\to R} J = \int_{t_0}^{t_f} g(x,\dot{x},..,x^{(n)},t)dt$$

subject to $t_0,\ x(t_0)=x_0,\ \dot{x}(t_0)=\dot{x}_0,..,x^{(n)}(t_0)=x_0^{(n)}$

and possibly $t_f,\ x(t_f)=x_f,\ \dot{x}(t_f)=\dot{x}_f,..,x^{(n)}(t_f)=x_f^{(n)}$

Note:

1) The final time t_f may or may not be specified.

If t_f is specified $\Rightarrow$ fixed-end-time problem

(or fixed-terminal-time)

If t_f is not specified $\Rightarrow$ variable-end-time problem

(or open-terminal-time, open horizon)

2) If t_f is specified, and if

$x(t_f)$ is also fixed $\Rightarrow$ fixed-end-point problem

$x(t_f)$ is constrained (i.e. $x(t_f)\in S$) $\Rightarrow$ constrained-terminal-point problem

$x(t_f)$ is not fixed or has no restriction $\Rightarrow$ free-end-point problem

Optimal Control Problems

$$\min_{\mathbf{u}:\,[t_0,t_f]\to R^m} J = h(\mathbf{x}(t_f),t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t),\mathbf{u}(t),t)dt$$

subject to

System Dynamics:

$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t),\mathbf{u}(t),t);$ where state $\mathbf{x}: [t_0,t_f] \to R^n$

Initial conditions: $\mathbf{x}(t_0) = \mathbf{x}_0$

Terminal or final conditions: $\mathbf{x}(t_f) = \mathbf{x}_f$

Find a control $\mathbf{u}(t)$ to take the system from the initial state $\mathbf{x}(t_0)$ at the initial time t_0 to the final state $\mathbf{x}(t_f)$ at the final time t_f.

Optimal Control Problems

Possible additional constraints:

Contraints on control:

$\mathbf{u}(t) \in U_t$ for all $t \in [t_0,t_f]$

e.g. $\mathbf{u}_{\min} \le \mathbf{u}(t) \le \mathbf{u}_{\max}$ for all $t \in [t_0,t_f]$

Contraints on state:

$\mathbf{x}(t) \in X_t$ for all $t \in [t_0,t_f]$

e.g. $\mathbf{x}_{\min} \le \mathbf{x}(t) \le \mathbf{x}_{\max}$ for all $t \in [t_0,t_f]$

Optimal Control Problems

Note:

1) As before, t_f may be specified (fixed-end-time)

or may not specified (variable-end-time

or open terminal time)

2) If t_f is specified,

$x(t_f)$ may be fixed (fixed-end-point or hard terminal constraint)

or constrained $x(t_f) \in S$ (soft terminal constraint)

or not fixed or no restriction (free-end-point)

6-Aug-09 Vira Chankong EECS, CWRU 9

Optimal Control Problems

Various types of J :

- g = 0, i.e. $J = h(\mathbf{x}(t_f), t_f)$ ------------------Mayer Problem

Special Mayer Problems:

(a) $J = c\mathbf{x}(t_f)$ (Linear Mayer)

(b) $J = \left(\mathbf{x}(t_f)\text{-}\mathbf{r}(t_f)\right)^T \mathbf{H}\left(\mathbf{x}(t_f)\text{-}\mathbf{r}(t_f)\right)$ (Terminal control problem)

- h = 0, i.e. $J = \int_{t_0}^{t_f} g(\mathbf{x}(t),\mathbf{u}(t),t)dt$--------Lagrange Problem

Special Lagrange Problems:

(a) g = 1, i.e. $J = \int_{t_0}^{t_f} 1dt = t_f\text{-}t_0$ (Minimum time)

(b) $g = \|\mathbf{u}(t)\|$, i.e. $J = \int_{t_0}^{t_f} \|\mathbf{u}(t)\| dt$ (Minimum fuel)

(c) $g = \|\mathbf{u}(t)\|^2$, i.e. $J = \int_{t_0}^{t_f} \mathbf{u}(t)^T \mathbf{u}(t) dt$ (Minimum energy)

6-Aug-09 Vira Chankong EECS, CWRU 10

Optimal Control Problems

- $J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)dt$--------Bolza Problems

Special Bolza Problems:

(a) Tracking problem:

$$h = (\mathbf{x}(t_f)-\mathbf{r}(t_f))^T \mathbf{H}(\mathbf{x}(t_f)-\mathbf{r}(t_f))$$

$$g = (\mathbf{x}(t)-\mathbf{r}(t))^T \mathbf{Q}(\mathbf{x}(t)-\mathbf{r}(t)) + \mathbf{u}(t)^T \mathbf{R}\mathbf{u}(t)$$

$$\Rightarrow J = (\mathbf{x}(t_f)-\mathbf{r}(t_f))^T \mathbf{H}(\mathbf{x}(t_f)-\mathbf{r}(t_f)) + \int_{t_0}^{t_f} (\mathbf{x}(t)-\mathbf{r}(t))^T \mathbf{Q}(\mathbf{x}(t)-\mathbf{r}(t)) + \mathbf{u}(t)^T \mathbf{R}\mathbf{u}(t)dt$$

Note: **H**, **Q** and **R** are weighting matrices

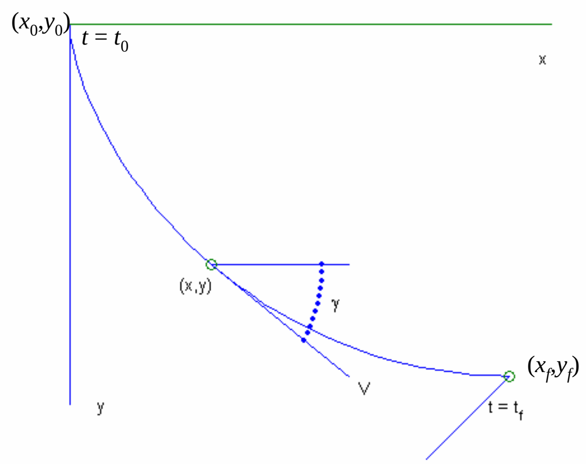
(b) Regulator problem:

$$h = \mathbf{x}(t_f)^T \mathbf{H}\mathbf{x}(t_f)$$

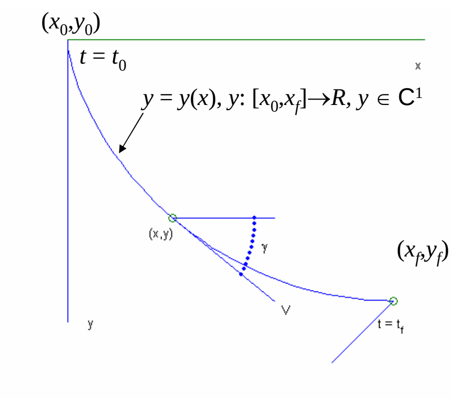
$$g = \mathbf{x}(t)^T \mathbf{Q}\mathbf{x}(t) + \mathbf{u}(t)^T \mathbf{R}\mathbf{u}(t)$$

$$\Rightarrow J = \mathbf{x}(t_f)^T \mathbf{H}\mathbf{x}(t_f) + \int_{t_0}^{t_f} \mathbf{x}(t)^T \mathbf{Q}\mathbf{x}(t) + \mathbf{u}(t)^T \mathbf{R}\mathbf{u}(t)dt$$

Example 1: (Brachistochrome) A bead of mass 1 unit descends along a wire joining two fixed points (x_0, y_0) and (x_f, y_f). We wish to find the shape of the wire so that the bead completes its slide in minimum time.



Example 1: (Brachistochrome) Model:



Dynamic Optimization Model:

$$\min t_f = \int_{x_0}^{x_f} \sqrt{\frac{1+y'(x)}{2g\left(y_0 - y(x)\right)}}\,dx$$

subject to

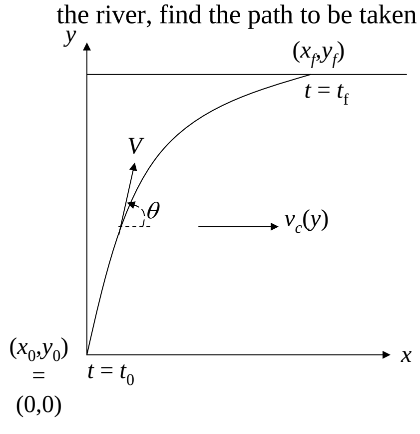
$$y(x_0) = y_0$$
$$y(x_f) = y_f$$

Minimum time

Fixed terminal point problem

(Hard terminal constraint)

Example 2: (River-Crossing 1) A boat travels with constant velocity V with respect to the water. In the region the velocity of the current is parallel to x-axis but varies with y. Given the destination (x_f,y_f) on the other side of the river, find the path to be taken by the boat to minimize the travel time,



Optimal Control Model:

$$\min t_f = \int_{t_0}^{t_f} 1dt + t_0$$

subject to

Dynamics (Equation of Motion):

$$\dot{x} = V\cos\theta + v_c(y)$$
$$\dot{y} = V\sin\theta$$

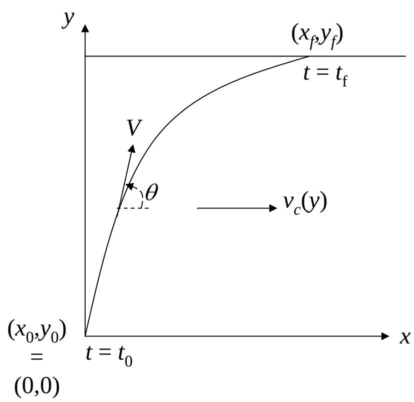
Initial conditions: $x(t_0) = x_0; y(t_0) = y_0$

Final conditions: $x(t_f) = x_f; y(t_f) = y_f$

Minimum time

Fixed terminal point problem

Example 3: (River-Crossing 2) A boat travels with constant velocity V with respect to the water. In the region the velocity of the current is parallel to x-axis but varies with y. Given the final time t_f, find the path to be taken by the boat to maximize the landing distance on the other side of the river,



Optimal Control Model:

$$\max J = x(t_f)$$

subject to

Dynamics (Equation of Motion):

$$\dot{x} = V\cos\theta + v_c(y)$$

$$\dot{y} = V\sin\theta$$

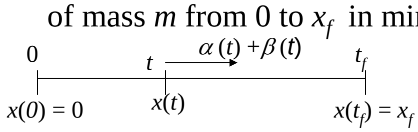
Initial conditions: $x(t_0) = x_0;\ y(t_0) = y_0$

Final conditions: $y(t_f) = y_f$

Linear Mayer problem with semi-hard terminal constraint

Example 4: (Braking and Acceleration): Want to move a car of mass m from 0 to x_f in minimum time.

Minimum time
Hard terminal constraint
with bound constraints



Optimal Control:

$$\min J = t_f = \int_0^{t_f} 1\,dt$$

subject to

Dynamics (Equation of Motion):

$$\dot{x}_1 = x_2(t)$$

$$\dot{x}_2 = \alpha(t) + \beta(t)$$

Initial conditions: $x_1(0) = 0;\ x_2(0) = 0$

Terminal conditions: $x_1(t_f) = x_f;\ x_2(t_f) = 0$

Contraints: For each $t \in [0, t_f]$:

On control: $0 \le u_1(t) \le M_1;\ 0 \le u_2(t) \le M_2$

On state: $0 \le x_1(t) \le x_f;\ 0 \le x_2(t)$

EOM: $\ddot{x}(t) = \alpha(t) + \beta(t)$

States: $x_1(t) = x(t)$

$x_2(t) = \dot{x}_1(t) = \dot{x}(t)$

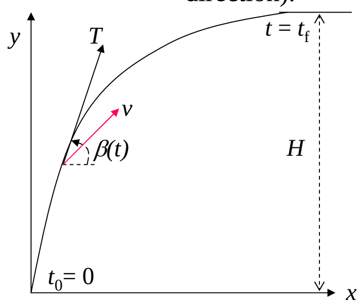
Compact Dynamics:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\ \mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$$

Example 5: (**Taking-off**) An aircraft of mass m (assumed point mass) is to be lifted by a constant trust T to reach the cruising altitude H at time t_f, at maximum speed (along x-direction).



Optimal Control Model:

$$\max\ J = x_2(t_f)$$

Dynamics (Equation of Motion):

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{T}{m}\cos\beta$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{T}{m}\sin\beta - g$$

EOM: $m\ddot{x}(t) = T\cos\beta(t)$

$m\ddot{y}(t) = T\sin\beta(t) - mg$

States: $x_1(t) = x(t); x_2(t) = \dot{x}_1(t) = \dot{x}(t)$

$x_3(t) = y(t); x_4(t) = \dot{x}_3(t) = \dot{y}(t)$

Control $u(t) = \beta(t)$

Initial conditions: $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0$

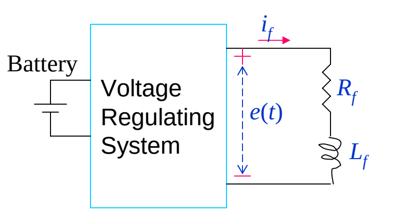
Final conditions: $x_3(t_f) = H; x_4(t_f) = 0$

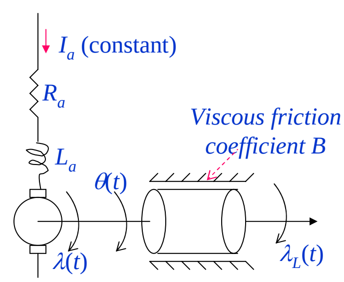
Constraints: On control $0 \le u(t) \le \pi/2,\ t \in [0, t_f]$

On state: $x_i(t) \ge 0,\ i = 1,..4;\ x_3(t) \le H$

Linear Mayer problem with semi-hard terminal constraint

Example 6: (**Rover Control**) Want to control the speed of a Mariner Mars rover at about 5 mph using as little energy as possible. The controller is the output voltage of a battery-operated voltage regulating system.





Dynamics: $R_f i_f(t) + L_f \frac{di_f(t)}{dt} = e(t)$

$$\lambda(t) = K_t i_f(t) --- torque$$

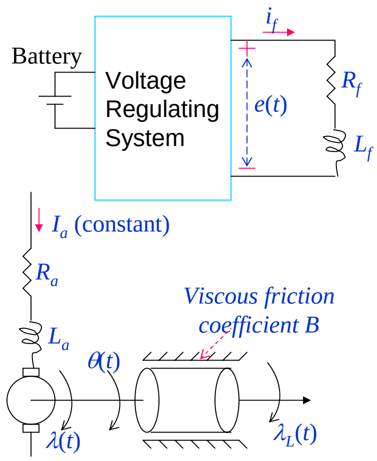
$$\lambda(t) = I\ddot{\theta}(t) + B\dot{\theta}(t) + \lambda_L(t)$$

States: $x_1(t) = i_f(t);\ x_2(t) = \dot{\theta}(t)$

Controls: $u_1(t) = e(t);\ u_2(t) = \lambda_L(t)$

Example 6: (Rover Control)

Min $J = \int_0^{t_f} \left(k(x_2(t)-5)^2 + w x_1(t) u_1(t) \right) dt$



Dynamics:

$$\dot{x}_1 = \frac{R_f}{L_f} x_1(t) \qquad + \frac{1}{L_f} u_1(t)$$

$$\dot{x}_2 = \frac{K_t}{I} x_1(t) - \frac{B}{I} x_2(t) - \frac{1}{I} u_2(t)$$

Initial conditions: $x_1(0) = x_2(0) = 0$

Final conditions: None

Constraints:

On control: $|u_1(t)| \le e_{\max},\ t \in [0, t_f]$

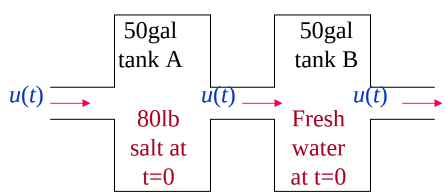
$|u_2(t)| \le \lambda_{\max},\ t \in [0, t_f]$

On state: $|x_1(t)| \le I_{\max},\ t \in [0, t_f]$

$|x_2(t)| \le \Omega_{\max},\ t \in [0, t_f]$

Minimum energy problem with free terminal conditions but with bound constraints

Example 7: (Mixture) Want to find inflow rate of fresh water (control) to the two-tank system so that the salt concentration in tanks A and B are equal using minimum amount of fresh water.



Optimal Control Model:

min $J = \int_0^{t_f} u(t)\,dt$

Assume: 1) Well-mixed at all time

2) Incompressible fluid

States: $x_1(t) =$ lbs of salt in A at time t

$x_2(t) =$ lbs of salt in B at time t

Controls: $u(t) =$ gal/m of fresh water passing thru at time t

Dynamics:

$$\dot{x}_1 = -\frac{u(t)}{50} x_1(t)$$

$$\dot{x}_2 = \frac{u(t)}{50} x_1(t) - \frac{u(t)}{50} x_2(t)$$

Initial conditions: $x_1(0) = 80;\ x_2(0) = 0$

Terminal conditions: $x_1(t_f) - x_2(t_f) = 0$

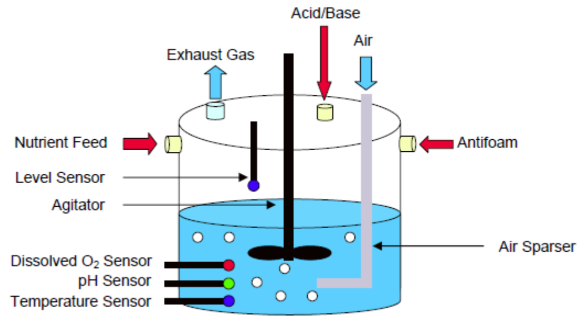
Contraints: For each :

On control: $0 \le u(t) \le u_{\max},\ t \in [0, t_f]$

On state: $0 \le x_1(t) \le 80;\ 0 \le x_2(t)$

Minimum control effort with Soft terminal constraint and bound constraints

Optimizing Yeast or Ethanol Production in a Bioreactor



Bioreactors are large vessels that serve as an environment for biochemical reactions to occur. Typical uses include the growth of microorganisms and the breakdown of products.

Source: D. Moore, MS Thesis, EECS, CWRU, 2007

The environment within the vessel is controlled to optimize performance. Typical control variables include nutrient feed rate, oxygen air flow rate, and temperature. There is large economic incentive to develop control strategies to maximize the production of baker's yeast and ethanol, two important commercial products produced in bioreactors. Yeast is typically grown off a solution containing glucose and other nutrients essential for cellular growth. When glucose concentration in the medium is high or when there is a limited supply of oxygen, the yeast microorganism excretes ethanol.

Optimizing Yeast or Ethanol Production in a Bioreactor

$$\frac{dX}{dt} = (\mu_1 + \mu_2 + \mu_3)X - DX \qquad X(0) = 0.1$$

$$\frac{dS}{dt} = -(k_1\mu_1 + k_2\mu_2)X + D(S_{in} - S) - mX \qquad S(0) = 0.02$$

$$\frac{dE}{dt} = (k_3\mu_2 - k_4\mu_3)X - DE \qquad E(0) = 0.15$$

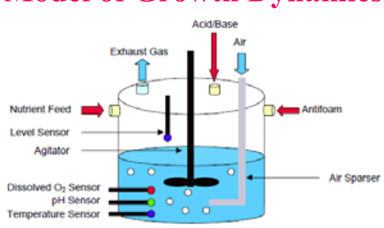
$$\frac{dO}{dt} = -(k_5\mu_1 + k_6\mu_3)X - DO + k_L a(O_S - O) \qquad O(0) = 0.0066$$

$$\frac{dC}{dt} = (k_7\mu_1 + k_8\mu_2 + k_9\mu_3)X - DC - k_V k_L aC \qquad C(0) = 0.008$$

$$\frac{dV}{dt} = F_{in} \qquad V(0) = 3.5$$

We wish to maximize production of yeast, or ethanol or both by controlling the substrate feed rate, airflow (O_2) and temperature

Model of Growth Dynamics



X = yeast concentration (g/l)
S = substrate (glucose) concentration (g/l)
E = ethanol concentration (g/l)
O = dissolved oxygen (O_2) concentration (g/l)
C = dissolved carbon dioxide (CO_2) concentration (g/l)
V = liquid volume (l)
F_{in} = Substrate feed rate (l/h)
S_{in} = Influent Substrate Concentration (g/l)
$D = F_{in}/V$ = Dilution rate (1/h)
$OTR = k_L a(O_S - O)$ = O_2 transfer rate (g/L h^{-1})
$CER = k_V k_L aC$ = CO_2 evolution rate (g/L h^{-1})
m = Maintenance term (g of S /g of X h^{-1})

Two Ways to Solve Dynamic Optimization/Optimal Control Problems

1. Indirect Method:

- Solve the necessary conditions for optimality derived through variational principles rooted in Calculus of Variations
- That is: Solve two-point boundary value problems (TPBVP)

Two Ways to Solve Dynamic Optimization/Optimal Control Problems

2. Direct Method:

- Optimize the functional directly as constrained optimization
- Require conversion to nonlinear programs through transcription of the ODEs (dynamics of system)
- Often possesses high sparsity and special structure

CASE SCHOOL OF ENGINEERING

Direct Method for Dynamic Optimization

- Convert to nonlinear programs through direct transcription of the ODEs (dynamics of system) or the corresponding DAEs
- Use nonlinear optimizer such SQP to solve the resulting nonlinear programs (Software such as SNOPT by Boeing, etc.
- Fine-tune the result through mesh-refinement techniques

CASE SCHOOL OF ENGINEERING

Direct Method: Transcription methods

Euler: $\mathbf{x}_{k+1} = \mathbf{x}_k + h_k \mathbf{a}(\mathbf{x}_k, \mathbf{u}_k, t_k)$

Classical Runge-Katta: $\mathbf{x}_{k+1} = \mathbf{x}_k + h_k \mathbf{s}_k$

where $\mathbf{s}_k = \frac{1}{6}\left(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4\right)$

$$\mathbf{k}_1 = h_k \mathbf{a}(\mathbf{x}_k, \mathbf{u}_k, t_k)$$

$$\mathbf{k}_2 = h_k \mathbf{a}(\mathbf{x}_k + \frac{1}{2}\mathbf{k}_1, \bar{\mathbf{u}}_{k+1}, t_k + \frac{h_k}{2})$$

$$\mathbf{k}_3 = h_k \mathbf{a}(\mathbf{x}_k + \frac{1}{2}\mathbf{k}_2, \bar{\mathbf{u}}_{k+1}, t_k + \frac{h_k}{2})$$

$$\mathbf{k}_4 = h_k \mathbf{a}(\mathbf{x}_k + \mathbf{k}_3, \bar{\mathbf{u}}_{k+1}, t_{k+1})$$

Trapezoidal: $\mathbf{x}_{k+1} = \mathbf{x}_k + h_k \left(\mathbf{a}(\mathbf{x}_k, \mathbf{u}_k, t_k) + \mathbf{a}(\mathbf{x}_k + h_k \mathbf{a}(\mathbf{x}_k, \mathbf{u}_k, t_k), \mathbf{u}_{k+1}, t_{k+1})\right)$

Hermit-Simpson:

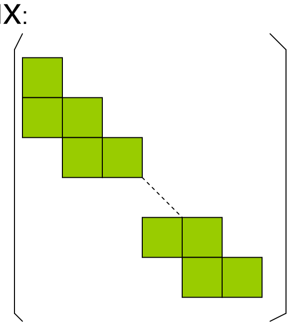
$$\mathbf{x}_{k+1} = \mathbf{x}_k + \frac{h_k}{6}\left(\mathbf{a}(\mathbf{x}_k, \mathbf{u}_k, t_k) + \mathbf{a}(\mathbf{x}_k + h_k \mathbf{a}(\mathbf{x}_k, \mathbf{u}_k, t_k), \bar{\mathbf{u}}_{k+1}, t_{k+1}) + \bar{\mathbf{a}}_{k+1}\right)$$

where $\bar{\mathbf{x}}_{k+1} = \frac{1}{2}\left(\mathbf{x}_k + \mathbf{x}_k + h_k \mathbf{a}(\mathbf{x}_k, \mathbf{u}_k, t_k)\right) + \frac{h_k}{8}\left(\mathbf{a}(\mathbf{x}_k, \mathbf{u}_k, t_k) - \mathbf{a}(\mathbf{x}_k + h_k \mathbf{a}(\mathbf{x}_k, \mathbf{u}_k, t_k), \mathbf{u}_{k+1}, t_{k+1})\right)$

and $\bar{\mathbf{a}}_{k+1} = \mathbf{a}(\bar{\mathbf{x}}_{k+1}, \bar{\mathbf{u}}_{k+1}, t_k + \frac{h_k}{2})$

Direct Method: Transcription methods

Band, Stair-case Structure and Sparsity of resulting matrix:



Employ special numerical tricks to take advantage of the special structure and sparsity of the resulting problem

Indirect Method: Derivation of Optimality Conditions

- Euler-Lagrange Equations and all Boundary Conditions
- Hamilton-Jacobi-Bellman Conditions
- Pontryagin's Minimum Principle

Indirect Method: Fundamentals

Variations of a Functional:

Path optimization:

Functional $J(\mathbf{y}(x))$: where $x \in [x_1, x_2]$ and $\mathbf{y}: [x_1, x_2] \to R^n$

$$\underbrace{\Delta J(\mathbf{y}^*, \delta\mathbf{y})}_{increment} = \underbrace{\delta J(\mathbf{y}^*, \delta\mathbf{y})}_{variation} + \underbrace{o(\|\delta\mathbf{y}\|^2)}_{error\ term}$$

where $o(\|\delta\mathbf{y}\|^2) \to 0$ as $\|\delta\mathbf{y}\| \to 0$

Optimal control:

Function $J(\mathbf{x}(t_f))$ or $J(\mathbf{x}(t_f), t_f)$: where $t \in [t_0, t_f]$ and $\mathbf{x}: [t_0, t_f] \to R^n$

$$\underbrace{\Delta J(\mathbf{x}^*, \delta\mathbf{x})}_{increment} = \underbrace{\delta J(\mathbf{x}^*, \delta\mathbf{x})}_{variation} + \underbrace{o(\|\delta\mathbf{x}\|^2)}_{error\ term}$$

$$\underbrace{\Delta J(\mathbf{x}^*, \delta\mathbf{x}, \delta t_f)}_{increment} = \underbrace{\delta J(\mathbf{x}^*, \delta\mathbf{x}, \delta t_f)}_{variation} + \underbrace{o(\|\delta\mathbf{x}\|^2, |\delta t_f|^2)}_{error\ term}$$

Indirect Method: Fundamentals

1) Key variation formular: $\delta J(\mathbf{x}, \delta\mathbf{x}) = \frac{\partial J}{\partial \mathbf{x}}^T \delta\mathbf{x}$

$$= \frac{\partial J}{\partial x_1}\delta x_1 + \frac{\partial J}{\partial x_2}\delta x_2 + .. + \frac{\partial J}{\partial x_n}\delta x_n$$

For example: $J(\mathbf{x}(t_f)) = \int_{t_0}^{t_f} g(x, \dot{x}, .., x^{(n)}, t)dt$

Then $\delta J(\mathbf{x}, \delta\mathbf{x}) = \frac{\partial J}{\partial x}\delta x + \frac{\partial J}{\partial \dot{x}}\delta \dot{x} + .. + \frac{\partial J}{\partial x^{(n)}}\delta x^{(n)}$

$$= \int_{t_0}^{t_f} \left(\frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial \dot{x}}\delta \dot{x} + .. + \frac{\partial g}{\partial x^{(n)}}\delta x^{(n)} \right) dt$$

2) Term like $\int_{t_0}^{t_f} \left(\frac{\partial g}{\partial \dot{x}}\delta \dot{x} \right) dt$ are dealt with thru integration by part

i.e. $\int_{t_0}^{t_f} \left(\frac{\partial g}{\partial \dot{x}}\delta \dot{x} \right) dt = \left(\frac{\partial g}{\partial \dot{x}}\delta x \right)_{t_0}^{t_f} - \int_{t_0}^{t_f} \left(\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}} \right) \right) \delta x dt$

Indirect Method: Fundamentals

3) Fundamental Theorem of Calculus of Variations:

$\mathbf{x}^*$ is an extremal of $J(\mathbf{x})$ only if $\delta J(\mathbf{x}^*, \delta\mathbf{x}) = 0$ for all admissible $\delta\mathbf{x}$.

4) Fundamental Lemmas of Calculus of Variations:

a) Let $\alpha(x) \in C[a,b]$.

If $\int_a^b \alpha(x)h(x)dx = 0$ for all $h(x) \in C[a,b]$ with $h(a) = h(b) = 0$

then $\alpha(x) = 0$ for all $x \in [a,b]$

b) Let $\alpha(x) \in C^1[a,b]$.

If $\int_a^b \alpha(x)h'(x)dx = 0$ for all $h(x) \in C^1[a,b]$ with $h(a) = h(b) = 0$

then $\alpha(x) = c$ for all $x \in [a,b]$

c) Let $\alpha(x)$ and $\beta(x) \in C^1[a,b]$.

If $\int_a^b \left(\alpha(x)h(x) + \beta(x)h'(x)\right)dx = 0$ for all $h(x) \in C^1[a,b]$ with $h(a) = h(b) = 0$

then $\beta'(x) = \alpha(x)$ for all $x \in [a,b]$

Indirect Method: Necessary Conditions

Now consider $J(x) = \int_{t_0}^{t_f} g(x, \dot{x}, t)dt, \quad x:[t_0, t_f] \to R$

a) Case: t_f and $x(t_f)$ are fixed (fixed end point)

$\mathbf{x}^*$ is an extremal of $J(x)$ only if $\delta J(x^*, \delta x) = 0$ for all admissible δx.

$$\Rightarrow 0 = \delta J(x^*, \delta x) = \frac{\partial J}{\partial x}\delta x + \frac{\partial J}{\partial \dot{x}}\delta \dot{x} \quad \text{for all admissible } \delta x$$

$$= \int_{t_0}^{t_f} \left(\frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial \dot{x}}\delta \dot{x} \right) dt \quad \text{for all admissible } \delta x$$

$$= \left(\frac{\partial g}{\partial \dot{x}}\delta x \right)\Bigg|_{t_0}^{t_f} + \int_{t_0}^{t_f} \left(\frac{\partial g}{\partial x}\delta x - \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}} \right)\delta x \right) dt \quad \text{(integration by part)}$$

$$= \int_{t_0}^{t_f} \left(\frac{\partial g}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}} \right) \right)\delta x dt \quad \text{for all admissible } \delta x$$

(since $\delta x(t_0) = 0$ and $\delta x(t_f) = 0$)

$\Rightarrow \frac{\partial g}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}} \right) = 0$ (by Fundamental Lemmas of Calculus of Variations 4a)

This is Euler-Lagrange Equation

Indirect Method: Necessary Conditions

For $J(x)=\int_{t_0}^{t_f} g(x,\dot{x},t)dt,\ \ x:[t_0,t_f]\rightarrow R$

with t_f and $x(t_f)$ fixed (fixed end point)

Necessary Conditions: Euler-Lagrange Equation

$\mathbf{x}^*$ is an extremal of $J(x)$ only if

$$\frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)=0 \quad \text{(second-order ODE)}$$

with 2-boundary point

$x(t_0)=x_0$ and

$x(t_f)=x_f$

Solved by shooting method (for example)

Indirect Method: Necessary Conditions

Example: $J(x)=\int_0^{\pi/2}\overbrace{\left(\dot{x}^2(t)-x^2(t)\right)}^{g(x,\dot{x})}dt,\ \ x:[t_0,t_f]\rightarrow R$

with $x(0)=0,\ x(\pi/2)=1$ (fixed end point)

Euler-Lagrange Equation

$$\frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)=-2x-\frac{\partial}{\partial t}\left(2\dot{x}\right)=2\ddot{x}+2x=0$$

$\Rightarrow x(t)=c_1\cos t+c_2\sin t$

With $x(0)=0$ and $x(\pi/2)=1$

$\Rightarrow x(t)=0\cos t+1\sin t = \sin t$

Solved numerically by the shooting method

Indirect Method: Necessary Conditions

For $J(x)=\int_{t_0}^{t_f} g(x,\dot{x},t)dt,\ \ x:[t_0,t_f]\to R$

b) Case: t_f is fixed and $x(t_f)$ is free (free end point)

$\mathbf{x}^*$ is an extremal of $J(x)$ only if $\delta J(x^*,\delta x,\delta x_f)=0$ for all admissible $(\delta x,\delta x_f)$.

$$\Rightarrow 0=\delta J(x^*,\delta x)=\frac{\partial J}{\partial x}\delta x+\frac{\partial J}{\partial \dot{x}}\delta\dot{x} \quad \text{for all admissible } \delta x$$

$$=\int_{t_0}^{t_f}\left(\frac{\partial g}{\partial x}\delta x+\frac{\partial g}{\partial \dot{x}}\delta\dot{x}\right)dt \quad \text{for all admissible } \delta x$$

$$=\left(\frac{\partial g}{\partial \dot{x}}\delta x\right)_{t_0}^{t_f}+\int_{t_0}^{t_f}\left(\frac{\partial g}{\partial x}\delta x-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)\delta x\right)dt \quad \text{(integration by part)}$$

$$=\left.\frac{\partial g}{\partial \dot{x}}\right|_{x^*,t_f}\delta x(t_f)+\int_{t_0}^{t_f}\left(\frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)\right)\delta x dt \quad \text{for all admissible } \delta x, \text{ and } \delta x(t_f)$$

(since $\delta x(t_0)=0$ and $\delta x(t_f)\neq 0$)

$$\Rightarrow \frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)=0 \text{ (Euler-Lagrange)}$$

$$\left.\frac{\partial g}{\partial \dot{x}}\right|_{x^*,t_f}=0$$

$$x(t_0)=0$$

Indirect Method: Necessary Conditions

Now consider $J(x,t_f)=\int_{t_0}^{t_f} g(x,\dot{x},t)dt,\ \ x:[t_0,t_f]\to R$

and t_f is free (variable end time, free terminal time or horizon)

a) Case: $x(t_f)$ is free and is independent of t_f (free terminal conditions)

$\Rightarrow 0=\delta J(x^*,\delta x,\delta x_f,\delta t_f)$ for all admissible $(\delta x,\delta x_f,\delta t_f)$

$$=\int_{t_0}^{t_f}\delta g(x,\dot{x},t)dt+\int_{t_f}^{t_f+\delta t_f} g(x^*,\dot{x}^*,t)dt \quad \text{for all admissible } \delta x, \delta x_f \text{ and } \delta t_f$$

$$=\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\delta x(t_f)+\int_{t_0}^{t_f}\left(\frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)\right)\delta x dt+g(x^*,\dot{x}^*,t_f)\delta t_f$$

$$=\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\left(\delta x_f-\dot{x}^*(t_f)\delta t_f\right)+\int_{t_0}^{t_f}\left(\frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)\right)\delta x dt+g(x^*,\dot{x}^*,t_f)\delta t_f$$

(since $\delta x(t_f)=\ \delta x_f-\dot{x}^*(t_f)\delta t_f$)

$$=\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\delta x_f+\int_{t_0}^{t_f}\left(\frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)\right)\delta x dt+\left(g(x^*,\dot{x}^*,t_f)-\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\dot{x}^*(t_f)\right)\delta t_f$$

for all admissible $\delta x,\delta x_f$ and δt_f

$$\Rightarrow \frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)=0 \text{ (Euler-Lagrange)}$$

$$\left.\frac{\partial g}{\partial \dot{x}}\right|_{x^*,t_f}=0,\quad g(x^*,\dot{x}^*,t_f)-\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\dot{x}^*(t_f)=0, \text{ and } x(t_0)=0$$

Indirect Method: Necessary Conditions

Now consider $J(x,t_f)=\int_{t_0}^{t_f} g(x,\dot{x},t)dt,\ \ x:[t_0,t_f]\to R$

and t_f is free (variable end time, free terminal time or horizon)

b) Case: $x(t_f)=x_f$ (Hard terminal contraint)

$$\Rightarrow \frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)=0 \text{ (Euler-Lagrange)}$$

$$x(t_f)=x_f,\ g(x^*,\dot{x}^*,t_f)-\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\dot{x}^*(t_f)=0, \text{ and } x(t_0)=0$$

c) Case: $x(t_f)=\theta(t_f)$ (Soft terminal contraint) --$\delta x_f=\dot{\theta}(t_f)\delta t_f$

$$\Rightarrow \frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)=0 \text{ (Euler-Lagrange)}$$

$$\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\dot{\theta}(t_f)+g(x^*,\dot{x}^*,t_f)-\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\dot{x}^*(t_f)=0,$$

and $x(t_0)=0,\ x(t_f)=\theta(t_f)$

Example Derivation of Case (c)

Now consider $J(x,t_f)=\int_{t_0}^{t_f} g(x,\dot{x},t)dt,\ \ x:[t_0,t_f]\to R$

and t_f is free (variable end time, free terminal time or horizon)

c) Case: $x(t_f)=\theta(t_f)$ (Soft terminal contraint)

$\Rightarrow 0=\delta J(x^*,\delta x,\delta x_f,\delta t_f)$ for all admissible δx

$$=\int_{t_0}^{t_f}\delta g(x^*,\dot{x}^*,t)dt+\int_{t_f}^{t_f+\delta t_f} g(x^*,\dot{x}^*,t)dt \quad \text{for all admissible } \delta x$$

$$=\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\delta x(t_f)+\int_{t_0}^{t_f}\left(\frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)\right)\delta x dt+g(x^*,\dot{x}^*,t_f)\delta t_f$$

$$=\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\left(\delta x_f-\dot{x}^*(t_f)\delta t_f\right)+\int_{t_0}^{t_f}\left(\frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)\right)\delta x dt+g(x^*,\dot{x}^*,t_f)\delta t_f$$

(since $\delta x(t_f)=\delta x_f-\dot{x}^*(t_f)\delta t_f$)

$$=\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\delta x_f+\int_{t_0}^{t_f}\left(\frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)\right)\delta x dt+\left(g(x^*,\dot{x}^*,t_f)-\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\dot{x}^*(t_f)\right)\delta t_f$$

for all admissible $\delta x,\delta x_f$ and δt_f

$$\Rightarrow \frac{\partial g}{\partial x}-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{x}}\right)=0 \text{ (Euler-Lagrange)}$$

$$\left.\frac{\partial g}{\partial \dot{x}}\right|_{x^*,t_f}=0,\quad g(x^*,\dot{x}^*,t_f)-\left.\frac{\partial g^T}{\partial \dot{x}}\right|_{x^*,t_f}\dot{x}^*(t_f)=0, \text{ and } x(t_0)=0$$

Indirect Method: Necessary Conditions

Further extensions:

1) Piecewise Continuous Solution:

Require additional Weistrass Erdman Corner Conditions or Tranversatility Conditions

2) Constraints on $\mathbf{x}(t)$ require the use of multipliers

Most important cases are in optimal control problems:

Indirect Method: Necessary Conditions

Optimal Control Problems:

Optimize $J(\mathbf{x},\mathbf{u},t_f) = h(\mathbf{x}(t_f),t_f) + \int_{t_0}^{t_f} g(\mathbf{x},\mathbf{u},t)dt$

s.t. $\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x},\mathbf{u},t)$

Boundary conditions: $\mathbf{x}(t_0) = \mathbf{x}_0$

Cases:

I) t_f is fixed and $\mathbf{x}(t_f)$ is fixed

II) t_f is fixed and $\mathbf{x}(t_f)$ is free

III) t_f is fixed, and $\mathrm{m}(\mathbf{x}(t_f)) = 0$

IV) t_f is free and $\mathbf{x}(t_f)$ is free

V) t_f is free and $\mathbf{x}(t_f)$ is fixed

VI) t_f is free and $\mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$

VII) t_f is free, and $\mathrm{m}(\mathbf{x}(t_f)) = 0$

VIII) t_f is free, and $\mathrm{m}(\mathbf{x}(t_f),t_f) = 0$

Indirect Method: Necessary Conditions

Optimal Control Problems:

Optimize $J(\mathbf{x},\mathbf{u},t_f) = h(\mathbf{x}(t_f),t_f) + \int_{t_0}^{t_f} g(\mathbf{x},\mathbf{u},t)dt$

s.t. $\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x},\mathbf{u},t)$

Boundary conditions: $\mathbf{x}(t_0) = \mathbf{x}_0$

Case IV: $h(\mathbf{x}(t_f),t_f) = \int_{t_0}^{t_f} \dot{h}(\mathbf{x}(t),t)dt + h(\mathbf{x}(t_0),t_0)$ – –ignored (constant)

$$= \int_{t_0}^{t_f} \left(\left(\frac{\partial h(\mathbf{x}(t),t)}{\partial \mathbf{x}} \right)^T \dot{\mathbf{x}} + \frac{\partial h(\mathbf{x}(t),t)}{\partial t} \right) dt$$

$$\mathrm{g_a}(\mathbf{x},\dot{\mathbf{x}},\mathbf{u},\mathbf{p},t) = \underbrace{g(\mathbf{x},\mathbf{u},t) + +\mathbf{p}^T (\mathbf{a}(\mathbf{x},\mathbf{u},t) - \dot{\mathbf{x}})}_{H(\mathbf{x},\mathbf{u},\mathbf{p},t)--Hamiltonian} + \left(\frac{\partial h(\mathbf{x}(t),t)}{\partial \mathbf{x}} \right)^T \dot{\mathbf{x}} \frac{\partial h(\mathbf{x}(t),t)}{\partial t}$$

$$\Rightarrow J_a(\mathbf{x},\dot{\mathbf{x}},\mathbf{u},\mathbf{p},t_f) = \int_{t_0}^{t_f} \mathrm{g_a}(\mathbf{x},\dot{\mathbf{x}},\mathbf{u},\mathbf{p},t)dt$$

$$\Rightarrow \delta J_a(\mathbf{x}^*,\delta\mathbf{x},\mathbf{u}^*,\delta\mathbf{u},\mathbf{p}^*,\delta\mathbf{p},\delta\mathbf{x}_f,\delta t_f)$$

$$= \int_{t_0}^{t_f} \left(\frac{\partial H^*}{\partial \mathbf{u}} \delta\mathbf{u} + \left(\frac{\partial H^*}{\partial \mathbf{x}} + \dot{\mathbf{p}} \right) \delta\mathbf{x} \right) dt$$

$$+ \left(\frac{\partial h(\mathbf{x}^*,t_f)}{\partial \mathbf{x}} - \mathbf{p}(t_f) \right)^T \delta\mathbf{x}_f + \left(g^* + \mathbf{p}^T \mathbf{a}(\mathbf{x}^*,\mathbf{u}^*,t_f) + \frac{\partial h(\mathbf{x}^*(t_f),t_f}{\partial \mathbf{x}} \right) \delta t_f$$

$= 0$ for all admissible $\delta\mathbf{x},\ \delta\mathbf{u},\ \delta\mathbf{x}_f, \delta t_f$

Indirect Method: Necessary Conditions

Necessary Conditions for Optimal Control Problems:

Case IV: Free-end point problems: t_f free and $\mathbf{x}(t_f)$ free

Hamiltonian-Jacobi Conditions:

$\Rightarrow$ $(\mathbf{x}^*,\mathbf{u}^*)$ is extremal, then there exists co-states $\mathbf{p}^*$ such that:

$$\frac{\partial H^*}{\partial \mathbf{u}} = 0 \qquad (1)$$

$$\dot{\mathbf{p}} = -\frac{\partial H^*}{\partial \mathbf{x}} \qquad (2)$$

$$\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x},\mathbf{u},t) \qquad (3)$$

$$\left(\frac{\partial h(\mathbf{x}^*,t_f)}{\partial \mathbf{x}} - \mathbf{p}(t_f) \right) \delta\mathbf{x}_f + \left(H^* + \frac{\partial h(\mathbf{x}^*(t_f),t_f}{\partial \mathbf{x}} \right) \delta t_f = 0 \qquad (4)$$

and $\mathbf{x}(t_0) = \mathbf{x}_0$

OTHER CASES CAN BE SIMILARILY DERIVED