CALCULUS I

206111

Academic year 2024

MIDTERM

Contents

Part I

Reviews

1 Limits and Continuity

THE TANGENT LINE PROBLEM Given a function f and a point $P(x_0, y_0)$ on the graph of f, find an equation of the line that is tangent to the graph of f at P. (Figure 1.1)

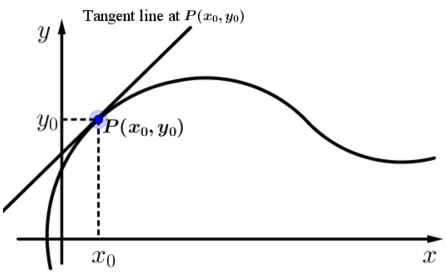


Figure 1.1: A picture of tangent line at point P

Example 1.1 Find an equation for the tangent line to the parabola $y = x^2 + 1$ at the point $P(1, 2)$.

Solution.

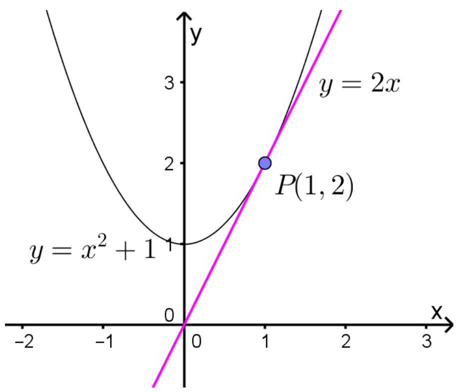


Figure 1.2: Graph of $y = x^2 + 1$

1.1 Limits

Definition 1 (Limits) If the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches a is L", or "$f(x)$ approaches L as x approaches a".

Example 1.2 Use numerical evidence to make a conjecture about the value of $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$.

Solution.

x	1.9	1.99	1.999	1.9999		2.0001	2.001	2.01	2.1
$f(x)$									

1.1.1 One-sided Limits

Definition 2 (One-sided Limits) If the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L$$

("the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right".)

and if the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x \to a^-} f(x) = L$$

("the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left".)

Example 1.3 Explain why $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.

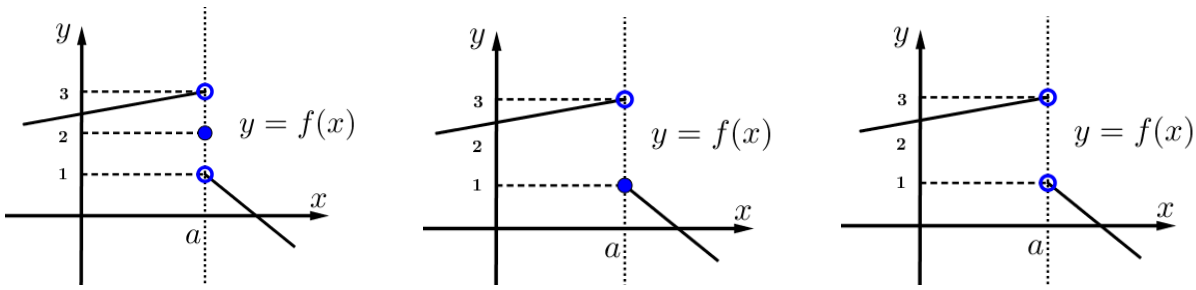
Solution.

The Relationship between One-Sided and Two-Sided Limits The two-sided limit of a function $f(x)$ exists at $x = a$ if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Example 1.4 For the functions in Figure 1.3 , find the one-sided and two-sided limits at $x = a$ if they exist.

Solution.

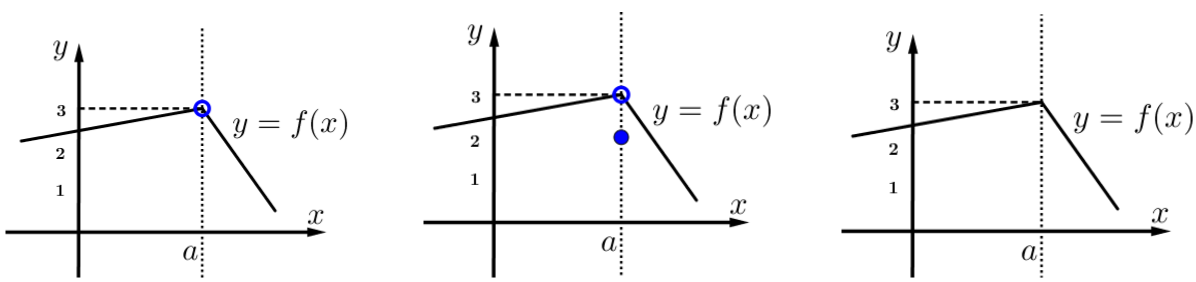


(a) (b) (c)

Figure 1.3: A picture for Exercise 1.4

Example 1.5 For the functions in Figure 1.4 , find the one-sided and two-sided limits at $x = a$ if they exist.

Solution.



(a) (b) (c)

Figure 1.4: A picture for Exercise 1.5

1.1.2 Infinite Limits

Sometimes the values of the function increase or decrease without bound.

x	-10	-1	-0.1	-0.01	-0.001	-0.0001	$\cdots$	0
$\frac{1}{x}$	-0.1	-1	-10	-100	-1000	-10,000	$\cdots$	

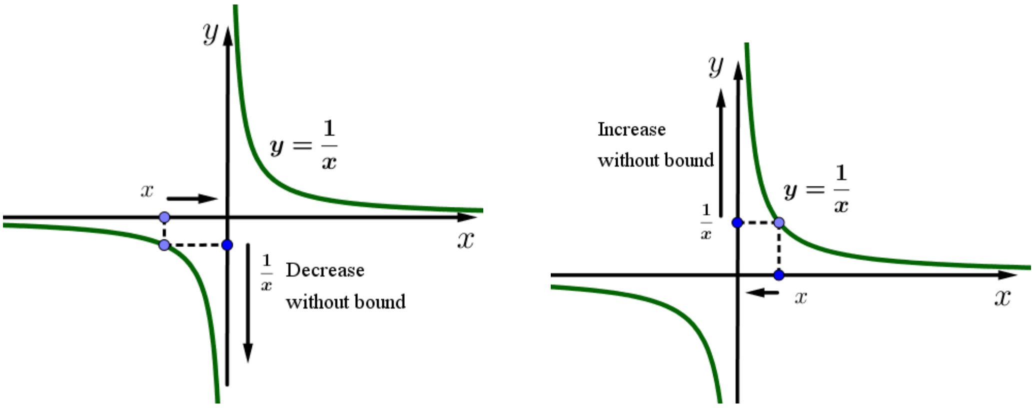


Figure 1.5: (Left) $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ (Right) $\lim_{x\to 0^+} \frac{1}{x} = \infty$

x	0	$\cdots$	0.0001	0.001	0.01	0.1	1	10
$\frac{1}{x}$		$\cdots$	10,000	1000	100	10	1	0.1

The above discussion represents the limit of $1/x$ as $x \to 0^-$ and $x \to 0^+$ graphically. These can be summarized that

$$\lim_{x\to 0^-} \frac{1}{x} = \ldots\ldots\ldots\ldots \qquad \text{and} \qquad \lim_{x\to 0^+} \frac{1}{x} = \ldots\ldots\ldots\ldots$$

Let us next consider the limit of $1/(x-a)$ as $x \to a^-$ and $x \to a^+$.

Example 1.6 Fill in the blank and guess what are $\lim_{x\to a^-} \frac{1}{x-a}$ and $\lim_{x\to a^+} \frac{1}{x-a}$

x	$a-1$	$a-0.1$	$a-0.01$	$a-0.001$	$a-0.0001$	$\cdots$	a
$\frac{1}{x-a}$	-1	-10	-100	-1000	-10,000	$\cdots$	

x	a	$\cdots$	$a+0.0001$	$a+0.001$	$a+0.01$	$a+0.1$	$a+1$
$\frac{1}{x-a}$		$\cdots$	10,000	1000	100	10	1

Definition 3 (Infinite Limits) The expressions

$$\lim_{x\to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x\to a} f(x) = +\infty.$$

Similarly, the expressions

$$\lim_{x\to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x\to a} f(x) = -\infty$$

These cases can be considered as limit does not exist.

Example 1.7 For the functions in Figure 1.6 , describe the limits at $x = a$ in appropriate limit notation.

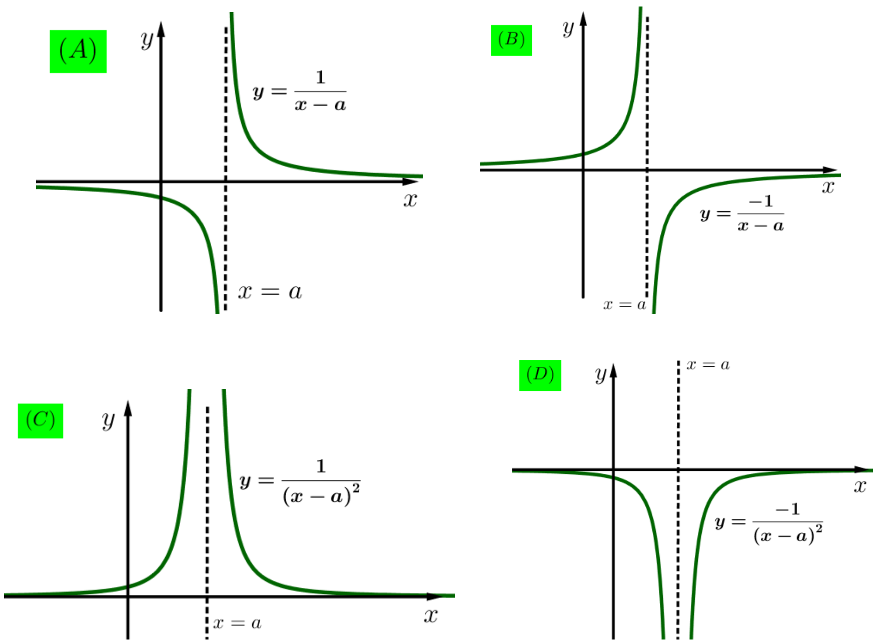


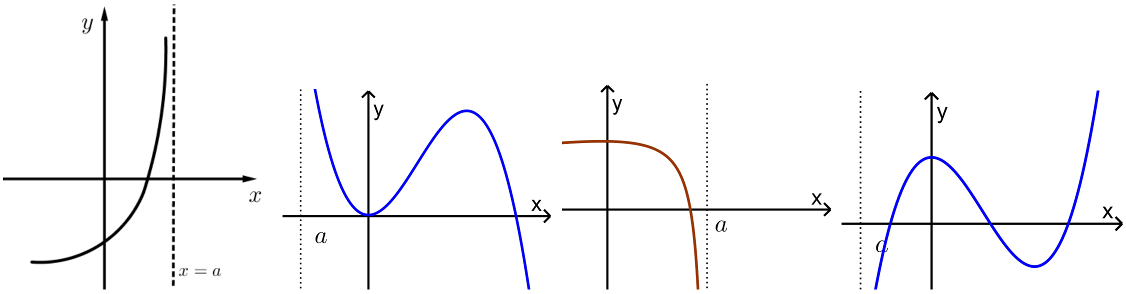
Figure 1.6: A picture for Exercise 1.7

Solution.

1.1.3 Vertical Asymptotes

Definition 4 If the graph of $f(x)$ either rises or falls without bound, squeezing closer and closer to the vertical line $x = a$ as x approaches a from the side indicated in the limit, we call the line $x = a$ ***vertical asymptote*** of the curve $y = f(x)$. In other word, the line $x = a$ is ***vertical asymptote*** of $f(x)$ if either $\lim_{x\to a^-} f(x) \to \pm\infty$ or $\lim_{x\to a^+} f(x) \to \pm\infty$.

Figure 1.7 illustrates geometrically what happen when any of the following situations occur:



$\lim_{x\to a^-} f(x) = +\infty$ $\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to a^-} f(x) = -\infty$ $\lim_{x\to a^+} f(x) = -\infty$

Figure 1.7: Examples of vertical asymptotes

1.2 Limits

1.2.1 Basic Limits

Theorem 1.1 Let a and k be real numbers.

(a) $\lim_{x\to a} k = k$ (b) $\lim_{x\to a} x = a$ (c) $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ (d) $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

Example 1.8 If $f(x) = k$ is a constant function, then the values of $f(x)$ remain fixed at k as x varies, which explains why $\lim_{x\to a} k = k$ for all value of a. For example,

Solution.

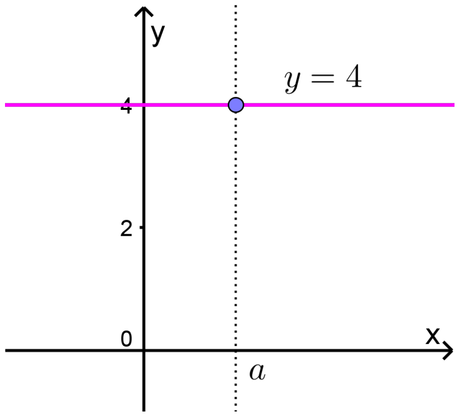


Figure 1.8: Graph of $f(x) = 4$

$\lim_{x\to -5} 4 =$

$\lim_{x\to 0} 4 =$

$\lim_{x\to 15} 4 =$

Example 1.9 If $f(x) = x$, then the values of $f(x)$ always equals to x varies, which explains why $\lim_{x\to a} x = a$ for all value of a. For example,

Solution.

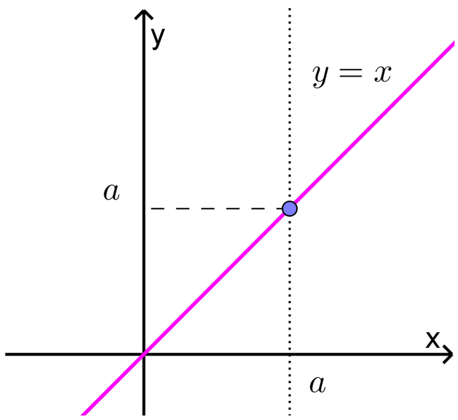


Figure 1.9: Graph of $f(x) = x$

$\lim_{x\to 0} x =$

$\lim_{x\to 3} x =$

$\lim_{x\to -100} x =$

Theorem 1.2 Let a be a real number, and suppose that

$$\lim_{x \to a} f(x) = L_1 \quad \text{and} \quad \lim_{x \to a} g(x) = L_2.$$

That is, the limits exist and have values L_1 and L_2, respectively, Then:

(a) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$

(b) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L_1 - L_2$

(c) $\lim_{x \to a} [f(x)g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = L_1 L_2$

(d) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L_1}{L_2}$, provided $L_2 \neq 0$

(e) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L_1}$, provided $L_1 > 0$ if n is even.

Moreover, these statements are also true for the one-sided limits as $x \to a^-$ or as $x \to a^+$.

1.2.2 Limits of Polynomials and Rational Functions as $x \to a$

Example 1.10 Find $\lim_{x \to 2} (x^3 - 3x + 5)$.

Solution.

Theorem 1.3 For any polynomial

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

and any real number a,

$$\lim_{x \to a} p(x) = c_0 + c_1 a + \cdots + c_n a^n = p(a)$$

Example 1.11 Find

(a) $\displaystyle\lim_{x\to 5^+} \frac{3-x}{(x-5)(x+1)}$ (b) $\displaystyle\lim_{x\to 5^-} \frac{3-x}{(x-5)(x+1)}$ (c) $\displaystyle\lim_{x\to 5} \frac{3-x}{(x-5)(x+1)}$

Solution.

Limits of Rational Functions $\dfrac{f(x)}{g(x)}$ as $x \to a$ when $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$

Example 1.12 Find

(a) $\displaystyle\lim_{x\to 5} \frac{x^2-10x+25}{x-5}$ (b) $\displaystyle\lim_{x\to 2} \frac{3x-6}{x^2-x-2}$ (c) $\displaystyle\lim_{x\to 3} \frac{x^2-5x+6}{x^2-6x+9}$.

Solution.

Theorem 1.4 a be any real number.

(a) If $q(a) \neq 0$, then $\lim_{x\to a} f(x) = f(a)$.

(b) If $q(a) = 0$ but $p(a) \neq 0$, then $\lim_{x\to a} f(x)$ does not exist.

Limits Involving Radicals

Example 1.13 Find $\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}$.

Solution.

Limits of Piecewise-Defined Functions

Example 1.14 Let

$$f(x) = \begin{cases} \dfrac{2}{x+3}, & x < -3 \\ x^2, & -3 < x \leq 2 \\ 6 - x, & x > 2 \end{cases} \cdot$$

Find (a) $\lim\limits_{x \to -3} f(x)$ (b) $\lim\limits_{x \to 0} f(x)$ (c) $\lim\limits_{x \to 2} f(x)$.

Solution.

1.3 Continuity

Definition 5 A function f is said to be ***continuous at*** $x = c$ provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

Example 1.15 Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2 - 4}{x - 2}, \qquad g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 2, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2. \end{cases}$$

Solution.

Continuity in Applications

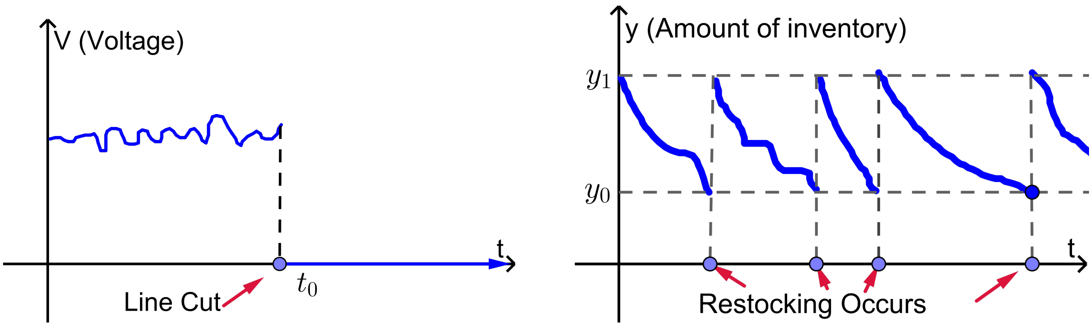


Figure 1.10: Examples of discontinuity

1.3.1 Continuity on an Interval

We say a function f is ***continuous from the left*** at c if

$$\lim_{x \to c^-} f(x) = f(c)$$

and is ***continuous from the right*** at c if

$$\lim_{x \to c^+} f(x) = f(c).$$

Definition 6 A function f is said to be ***continuous on a closed interval*** $[a, b]$ if the following conditions are satisfied:

1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

Example 1.16 Explain the continuity of the function $f(x) = \sqrt{9 - x^2}$ on the interval $[-3, 3]$

Solution.

Some Properties of Continuous Functions

Theorem 1.5 If the functions f and g are continuous at c, then

(a) $f+g$ is continuous at c.

(b) $f-g$ is continuous at c.

(c) fg is continuous at c.

(d) f/g is continuous at c if $g(c) \neq 0$ and has a discontinuity at c if $g(c)=0$.

Continuity of Polynomials and Rational Functions

Theorem 1.6 (a) A polynomial is continuous everywhere.

(b) A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.

Example 1.17 For what values of x is there a discontinuity in the graph of

$$y = \frac{x^2-4}{x^2-5x+6}?$$

Solution.

NOTE: Exponential, Logarithm, Trigonometric and Inverse trigonometric functions are continuous on ***their domains*** .

Example 1.18 Show that $f(x) = |x|$ is continuous everywhere.

Solution.

1.3.2 Continuity of Compositions

Theorem 1.7 If $\lim_{x\to c} g(x) = L$ and if the function f is continuous at L, then $\lim_{x\to c} f(g(x)) = f(L)$. That is,

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$$

Example 1.19 Given that $\lim_{x\to 2} x^2 - 9 = -5$ find, $\lim_{x\to 2} |x^2 - 9|$.

Solution.

Theorem 1.8 (a) If the function g is continuous at c, and the function f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

(b) If the function g is continuous everywhere, and the function f is continuous everywhere, then the composition $f \circ g$ is continuous everywhere.

1.4 Intermediate-Value Theorem

Theorem 1.9 (Intermediate-Value Theorem) If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, inclusive, then there is at least one number x in the interval $[a, b]$ such that $f(x) = k$.

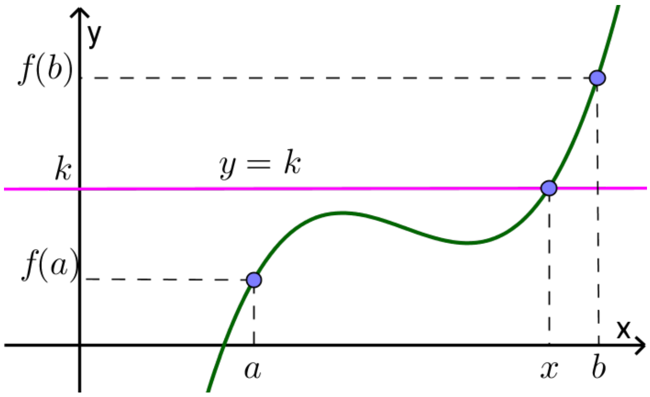


Figure 1.11: Concept of Intermediate-Value Theorem

Example 1.20 Verify that there exists at least one root of the equation $x^3 - x - 1 = 0$ in the closed interval $[1, 2]$. Then, approximate this root to two decimal-place accuracy.

Solution. Since $f(1) = -1$ and $f(2) = 5$, we have $f(1) \leq 0 \leq f(2)$. Therefore the root is between 1 and 2.

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$f(x)$	-1	-0.77	-0.47	-0.1	0.34	0.88	1.5	2.21	3.03	3.96	5

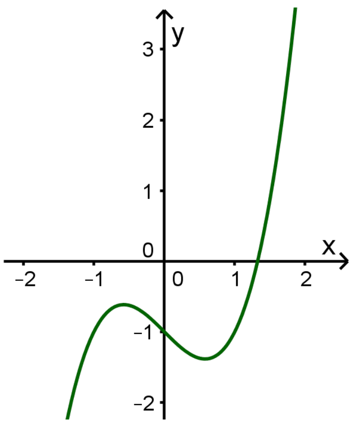


Figure 1.12: Graph of $y = x^3 - x - 1$

Since $f(1.3) < 0$ and $f(1.4) > 0$,
the root is between _____ and _____.

x	1.30	1.31	1.32	1.33	1.34	1.35
$f(x)$	-0.1	-0.06	-0.02	0.02	0.07	0.11

x	1.36	1.37	1.38	1.39	1.40
$f(x)$	0.16	0.2	0.25	0.3	0.34

The root of the equation $x^3 - x - 1 = 0$, that is between 1 and 2, is approximately __________.

Exercise 1

1. Given that

$$\lim_{x\to a} f(x) = 2, \qquad \lim_{x\to a} g(x) = -4, \qquad \lim_{x\to a} h(x) = 0$$

find the limits.

(a) $\lim_{x\to a} [f(x) + 2g(x)]$

(b) $\lim_{x\to a} [h(x) - 3g(x) + 2]$

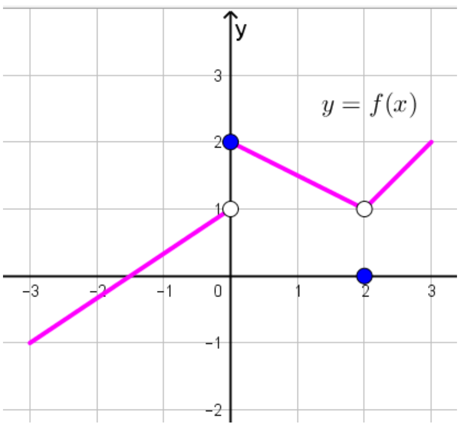
(c) $\lim_{x\to a} [f(x)g(x)]$

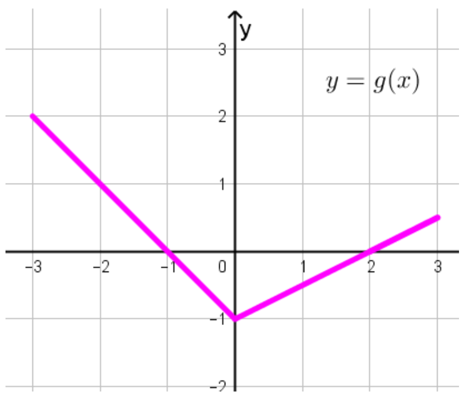
(d) $\lim_{x\to a} [g(x)]^2$

(e) $\lim_{x\to a} \sqrt[3]{6 + f(x)}$

(f) $\lim_{x\to a} \dfrac{2}{g(x)}$

2. use the following graphs of f and g to find the limits that exists. If the limit does not exist, explain why.





(a) $\lim_{x\to 2} [f(x) + g(x)]$

(b) $\lim_{x\to 0} [f(x) + g(x)]$

(c) $\lim_{x\to 0^+} [f(x) + g(x)]$

(d) $\lim_{x\to 0^-} [f(x) + g(x)]$

(e) $\lim_{x\to 2} \dfrac{f(x)}{1 + g(x)}$

(f) $\lim_{x\to 2} \dfrac{1 + g(x)}{f(x)}$

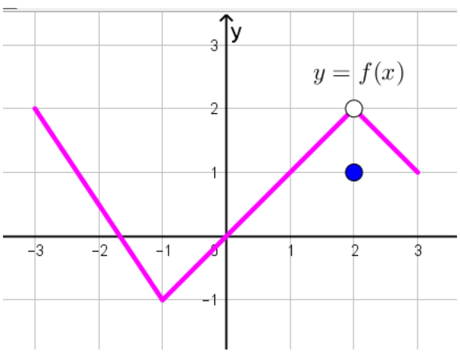
(g) $\lim_{x\to 0^+} \sqrt{f(x)}$

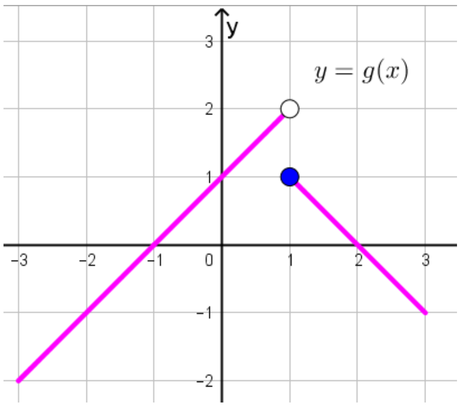
(h) $\lim_{x\to 0^-} \sqrt{f(x)}$

3. Find each limit.

(a) $\lim_{x\to 2}(3x+5)$ (b) $\lim_{x\to -1}\left(\frac{2x}{3x+2}\right)$ (c) $\lim_{x\to 3}\sqrt[3]{3x-1}$

4. Given the graphs of functions $f(x)$ and $g(x)$ as the following figures. Find each limit.





(a) $\lim_{x\to 2}[f(x)+g(x)]$

(b) $\lim_{x\to 0}[f(x)g(x)]$

(c) $\lim_{x\to 2}[2xf(x)]$

(d) $\lim_{x\to 1}\sqrt{3+f(x)}$

5. Find each limit.

(a) $\lim_{x\to -5}10$

(b) $\lim_{x\to 4}(x^2-4x+3)$

(c) $\lim_{x\to 3}(5x^3+4)$

(d) $\lim_{x\to -1}(x^4-2x+3)$

6. Find each limit.

(a) $f(x)=\begin{cases} 2x+3 & \text{if } x<5 \\ -x+12 & \text{if } x>5 \end{cases}$

i. $\lim_{x\to 5^-}f(x)=$ ii. $\lim_{x\to 5^+}f(x)=$ iii. $\lim_{x\to 5}f(x)=$ iv. $f(5)=$

(b) $f(x)=\begin{cases} x & \text{if } x<0 \\ x^2 & \text{if } 0<x\le 2 \\ 8-x & \text{if } x>2 \end{cases}$

i. $\lim_{x\to 0^-}f(x)=$ iii. $\lim_{x\to 0}f(x)=$ v. $\lim_{x\to 2^-}f(x)=$ vii. $\lim_{x\to 2}f(x)=$

ii. $\lim_{x\to 0^+}f(x)=$ iv. $\lim_{x\to 1}f(x)=$ vi. $\lim_{x\to 2^+}f(x)=$ viii. $f(2)=$

7. Find each limit.

(a) $\lim_{x\to 2} \frac{5x^3+4}{x-3} =$ (b) $\lim_{x\to -1} \frac{4}{3x^2-5} =$ (c) $\lim_{x\to -2} \frac{x^2}{x^2+1} =$

8. Find the following limits.

(a) $\lim_{x\to 2} \frac{x^2-4}{x-2}$

(b) $\lim_{x\to 3} \frac{x^2-6x+9}{x-3}$

(c) $\lim_{x\to -4} \left(\frac{2x+8}{x^2+x-12}\right)$

(d) $\lim_{x\to 5} \frac{x^2-3x-10}{x^2-10x+25}$

(e) $\lim_{x\to 2^-} \frac{3x^2-x-10}{x^2-4}$

(f) $\lim_{x\to 1^+} \frac{x^4-1}{x-1}$

9. Find the following limits.

(a) $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$

(b) $\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1}$

(c) $\lim_{x\to 4} \frac{3-\sqrt{x+5}}{x-4}$

(d) $\lim_{x\to 4} \frac{x-\sqrt{3x+4}}{4-x}$

10. Find the following limits.

(a) $\lim_{x\to -1} \frac{x|x+1|}{x+1}$

(b) $\lim_{x\to 4} \frac{x^2-16}{|x-4|}$

11. Find the following limits.

(a) $\lim_{x\to 0^-} \frac{1}{x} =$ (b) $\lim_{x\to 0^+} \frac{1}{x} =$ (c) $\lim_{x\to 0} \frac{1}{x} =$

12. Find the following limits.

(a) $\lim_{x\to 3^-} \frac{2x}{x-3} =$ (b) $\lim_{x\to 3^+} \frac{2x}{x-3} =$ (c) $\lim_{x\to 3} \frac{2x}{x-3} =$

13. Find the following limits.

(a) $\lim_{x\to 3^-} \frac{5x^3+4}{x-3}$ (b) $\lim_{x\to 3^+} \frac{5x^3+4}{x-3}$ (c) $\lim_{x\to 3} \frac{5x^3+4}{x-3}$

14. Find the following limits.

(a) $\lim_{x\to 4^-} \dfrac{2-x}{(x-4)(x+2)}$ (b) $\lim_{x\to 4^+} \dfrac{2-x}{(x-4)(x+2)}$ (c) $\lim_{x\to 4} \dfrac{2-x}{(x-4)(x+2)}$

15. Find the following limits.

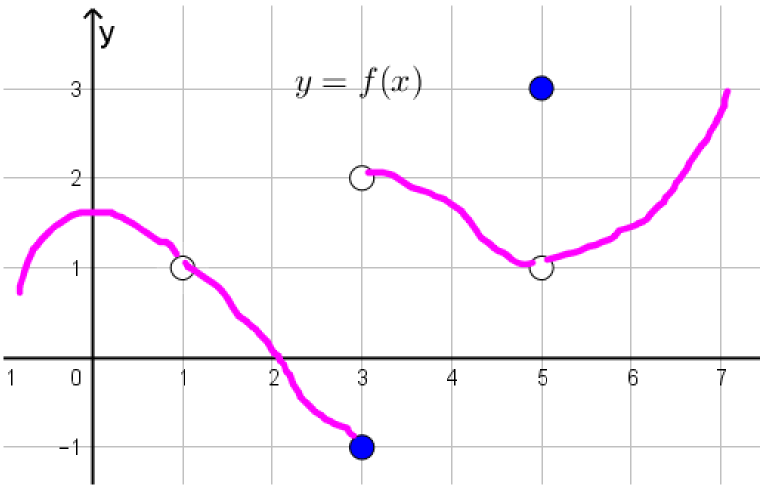
(a) $\lim_{x\to 0^+} \ln x$

(b) $\lim_{x\to \frac{\pi}{2}^-} \tan x$

(c) $\lim_{x\to \frac{\pi}{2}^+} \tan x$

(d) $\lim_{x\to 5} \dfrac{x^2-3x-10}{x^2-10x+25}$

16. Given the graph of a function $f(x)$ as the following figure. At what point is $f(x)$ discontinuous? Explain why? Where is $f(x)$ continuous?



17. Let $f(x)$ be a function. Determine if $f(x)$ is continuous at given points.

(a) $f(x) = \dfrac{x^2-x-2}{x-2}$ at x=2

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ at $x = 0$ and $x = 1$

(c) $f(x) = \dfrac{|x|}{x}$ at $x = 0$ and $x = 1$

18. Find the value of k so that the function $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ k & \text{if } x = 2 \end{cases}$ is continuous.

Part II

Midterm

2 Derivative

2.1 Tangent Lines and Rate of Change

In this section we will see how the concept of "tangent lines to a curve" and that of "the rate at which one variable changes relative to another" are related.

Tangent Lines

Definition 7 Suppose that x_0 is in the domain of the function f. The ***tangent line*** to the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the line with equation

$$y - f(x_0) = m_{tan}(x - x_0)$$

where

$$m_{tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{2.1}$$

provided that the limit exists.

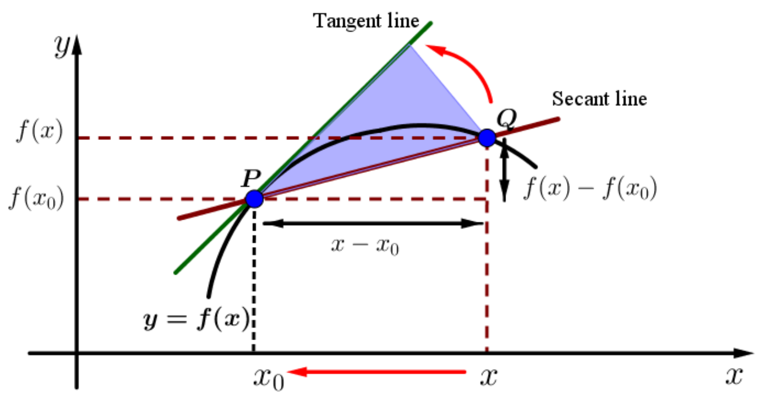


Figure 2.1: The slope of the tangent line as a limit of slopes of secant lines

Example 2.1 Use Formula (2.1) to find an equation for the tangent line to the parabola $y = x^2 + 1$ at the point $P(1, 2)$, and confirm the result agrees with that obtained in Example 1.1.

Solution.

If we let $h = x - x_0$, then $x \to x_0$ is equivalent to $h \to 0$ and

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{2.2}$$

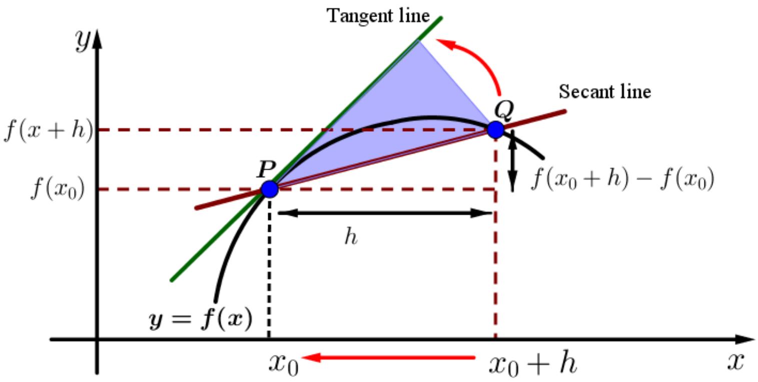


Figure 2.2: The slope of the tangent line as a limit of slopes of secant lines

Example 2.2 Compute the slope in Example 2.1 using Formula (2.2).

Solution.

Example 2.3 Find the slopes of the tangent lines to the curve $y = \sqrt{x}$ at $x_0 = 1$, $x_0 = 4$, and x_0=9.

Solution.

2.2 Calculus as a Rate of Change

In scientific works, we always deal with change of phenomena such as the number of population, the concentration of reactant after chemical reaction, or rectilinear motion. In this section, we consider an example of the rectilinear motion as a rate of change.

Rectilinear Motions

The position coordinate of a particle in rectilinear motion at time t is $s = f(t)$. The ***average velocity*** of the particle over a time interval $[t_0, t_0 + h]$ for $h > 0$ is

$$v_{ave} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t_0 + h) - f(t_0)}{h}$$

Example 2.4 Suppose that $s = f(t) = 1 + 5t - 2t^2$ is the position function of a particle, where s is in meters and t is in seconds. Find the average velocities of the particle over the time intervals (a) $[0, 2]$ and (b) $[2, 3]$.

Solution.

Taking the limit of v_{ave} as $h \to 0$, we get instantaneous velocity v_{inst} of the particle at time t_0.

$$v_{inst} = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}$$

Example 2.5 Consider particle in Example 2.4, where

$$s = f(t) = 1 + 5t - 2t^2$$

The position of the particle at time $t = 2$ *sec* is $s = 3$ m. Find the instantaneous velocity of the particle at time $t = 2$ *sec*.

Solution.

2.3 Derivative

2.3.1 Definition of Derivative

Definition 8 The function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

is called the ***derivative of f with respect to x*** . The domain of f' consists of all x in the domain of f where the limit exists.

The term "derivative" is used because the function f' is ***derived*** from the function f by a limiting process.

Example 2.6 Find the derivative with respect to x of $f(x) = x^2$, and use it to find an equation of the tangent line to $y = x^2$ at $x = 1$.

Solution.

Example 2.7 Find the derivative with respect to x of $f(x) = x^3 - x$

Solution.

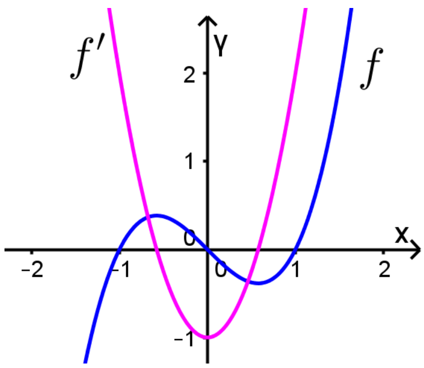


Figure 2.3: Graph of f and f' in Example 2.7

If we graph f and f' together as in Figure 2.3, we can see the relationship between them. Since $f'(x)$ is the slope of the tangent line to the graph of $y = f(x)$ at x, it follows that f' is positive, negative, and zero where the tangent line has positive slope, has negative slope, and is horizontal, respectively.

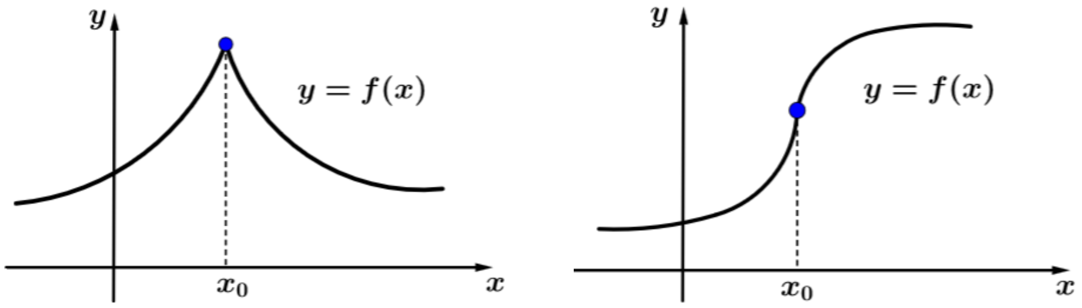
2.3.2 Differentiability

Definition 9 A function f is ***differentiable at*** x_0 if the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. If f is differentiable at each point of the open interval (a, b), then we say that it is ***differentiable on*** (a, b) , and similarly for open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the last case, we say that f is ***differentiable everywhere*** .

Figure 2.4 illustrates two common ways in which a function that is continuous at x_0 is not differentiable at x_0 since the slopes of the secant lines have different limits from the left and from the right.



Corner point   Point of vertical tangency

Figure 2.4: Continuous at x_0 but not differentiable at x_0

Example 2.8 (a) Prove that $f(x) = |x|$ is not differentiable at $x = 0$ by considering the limit.
(b) Find a formula for $f'(x)$.

Solution.

Example 2.9 (a) Prove that $f(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$ by considering the limit.
(b) Find a formula for $f'(x)$. (c) Does it has any vertical tangent ?

Solution.

The Relationship between Differentiability and Continuity

Theorem 2.1 If a function f is differentiable at x_0, then f is continuous at x_0.

Note that the converse of the theorem above is false; that is, a function may be continuous at a point but not differentiable at that point.

2.3.3 Other Derivative Notations

When the independent variable is x, the derivative is commonly denoted by

$$f'(x) \qquad \text{or} \qquad \frac{d}{dx}[f(x)] \qquad \text{or} \qquad D_x[f(x)]$$

In the case where the dependent variable is $y = f(x)$, the derivative is also denoted by

$$y' \qquad \text{or} \qquad y'(x) \qquad \text{or} \qquad \frac{dy}{dx}$$

With the above notations, the value of the derivative at a point x_0 can be expressed as

$$f'(x_0) \qquad \text{or} \qquad \left.\frac{d}{dx}[f(x)]\right|_{x=x_0} \qquad \text{or} \qquad D_x[f(x)]|_{x=x_0} \qquad \text{or} \qquad y'(x_0) \qquad \text{or} \qquad \left.\frac{dy}{dx}\right|_{x=x_0}$$

2.4 Basic Differentiation Formulas

RECAP Henceforth, we will use the following notations for the derivative of a function $y = f(x)$.

- $f'(x)$
- $\dfrac{d}{dx}[f(x)]$
- $\dfrac{dy}{dx}$
- y'

Derivative of a Constant

Theorem 2.2 (The Constant Rule) The derivative of a constant function is 0; that is, if c is any real number, then

$$\frac{d}{dx}[c] = 0$$

Proof

Example 2.10

$$\frac{d}{dx}[5] = \ldots\ldots\ldots\ldots \quad \frac{d}{dx}[-7] = \ldots\ldots\ldots\ldots \quad \frac{d}{dx}[5^2] = \ldots\ldots\ldots\ldots$$

$$\frac{d}{dx}[\pi] = \ldots\ldots\ldots\ldots \quad \frac{d}{dx}[\sqrt{2}] = \ldots\ldots\ldots\ldots \quad \frac{d}{dx}[\pi^2] = \ldots\ldots\ldots\ldots$$

Derivatives of Power Functions

Theorem 2.3 (The Power Rule) If n is any real number, then

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Example 2.11

$\frac{d}{dx}[x^5] = \dots\dots\dots\dots$ $\frac{d}{dx}[x] = \dots\dots\dots\dots$ $\frac{d}{dx}[x^{12}] = \dots\dots\dots\dots$

$\frac{d}{dx}[x^{-5}] = \dots\dots\dots\dots$ $\frac{d}{dx}[\frac{1}{x^5}] = \dots\dots\dots\dots$ $\frac{d}{dx}[x^{\sqrt{2}}] = \dots\dots\dots\dots$

$\frac{d}{dx}[x^{\frac{2}{5}}] = \dots\dots\dots\dots$ $\frac{d}{dx}[\sqrt{x}] = \dots\dots\dots\dots$ $\frac{d}{dx}[\frac{1}{\sqrt{x}}] = \dots\dots\dots\dots$

Derivative of a Constant Times a Function

Theorem 2.4 (The Constant Multiple Rule) If f is differentiable at x and c is any real number, then cf is also differentiable at x and

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Example 2.12 $\frac{d}{dx}(4x^8) = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\frac{d}{dx}(-x^{12}) = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\frac{d}{dx}\left(\frac{x}{\pi}\right) = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

Derivatives of Sums and Differences

Theorem 2.5 (The Sum and Difference Rules) If f and g are differentiable at x, then so are $f+g$ and $f-g$, and

$$\frac{d}{dx}\big[f(x) \pm g(x)\big] = \frac{d}{dx}\big[f(x)\big] \pm \frac{d}{dx}\big[g(x)\big]$$

Example 2.13 $\frac{d}{dx}\big[x^8 + x^2\big] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\frac{d}{dx}\big[6x^{11} - 9\big] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\frac{d}{dx}\left(\frac{1}{x} - x^{-9}\right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\frac{d}{dx}\left(\frac{\sqrt{x} - 3x}{\sqrt{x}}\right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Example 2.14 At what points, if any, does the graph of $y = x^3 - 3x + 4$ have a horizontal tangent line?

Solution.

2.4.1 Higher Derivatives

The derivative f' of a function f is itself a function and hence may have its own derivative. If f' is differentiable, then its derivative is denoted by f'' and is called the second derivative of f. As long as we have differentiability, we can continue the process of differentiating to obtain the third, the fourth, the fifth, and even higher derivatives of f. These successive derivatives are denoted by

the first derivative	y',	$f'(x)$,	$\dfrac{dy}{dx}$,	$\dfrac{d}{dx}[f(x)]$
the second derivative	y'',	$f''(x)$,	$\dfrac{d^2y}{dx^2}$,	$\dfrac{d^2}{dx^2}[f(x)]$
the third derivative	y''',	$f'''(x)$,	$\dfrac{d^3y}{dx^3}$,	$\dfrac{d^3}{dx^3}[f(x)]$
the fourth derivative	$y^{(4)}$,	$f^{(4)}(x)$,	$\dfrac{d^4y}{dx^4}$,	$\dfrac{d^4}{dx^4}[f(x)]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A general nth order derivative	$y^{(n)}$,	$f^{(n)}(x)$,	$\dfrac{d^ny}{dx^n}$,	$\dfrac{d^n}{dx^n}[f(x)]$

Example 2.15 If $f(x) = 6x^4 - 5x^3 + x^2 - 7x + 8$, then

$f'(x) =$

$f''(x) =$

$f'''(x) =$

$f^{(4)}(x) =$

$f^{(5)}(x) =$

$f^{(500)}(x) =$

Example 2.16 Let $y = \dfrac{x+1}{x}$. Find $y^{(111)}$.

Solution.

2.5 Product and Quotient Rules

2.5.1 Derivative of a Product

Theorem 2.6 (The Product Rule) If f and g are differentiable at x, then so is the product $f \cdot g$, and

$$\frac{d}{dx}\big[f(x)g(x)\big] = f(x)\frac{d}{dx}\big[g(x)\big] + g(x)\frac{d}{dx}\big[f(x)\big]$$

Example 2.17 1. Let $y = (4x^2 - 1)(7x^3 + x)$. Find dy/dx.

Solution.

2. Let $f(x) = (3x + 1)(2x^2 + 5)$. Find $f'(0)$.

Solution.

2.5.2 Derivative of a Quotient

Theorem 2.7 (The Quotient Rule) If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and

$$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Example 2.18 Find dy/dx for y.

1. $y = \dfrac{x^2 - 1}{x^4 + 1}$

   Solution.

2. $y = \dfrac{2x + 1}{\sqrt{x}}$

   Solution.

2.6 Derivatives of Trigonometric Functions

Before we actually get into the derivatives of the trigonometric functions we need to give a couple of limits that will show up in the derivation of two of the derivatives.

$$\lim_{h\to 0}\frac{\sin h}{h}=1 \quad \text{and} \quad \lim_{h\to 0}\frac{1-\cos h}{h}=0.$$

Since $\sin(x+h)=\sin x\cos h+\cos x\sin h$. We have

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\big(\sin x\cos h+\cos x\sin h\big)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\big(\cos h-1\big)+\cos x\sin h}{h}\\
&= \lim_{h\to 0}\left(\sin x\cdot\frac{\cos h-1}{h}\right)+\lim_{h\to 0}\left(\cos x\cdot\frac{\sin h}{h}\right)\\
&= \sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h} \quad =\sin x\cdot 0+\cos x\cdot 1=\cos x
\end{aligned}$$

Thus, we have $\frac{d}{dx}\sin x=\cos x$.

Similarly, using $\cos(x+h)=\cos x\cos h-\sin x\sin h$, we have

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\
&= \lim_{h\to 0}\frac{\big(\cos x\cos h-\sin x\sin h\big)-\cos x}{h}\\
&= \lim_{h\to 0}\frac{\cos x\big(\cos h-1\big)-\sin x\sin h}{h}\\
&= \cos x\lim_{h\to 0}\frac{\cos h-1}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h} \quad =\cos x\cdot 0-\sin x\cdot 1=-\sin x
\end{aligned}$$

Here are the derivatives of all six of the trigonometric functions.

Theorem 2.8

1. $\frac{d}{dx}[\sin x]=\cos x$
2. $\frac{d}{dx}[\cos x]=-\sin x$
3. $\frac{d}{dx}[\tan x]=\sec^2 x$
4. $\frac{d}{dx}[\cot x]=-\csc^2 x$
5. $\frac{d}{dx}[\sec x]=\sec x\tan x$
6. $\frac{d}{dx}[\csc x]=-\csc x\cot x$

Example 2.19 1. Let $f(x) = \dfrac{\sin x}{1+\cos x}$. Find $f'(x)$.

Solution.

2. Let $f(x) = \sec x$. Find $f''(\frac{\pi}{4})$.

Solution.

2.7 The Chain Rule

Theorem 2.9 (Chain Rule) If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 2.20 Find $\dfrac{dw}{dt}$ if $w = \tan x$ and $x = 4t^3 + t$.

Solution.

Example 2.21 Find $\dfrac{dy}{dt}$ if $y = \sin(4t^3)$.

Solution.

Example 2.22 Find $\dfrac{dy}{dx}$ if $y = \sqrt[5]{(x^2 - 9x + 1)^2}$.

Solution.

Example 2.23 Find $\dfrac{dw}{dz}$ if $w(z) = \dfrac{1}{\sqrt{9z + 1}}$.

Solution.

Example 2.24 Find $\dfrac{df}{dx}$ if $f(x) = (x^2 - 5)^{-8}$.

Solution.

Example 2.25 Find $\dfrac{dg}{dt}$ if $g(t) = \dfrac{-3}{\cos^2(t-7)}$.

Solution.

Example 2.26 Find $\dfrac{ds}{dt}$ if $s(t) = \cos^3\left(\sin\sqrt{t}\,\right)$.

Solution.

Exercise 2

1. Given $f(x) = x^2$, use the definition of the derivative to find

   (a) $f'(x)$

   (b) $f'(2)$

   (c) the equation of the tangent line of $f(x)$ at $x = 2$

2. (a) Use the definition to find the derivative of $f(x) = x^2 - 3x$.

   (b) Find the equations of the tangent lines of $f(x)$ at $x = 0$, $x = 2$ and $x = 3$

3. Let $f(x) = |x|$. Determine whether f is differentiable at $x = 0$.

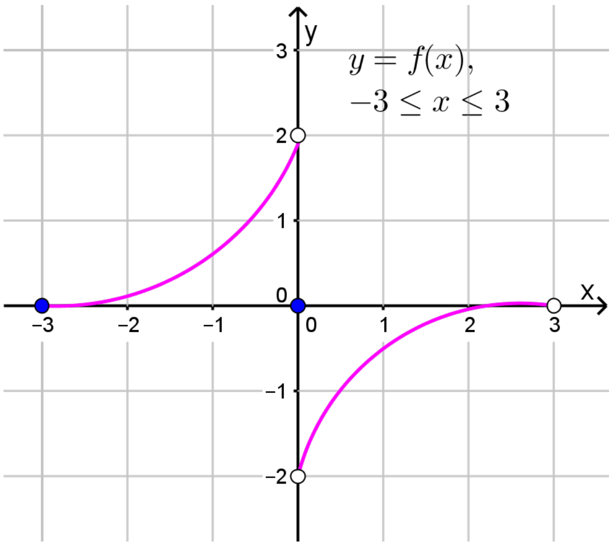
4. Let

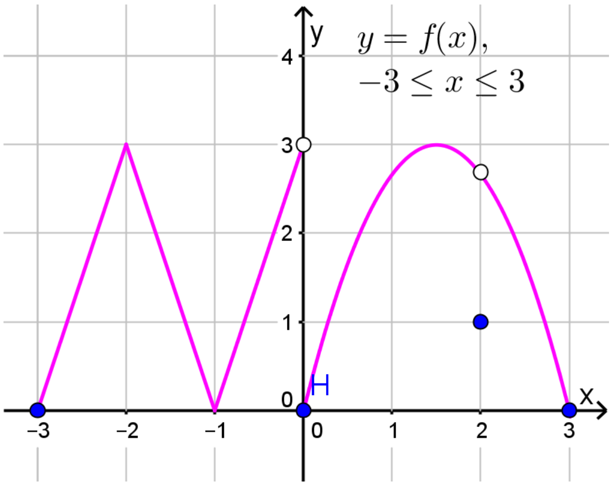
$$f(x) = \begin{cases} 2x - 1 & \text{if } x \leq 3 \\ x^2 - 4 & \text{if } x > 3 \end{cases}$$

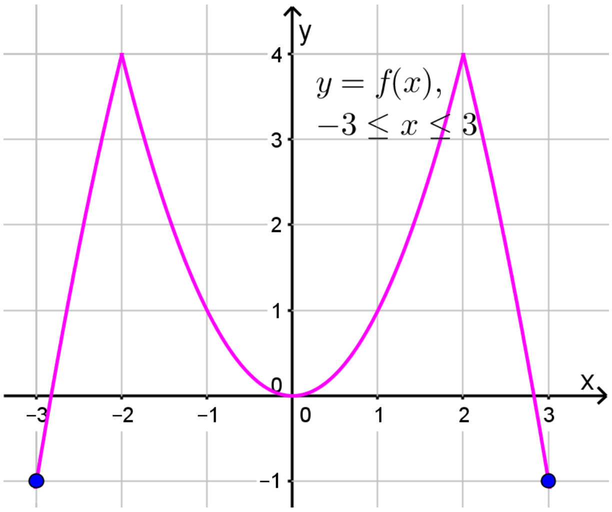
   Determine whether f is differentiable at $x = 3$.

5. From the following graphs of f defined on $[-3, 3]$, determine where each f is

   1. differentiable 2. continuous and non-differentiable 3. discontinuous and non-differentiable







6. Let $f(x) = x^4 - 6x^2 + 10$.

   (a) Find $f'(x)$.

   (b) Find the equation of the tangent line at $x = 1$.

   (c) Find the values of x where the tangent line is horizontal.

7. Find the derivative of the following functions:

   (a) $f(x) = \pi^3$

   (b) $f(u) = u^7$

   (c) $f(t) = \dfrac{5}{t^8}$

   (d) $f(w) = \sqrt[3]{w}$

   (e) $f(x) = -3x^8 + 2x + \cos x - \dfrac{\sin x}{18}$

8. Find $y'(1)$.

   (a) $y = 5x^2 - 3x + 1$

   (b) $y = 1 + \dfrac{4}{\sqrt{u}}$

9. Find an equation of the tangent line to the graph of $y = \dfrac{1}{2}x^2 + 5x - 2$ at $(0, -2)$.

10. Find $y^{(5)}$ where $y = x^4 - 7x^3 + 8x + 500$.

11. Find the derivative of the following functions:

    (a) $f(x) = (x+1)(2x^{-3} + x^{-4})$

    (b) $f(u) = (3u^2 + 6)(2u - \dfrac{1}{16})$

    (c) $f(t) = \dfrac{t^2}{3t - 5}$

    (d) $f(w) = \dfrac{4w - 1}{w^2 + 5}$

12. Find $y''(x)$.

    (a) $y = -4x^3 + 5x^2 - 3x + 1$

    (b) $y = 5 + \dfrac{4}{\sqrt{u}}$

13. Compute the derivatives of the following functions.

(a) $y = 2\cot(\pi x)$

(b) $y = \dfrac{\cos t}{2 - \sin t}$

(c) $y = \tan\left(\dfrac{x}{x+1}\right)$

(d) $y = \dfrac{\sin^2(x^3)}{\tan^3(x^4)}$

(e) $y = 2\csc x + \cot x$

14. Compute the derivatives of the following functions.

(a) $y = (6x-1)^4$

(b) $f(x) = (9 - x^2)^{3/5}$

(c) $f(t) = \sqrt{1 + t^2}$

(d) $y = \dfrac{1}{x-8}$

(e) $f(x) = x^6(3x-2)^3$

(f) $y = t^3\sqrt{1+t}$

(g) $f(x) = \left(\dfrac{x+2}{x^3 - 5x + 9}\right)^2$

(h) $y = \cos^2(3x)$

(i) $w(t) = \sec(4t^2 - 1)$

(j) $y = \cos(\sin x)$

3

Topics of Differentiation

3.1 Implicit Function

Sometimes we deal with an expression that is not explicitly given but instead implicitly given in the form $f(x,y) = 0$ or $f(x,y) = g(x,y)$, e.g., $2y^2 = 3x + 8$. To find the derivative in this case,

1. first differentiate both sides of the expression,
2. then solve for $\dfrac{dy}{dx}$.

Example 3.1 Find $\dfrac{dy}{dx}$ where $y^2 - x + 1 = 0$.

Solution.

Example 3.2 Find $\dfrac{dy}{dx}$ where $4xy^2 = 9$.

Solution.

Example 3.3 Find $\dfrac{dy}{dx}$ where $4x^4y - y^3 = 4\sin y$.

Solution.

Example 3.4 Find $\dfrac{d^2y}{dx^2}$ where $2x^3 - 3y^2 = 8$.

Solution.

Example 3.5 Find the slopes of the tangent lines to the curve $y^2 - x + 1 = 0$ at the points $(2, -1)$ and $(2, 1)$.

Solution.

3.2 Derivatives of Exponential and Logarithmic Functions

3.2.1 Derivatives of Exponential and Logarithmic Functions

Here is a summary of the derivative formulas in this section.

Theorem 3.1

1. $\frac{d}{dx}[a^x] = a^x \ln a$
2. $\frac{d}{dx}[e^x] = e^x$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$
4. $\frac{d}{dx}[\ln x] = \frac{1}{x}$

Example 3.6

$\frac{d}{dx}[e^x] =$.............

$\frac{d}{dx}[2^x] =$.............

$\frac{d}{dx}[5^x] =$.............

$\frac{d}{dx}[\pi^x] =$.............

$\frac{d}{dx}[2 \cdot 5^x] =$.............

$\frac{d}{dx}[\ln x] =$.............

$\frac{d}{dx}[\log_3 x] =$.............

$\frac{d}{dx}[4 \ln x] =$.............

$\frac{d}{dx}[\ln x^2] =$.............

Example 3.7 Find the derivative of $y = 2^{\ln 5x}$.

Solution.

Example 3.8 Find the derivative of $y = e^{\cos^3 x}$.

Solution.

Example 3.9 Find the derivative of $y = (\log_3 x)^3$.

Solution.

3.2.2 Logarithm Differentiation

We now consider a technique called logarithmic differentiation. This technique is useful for functions that are composed of products, quotients, and powers.

Example 3.10 Find the derivative of $y = \dfrac{x \ln x \sqrt{x^2+1}}{(1+x^4)^2}$.

Solution.

Example 3.11 Find the derivative of $y = (x^2+1)^{\sin x}$.

Solution.

3.3 Derivatives of the Inverse Trigonometric Functions

Let $y = \arcsin x$. Then $x = \sin y$. Hence,

The method used to derive this formula can be used to obtain generalized derivative formulas for the remaining inverse trigonometric functions. The following is a complete list of these formulas, each of which is valid on the natural domain of the function.

Theorem 3.2

1. $\frac{d}{dx}[\arcsin u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
2. $\frac{d}{dx}[\arccos u] = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
3. $\frac{d}{dx}[\arctan u] = \frac{1}{1+u^2}\frac{du}{dx}$
4. $\frac{d}{dx}[\text{arccot } u] = \frac{-1}{1+u^2}\frac{du}{dx}$
5. $\frac{d}{dx}[\text{arcsec } u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$
6. $\frac{d}{dx}[\text{arccsc } u] = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

Example 3.12 Find the derivative of $y = \arcsin(x^3 + 1)$.

Solution.

Example 3.13 Find the derivative of $y = \arccos(\sqrt{x})$.

Solution.

Example 3.14 Find $\dfrac{dy}{dx}$ when $x^3 + x \arctan y = y$.

Solution.

3.4 Derivatives of Hyperbolic Functions

Definition 10 ***Basic hyperbolic functions*** can be seen in Figure 3.1 and are defined as follow:

1. Hyperbolic cosine of x: $\cosh x = \dfrac{e^x + e^{-x}}{2}$
2. Hyperbolic sine of x : $\sinh x = \dfrac{e^x - e^{-x}}{2}$
3. Hyperbolic tangent of x : $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
4. Hyperbolic cotangent of x : $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
5. Hyperbolic secant of x : $\text{sech } x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
6. Hyperbolic cosecant of x : $\text{csch } x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

For $u = u(x)$ we have ***derivatives of hyperbolic functions*** as following:

1. $\dfrac{d}{dx}[\sinh u] = \cosh u \dfrac{du}{dx}$
2. $\dfrac{d}{dx}[\tanh u] = \text{sech}^2 u \dfrac{du}{dx}$
3. $\dfrac{d}{dx}[\text{sech } u] = -\text{sech } u \ \tanh u \dfrac{du}{dx}$
4. $\dfrac{d}{dx}[\cosh u] = \sinh u \dfrac{du}{dx}$
5. $\dfrac{d}{dx}[\coth u] = -\text{csch}^2 u \dfrac{du}{dx}$
6. $\dfrac{d}{dx}[\text{csch } u] = \text{csch } u \coth u \dfrac{du}{dx}$

Some basic graphs of hyperbolic functions are

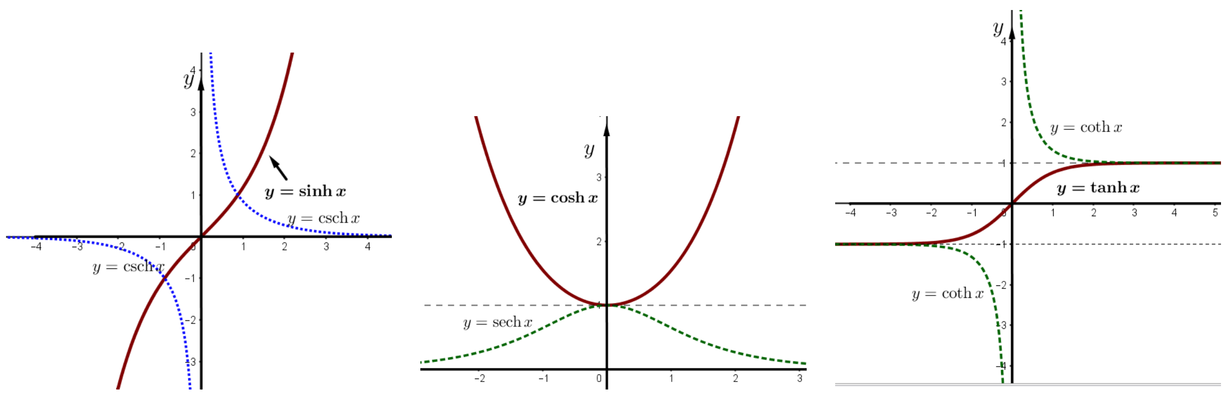


Figure 3.1: Graphs of Hyperbolic Functions

Example 3.15 Find $\frac{d}{dx}\sinh 5x$.

Solution.

Example 3.16 Find $\frac{d}{dx}\big(x\coth\sqrt{2-3x^4}\,\big)$.

Solution.

Example 3.17 Find $\frac{d}{dx}\big(e^{2x}\tanh x^4+5\big)$.

Solution.

Exercise 3a

1. Find $\dfrac{dy}{dx}$ when $(x, y) = (1, 1)$ where $x^2 + y^2 = 2$.
2. Find $\dfrac{dy}{dx}$ when $xy = x + y$.
3. Find $\dfrac{dy}{dx}$ when $x^2 + y^2 = \sqrt{7}$.
4. Find $\dfrac{dy}{dx}$ when $x^2 + xy - y^3 = xy^2$.
5. Find $\dfrac{dy}{dx}$ when $xy + x + y = 5$.
6. Find $\dfrac{dy}{dx}$ when $\sqrt{x} + \sqrt{y} = 25$.
7. Find $\dfrac{dy}{dx}$ when $\sin(xy) = 2x + 5$.
8. Find $\dfrac{dy}{dx}$ when $x^2y + 2xy^3 - y = 5$.
9. Find $\dfrac{dy}{dx}$ when $x^3 + y^3 - 5x^2y + xy = 1$.
10. Find $\dfrac{dy}{dx}$ when $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 5$.
11. Find $\dfrac{dy}{dx}$ when $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$.
12. Find $\dfrac{dy}{dx}$ when $y\cos(x^2) = x\cos(y^2)$.
13. Find $\dfrac{dy}{dx}$ when $xy = \sqrt{\cot x}$.

14. Find $\dfrac{dy}{dx}$ when

(a) $y = \ln(x^2 + 2)$

(b) $y = (1 + \ln x)x \ln x$

(c) $y = \dfrac{1}{\sqrt{\ln x}}$

(d) $y = x3^{x^2}$

(e) $y = \tan(\ln x)$

(f) $y = e^{\cot 5x}$

(g) $y = e^x \cos(e^x + 1)$

(h) $y = \ln|\cot x|$

(i) $y = \ln(\ln x^2)$

(j) $y = \ln(\sqrt{x})$

(k) $y = \sqrt{\ln 4x - x^2}$

(l) $y = \dfrac{\ln x}{\ln(x+5)}$

(m) $y = \log_3(x^5 + 3x - 1)$

(n) $y = e^{\ln x}$

(o) $y = x3^{x+5}$

(p) $y = \dfrac{e^x}{\ln x}$

(q) $y = \cos(e^{1/x})$

15. Find $\dfrac{dy}{dx}$ when

(a) $y = \arcsin(6x^2 - 1)$

(b) $y = \arccos(\dfrac{x}{5})$

(c) $y = \arctan(t^3)$

(d) $y = \sqrt{x}\arcsin(x)$

(e) $y = \ln(\arccos x)$

(f) $y = \text{arcsec} x + \text{arccsc } x$

(g) $y = e^{\text{arccot } x}$

(h) $y = \sqrt{\arcsin x}$

(i) $y = \arccos(e^x)$

(j) $y = \arccos(e^{1/x})$

16. Find $\dfrac{dy}{dx}$ when

(a) $y = 2\coth(\ln x)$

(b) $y = \cosh 5x - 4\sinh 2x$

(c) $y = \sinh(4 - 2e^{3x})$

(d) $y = \sqrt{x}\sinh(4x)$

(e) $y = \ln(\text{csch} x)$

(f) $y = \dfrac{1 + \sin 4x}{\cosh x}$

(g) $y = e^{\tanh x}$

(h) $y = \sqrt{\sinh(2/x)}$

3.5 Related Rates

In this section, one tries to find the rate at which some quantity is changing by relating itself to other quantities, whose rates of change are known.

Example 3.18 Suppose that x, y are differentiable functions of t and are related by $y = x^4 + 5$. Find $\frac{dy}{dt}|_{t=2}$ if $x(2) = 1$ and $\frac{dx}{dt}|_{t=2} = -2$.

Example 3.19 Air is being pumped into a spherical balloon at the rate of 6 cubic centimetres per second. What is the rate of change of the radius at the instant the volume equals 36π? The volume of a sphere of radius r is $\frac{4\pi r^3}{3}$.

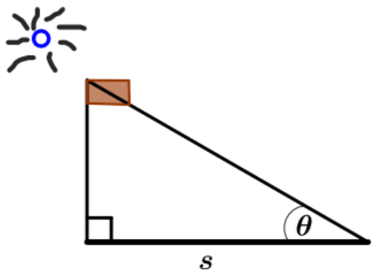
The steps in solving a related rates problem:

- Decide what the two variables are.
- Find an equation relating them.
- Take $\dfrac{d}{dt}$ of both sides.
- Plug in all known values at the instant in question.
- Solve for the unknown rate.

Example 3.20 A ladder 20 feet long is placed against a wall. The foot of the ladder begins to slide away from the wall at the rate of 1 ft/sec. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 12 feet from the wall?

Example 3.21 On a sunny day, a 50 ft flagpole casts a shadow that changes with the angle of elevation of the Sun. Let s be the length of the shadow and θ the angle of elevation of the Sun. Find the rate at which the length of the shadow is changing with respect to θ when $\theta = 45°$. Express your answer in units of feet/degree.

Solution.



3.5.1 Local Linear Approximation; Differentials

Local Linear Approximation

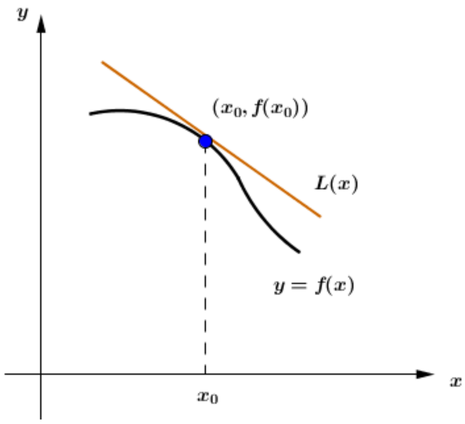


Figure 3.2: Linear approximation near x_0

Many functions are computable only for specific values. For other values, we need to estimate them from what is known. The line that best approximates the graph of $y = f(x)$ at $x = x_0$ is

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Thus, for values of x near x_0 we can approximate values of $f(x)$ by

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

This is called the ***local linear approximation*** of f at x_0. This formula can also be expressed in terms of the increment $\Delta x = (x - x_0)$ as $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$.

Example 3.22 Use the local linear approximation to approximate $\sqrt{4.01}$.

Differentials

Let $y = f(x)$ be a differentiable function. The differential of y with respect to x, denoted dy, is $f'(x)dx$, i.e.,

$$dy = f'(x)dx$$

Estimation with Differential

Recall that

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

So when Δx is small,

$$\Delta y \approx dy$$

$$\Delta y = f(x + \Delta x) - f(x) \approx f'(x)\Delta x \approx f'(x)dx$$

Example 3.23 Suppose that the side of a square is measured with a ruler to be 8 inches with a measurement error of at most $\pm\frac{1}{64}$ inches. Estimate the error in the computed area of the square.

3.6 Limits at Infinity

In this section, we will discuss the behavior of functions as $x \to -\infty$ and $+\infty$ (the ***end behavior*** of the function).

Example 3.24 Consider the function $f(x) = \frac{1}{x}$.

As $x \to -\infty$, we have $\lim_{x\to-\infty} \frac{1}{x} = $ ....

	values						conclusion
x	-1	-10	-100	-1000	-10000	$\cdots$	as $x \to -\infty$
$\frac{1}{x}$	-1	-0.1	-0.01	-0.001	-0.0001	$\cdots$	

As $x \to \infty$, we have $\lim_{x\to\infty} \frac{1}{x} = $ ....

	values						conclusion
x	1	10	100	1000	10000	$\cdots$	as $x \to \infty$
$\frac{1}{x}$	1	0.1	0.01	0.001	0.0001	$\cdots$	

In general, if the values of $f(x)$ eventually get as close as we like to a number L as x decreases without bound. Then we write

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad f(x) \to L \quad \text{as} \quad x \to -\infty$$

Similarly, if the values of $f(x)$ eventually get as close as we like to a number L as x increases without bound. Then we write

$$\lim_{x\to+\infty} f(x) = L \quad \text{or} \quad f(x) \to L \quad \text{as} \quad x \to +\infty$$

Definition 11 (Asymptotes) A line $y = L$ is a ***horizontal asymptote*** of the graph $y = f(x)$ if either

$$\lim_{x\to-\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to\infty} f(x) = L$$

and a line $x = a$ is a ***vertical asymptote*** of the graph $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty$$

Theorem 3.3 Let k, L, M be real numbers, and let f, g be functions such that $\lim_{x\to+\infty} f(x) = L$ and $\lim_{x\to+\infty} g(x) = M$. Then

1. $\lim_{x\to+\infty} k = k$,
2. $\lim_{x\to+\infty} \frac{1}{x^a} = 0$ when a is positive,
3. $\lim_{x\to+\infty} [f(x) \pm g(x)] = L \pm M$,
4. $\lim_{x\to+\infty} f(x)g(x) = LM$,
5. $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$ when $M \neq 0$,
6. $\lim_{x\to+\infty} [f(x)]^a = L^a$ when $a > 0$ and $L \geq 0$.

Note that the above theorem also holds in case of x approaching to $-\infty$.

Example 3.25 Find $\lim_{x\to-\infty} \left(\frac{5}{x} - 7\right)$.

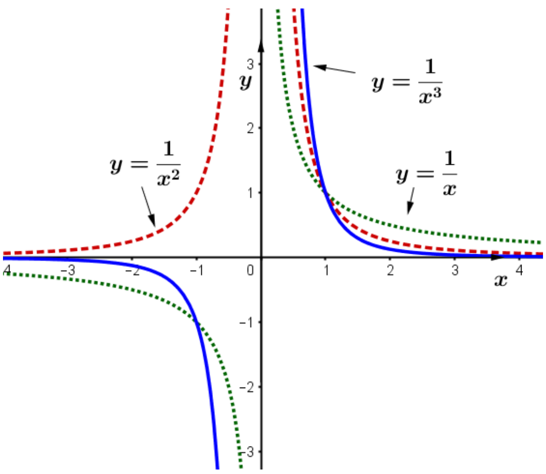


Figure 3.3: Graphs of rational functions $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}$.

Infinite Limits at Infinity

If the values of $f(x)$ decrease or increase without bound as $x \to -\infty$, then we write

$$\lim_{x\to-\infty} f(x) = -\infty \text{ or } \lim_{x\to-\infty} f(x) = +\infty,$$

as appropriate. Similarly, if the values of $f(x)$ decrease or increase without bound as $x \to +\infty$, then we write

$$\lim_{x\to+\infty} f(x) = -\infty \text{ or } \lim_{x\to+\infty} f(x) = +\infty,$$

as appropriate.

Example 3.26 Find the following limits.

1. $\lim_{x\to-\infty} x =$ $\qquad$ $\lim_{x\to+\infty} x =$
2. $\lim_{x\to-\infty} x^2 =$ $\qquad$ $\lim_{x\to+\infty} x^2 =$
3. $\lim_{x\to-\infty} x^3 =$ $\qquad$ $\lim_{x\to+\infty} x^3 =$
4. $\lim_{x\to-\infty} x^4 =$ $\qquad$ $\lim_{x\to+\infty} x^4 =$
5. $\lim_{x\to-\infty} 15x =$ $\qquad$ $\lim_{x\to+\infty} 15x =$
6. $\lim_{x\to-\infty} -3x^2 =$ $\qquad$ $\lim_{x\to+\infty} -3x^2 =$

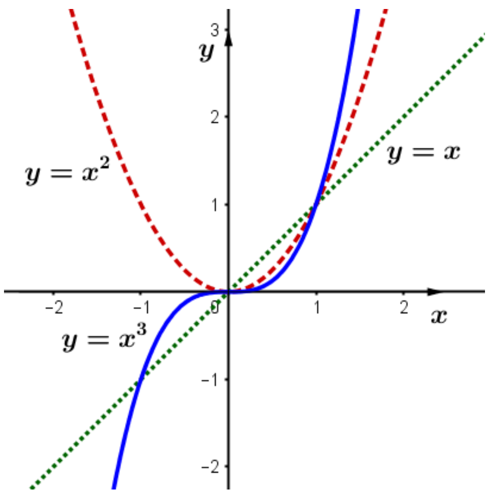


Figure 3.4: Graphs of polynomails x, x^2, x^3.

The end behavior of polynomials

The end behavior of a polynomial matches the end behavior of its highest degree term.

Example 3.27 Find the following limits.

1. $\lim\limits_{x \to -\infty} (5x^4 - 10x^3 + 897x + 84000) =$

2. $\lim\limits_{x \to +\infty} (-x^3 + 650x^2 + 7x - 1) =$

Example 3.28 Find the following limits.

1. $\lim\limits_{x \to -\infty} \dfrac{24x - 9}{8x + 11}$

2. $\lim\limits_{x \to +\infty} \dfrac{3x^2 - 5x}{8x^6 + 14x - 1}$

Limits Involving Radicals

Example 3.29 Find the following limits.

1. $\lim\limits_{x \to -\infty} \dfrac{\sqrt{4x^2 + 3x + 2}}{2x - 7}$

2. $\lim_{x\to+\infty}\left(x^2-\sqrt{x^4+9}\right)$

3. $\lim_{x\to+\infty}\dfrac{1}{\sqrt{x^4+3x^2}-x^2}$

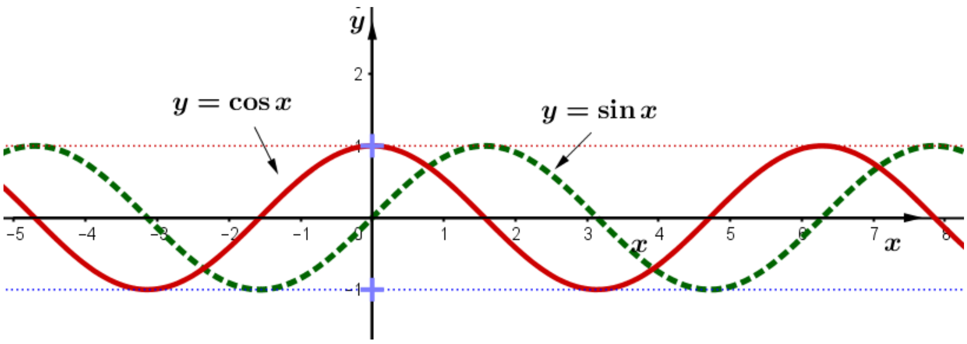
End behavior of Trigonometric, Exponential and Logarithmic Functions

$$\left.\begin{aligned} &\lim_{x\to-\infty} \sin x, \\ &\lim_{x\to+\infty} \sin x, \\ &\lim_{x\to-\infty} \cos x, \\ &\lim_{x\to+\infty} \cos x \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots$$

$\lim_{x\to-\infty} e^x =$ $\qquad$ $\lim_{x\to+\infty} e^x =$

$\lim_{x\to-\infty} e^{-x} =$ $\qquad$ $\lim_{x\to+\infty} e^{-x} =$

$\lim_{x\to 0^+} \ln x =$ $\qquad$ $\lim_{x\to+\infty} \ln x =$



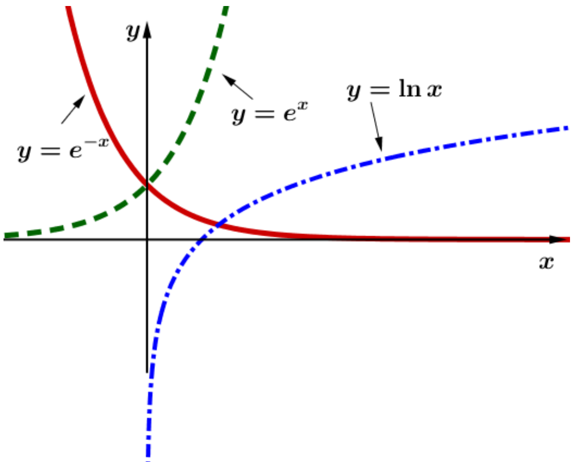


Figure 3.5: Graphs of $\sin x, \cos x, e^x, e^{-x}$ and $\ln x$.

3.7 Indeterminate Forms

3.7.1 L'Hôpital's Rule

Theorem 3.4 Let f, g be differentiable functions on an open interval containing $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Note that the theorem also works for one-sided limits.

Example 3.30 Find $\lim_{x\to 0} \frac{3x - \sin x}{x}$.

Theorem 3.5 Let f, g be differentiable functions on an open interval containing $x = a$ and $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Note that the theorem also works for one-sided limits.

Example 3.31 Find $\lim_{x\to\infty} \frac{2x + 3}{5x + 7}$.

Other Indeterminate Forms

Solving other indeterminate forms require us to change the limit so that we can use L'Hôpital's rule.

Example 3.32 Find $\lim\limits_{x\to\infty} x\tan\dfrac{1}{x}$.

Example 3.33 Find $\lim\limits_{x\to 0^+}\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$.

In case the limit of the form $\mathbf{0^0}, \infty^{\mathbf{0}}, \mathbf{1}^{\infty}$, apply logarithm to the limit.

Example 3.34 Find $\lim\limits_{x\to 0^+}(\sin x)^x$.

Failures of L'Hôpital's Rule

Sometimes using L'Hôpital's Rule will not solve the problem as in following example.

Example 3.35 Find $\lim\limits_{x\to(\pi/2)^-} \dfrac{\sec x}{\tan x}$.

Failed Solution First, let's find out what happens if we use L'Hôpital's rule.

Notice that $\lim\limits_{x\to(\pi/2)^-} \sec x = \infty = \lim\limits_{x\to(\pi/2)^-} \tan x$. Therefore, it is possible to use L'Hôpital's rule.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x} = \lim_{x\to(\pi/2)^-} \frac{\frac{d}{dx}\sec x}{\frac{d}{dx}\tan x} = \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x}$$

Using L'Hôpital's rule again, we will have

$$\lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x} = \lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x}$$

So we return to the problem at the beginning and nothing is solved!

Good Solution ***Try simplifying the expression first.***

$$\frac{\sec x}{\tan x} = \frac{1/\cos x}{\sin x/\cos x} = \frac{1}{\sin x}$$

Therefore,

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x} = \lim_{x\to(\pi/2)^-} \frac{1}{\sin x} = 1$$

Exercise 3b

1. A small steel ball dropped from a tower will fall a distance of y feet in x seconds, as given approximately by the formula

$$y = f(x) = 16x^2$$

(Note : Positive y direction is down.)

   (a) Find the position of the ball at $x = 0$, $x = 1$, $x = 2$ and $x = 3$ seconds.

   (b) Find the average velocity from $x = 2$ seconds to $x = 3$ seconds.

   (c) Find and simplify the average velocity from $x = 2$ seconds to $x = 2 + h$ seconds, $h \neq 0$.

   (d) Find the limit of the expression from part (3) as $h \to 0$ if that limit exists and discuss possible interpretations of the limit.

2. An object moves along the y axis (marked in feet) so that its position at time x (in seconds) is

$$y = f(x) = x^3 - 6x^2 + 9x$$

   (a) Find the instantaneous velocity function v.

   (b) Find the velocity at $x = 2$ and $x = 5$ seconds.

   (c) Find the time(s) when the velocity is 0.

3. Suppose an object is moving along a path described by $y = x^2$, that is, it is moving on a parabolic path. At a particular time, say $t = 5$, the x coordinate is 6 and we measure the speed at which the x coordinate of the object is changing and find that $\frac{dx}{dt} = 3$. At the same time, how fast is the y coordinate changing?

4. A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the

plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

5. You are inflating a spherical balloon at the rate of 7 cm^3/sec. How fast is its radius increasing when the radius is 4 cm?

6. Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm. How fast is the water level rising when the water is 4 cm deep (at its deepest point)?

7. A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometers to the north of P and traveling at 80 km/hr, while car B is 15 kilometers to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing?

8. A cylindrical tank standing upright (with one circular base on the ground) has radius 20 cm. How fast does the water level in the tank drop when the water is being drained at 25 cm^3/sec?

9. A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.6 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall?

10. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 mi^2/h. How fast is the radius of the spill increasing when the area is 9 mi^2?

11. Sand is poured onto a surface at 15 cm^3/sec, forming a conical pile whose base diameter is always equal to its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high?

12. A balloon is at a height of 50 meters, and is rising at the constant rate of 5 m/sec. A bicyclist passes beneath it, traveling in a straight line at the constant speed of 10 m/sec. How fast is the distance between the bicyclist and the balloon increasing 2 seconds later?

13. A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must air be removed when the radius is 9 cm?

14. A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 2 kilometers from the helicopter, and that this distance is decreasing at 250 kph. Find the speed of the car.

15. Use an appropriate local linear approximation to estimate the value of the following quantity.

(a) $(2.01)^{11}$ (c) $\sqrt{48.84}$ (e) $\sin 31^\circ$ (g) $\ln 0.99$

(b) $(1.98)^{100}$ (d) $\sqrt[4]{82}$ (f) $\cos 0.2$ (h) $\log_{10} 101$

16. Find the differential dy of following functions.

(a) $y = 3x^6 + 4x - 2$ (c) $\frac{2x+3}{x-e^x}$ (e) $\sin \pi x + 7\cos x$ (g) $\frac{x^2e^{-x}}{\sec x}$

(b) $y = \sqrt{x^2 - x + 5}$ (d) $8^x - \ln(11x - 1)$ (f) $\tan^{-1}\sqrt{x}$

17. Find dy when $y = \sin(2\pi x)$ and discuss the relationship between dy and dx at $x = 1$.

18. Let $f(x) = x^4$. If $x_0 = 1$ and $dx = \Delta x = 1/2$, what are Δy and dy?

19. Let $f(x) = \sqrt{x}$. If $x_0 = 1$ and $dx = \Delta x = 1/10$, what are Δy and dy?

20. Let $f(x) = \sin(2x)$. If $x_0 = \pi$ and $dx = \Delta x = \pi/100$, what are Δy and dy?

21. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.02 cm thick to a sphere with diameter 40 meters. (Recall that the volume of a sphere of radius r is $V = (4/3)\pi r^3$. Notice that you are given that $dr = 0.02$.)

22. The side of a cube is measured to be 25 cm, with a possible error of ± 1 cm. Use differentials to estimate the error in the calculated volume.

23. The hypotenuse of a right triangle is known to be 10 in exactly, and one of the acute angles is measured to be 30° with a possible error of $\pm 1^\circ$. Use differentials to estimate the errors in the sides opposite and adjacent to the measured angle.

24. The electrical resistance R of a certain wire is given by $R = k/r^2$, where k is a constant and r is the radius of the wire. Assuming that the radius r has a possible error of $\pm 5\%$, use differentials to estimate the percentage error in R. (Assume k is exact.)

25. Find the following limits.

(a) $\lim\limits_{x\to+\infty} \left(x^7 - 3x^4 + 2\right)$

(b) $\lim\limits_{x\to-\infty} \left(3x^4 + x^2 - x\right)$

(c) $\lim\limits_{x\to+\infty} \sqrt{6x - 100}$

(d) $\lim\limits_{x\to-\infty} \sqrt{4 - x^2 - 3x^5}$

(e) $\lim\limits_{x\to-\infty} \dfrac{2x+3}{2x-3}$

(f) $\lim\limits_{x\to+\infty} \dfrac{x^4 + 5x^3 - 2x^2}{-2x^4 + 6x - 7}$

(g) $\lim\limits_{x\to+\infty} \dfrac{x-8}{x^2+5}$

(h) $\lim_{x\to+\infty} \frac{x^3}{1000x^2+2x-10000}$

(i) $\lim_{x\to+\infty} \sqrt{\frac{3x^2-x+4}{4x^2+2x+9}}$

(j) $\lim_{x\to-\infty} \sqrt[7]{\frac{-5x^4+4x^3-6}{x^4+3x-11}}$

(k) $\lim_{x\to-\infty} \sqrt[3]{\frac{x^6+5x^2-1}{x^2+x+8}}$

(l) $\lim_{x\to-\infty} \frac{\sqrt{8x^2-3}}{x+7}$

(m) $\lim_{x\to+\infty} \frac{\sqrt{2x^4-5x^3-6}}{7x^2-300}$

(n) $\lim_{x\to-\infty} \frac{4x+1}{\sqrt{x^2-2x+3}}$

(o) $\lim_{x\to-\infty} \frac{-901x^3+x^2-5}{\sqrt{-3x^7-6x+4}}$

(p) $\lim_{x\to+\infty} \left(\sqrt{4x^2-6}-2x\right)$

(q) $\lim_{x\to+\infty} \left(\sqrt{4x^2-6x}-2x\right)$

(r) $\lim_{x\to-\infty} \left(\sqrt{4x^2-6x^3}-2x\right)$

(s) $\lim_{x\to-\infty} e^{2x+1818}$

(t) $\lim_{x\to+\infty} \frac{2e^x-6}{5e^x+6}$

(u) $\lim_{x\to+\infty} \frac{1}{x^5e^x}$

(v) $\lim_{x\to+\infty} e^x \ln x$

(w) $\lim_{x\to+\infty} \frac{-3}{\ln x}$

(x) $\lim_{x\to0^+} \frac{0.01x}{\ln x}$

(y) $\lim_{x\to+\infty} \ln\left(\frac{4}{e^x}\right)$

(z) $\lim_{x\to+\infty} \frac{\sin x}{x}$

26. Find the following limits.

(a) $\lim_{x\to0} \frac{\sin x}{x}$

(b) $\lim_{x\to0} \frac{1-\cos x}{x}$

(c) $\lim_{x\to0} \frac{x^2}{\sin x}$

(d) $\lim_{x\to\pi^+} \frac{\sin x}{x-\pi}$

(e) $\lim_{x\to\pi/2} \frac{1-\sin x}{\cos x}$

(f) $\lim_{x\to+\infty} \frac{6x}{\ln x}$

(g) $\lim_{x\to+\infty} \frac{x^{11}+45x^8-7x+4}{e^x}$

(h) $\lim_{x\to+\infty} \frac{e^x}{x^{111}}$

(i) $\lim_{x\to+\infty} xe^{-x}$

(j) $\lim_{x\to0^+} \frac{1-\ln x}{e^{1/x}}$

(k) $\lim_{x\to\pi^+} (x-\pi)\tan(x/2)$

(l) $\lim_{x\to0^+} \frac{\ln(\sin x)}{\ln(\tan x)}$

(m) $\lim_{x\to+\infty} x\sin(\pi/x)$

(n) $\lim_{x\to0^+} \tan x \ln x$

(o) $\lim_{x\to\pi} (x-\pi)\cot x$

(p) $\lim_{x\to0} \left(\csc x - \frac{1}{x}\right)$

(q) $\lim_{x\to0^+} \left(\frac{1}{x} - \frac{1}{e^x-1}\right)$

(r) $\lim_{x\to+\infty} \left(x-\ln(x^2+5)\right)$

(s) $\lim_{x\to+\infty} \left(\sqrt{x^2+2x}-x\right)$

(t) $\lim_{x\to+\infty} \left(1+\frac{2}{x}\right)^x$

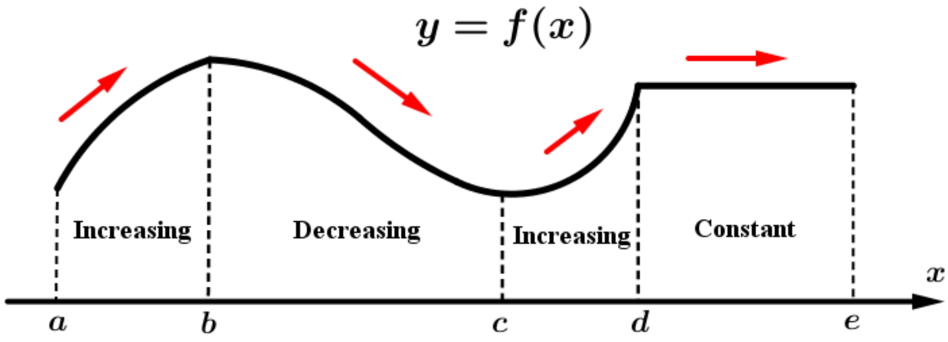
(u) $\lim_{x\to0} \left(x+e^{2x}\right)^{2/x}$

(v) $\lim_{x\to0^+} x^{\sin x}$

(w) $\lim_{x\to+\infty} (\ln x)^{1/x}$

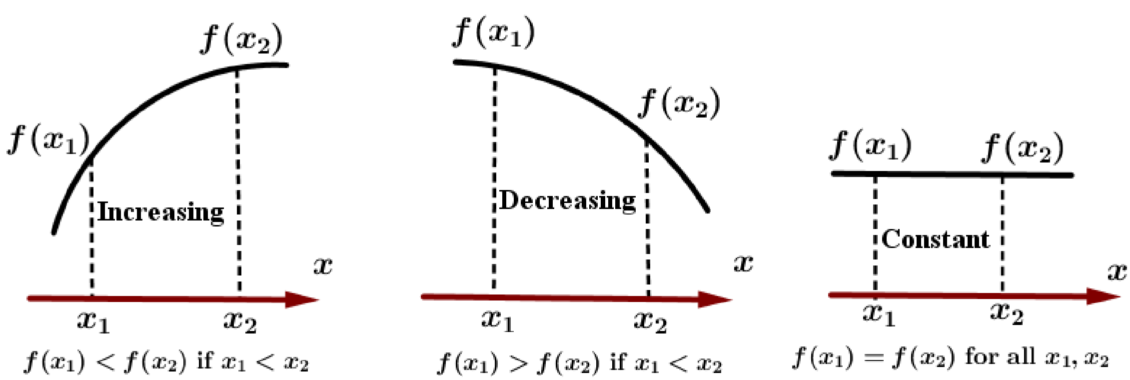
4 The Derivative in Graphing and Applications

4.1 Increasing and Decreasing Functions



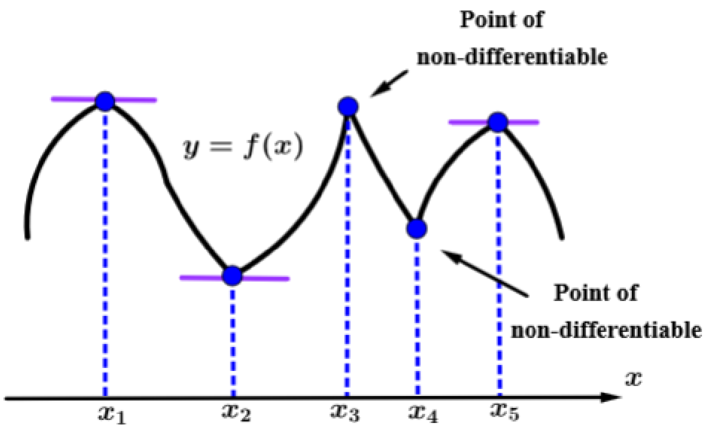
Definition 12 Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

1. f is ***increasing*** on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
2. f is ***decreasing*** on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
3. f is ***constant*** on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.



Definition 13 Suppose that f is a function defined on an open interval containing the point x_0. A point x_0 in the domain of f is said to be a ***critical point*** of f if either $f'(x_0) = 0$ or f is not differentiable at x_0.

Theorem 4.1 Assume that f is defined on an open interval containing x_0. If f has a relative extremum at x_0, then x_0 is a critical point of f, that is $f'(x_0) = 0$ or $f'(x_0)$ does not exist.



Example 4.1 Find all critical points of $f(x) = x^2 - 4x + 5$.

Solution.

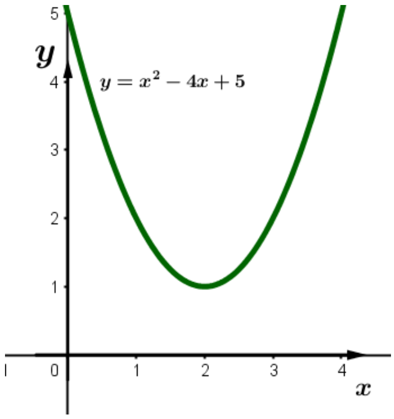
Example 4.2 Find all critical points of $f(x) = \dfrac{x-2}{x+2}$.

Solution.

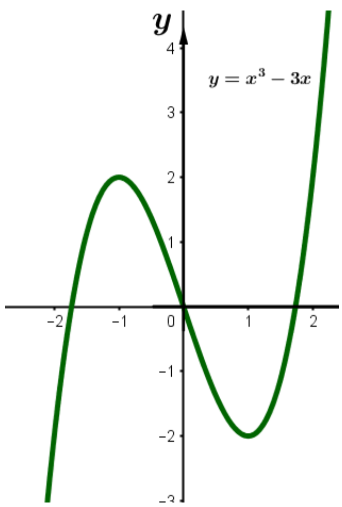
Theorem 4.2 Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.

Example 4.3 Find the intervals on which $f(x) = x^2 - 4x + 5$ is increasing and the intervals on which it is decreasing.

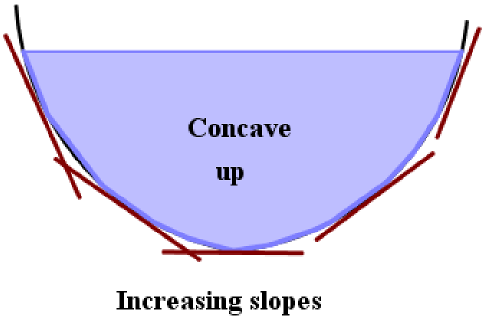


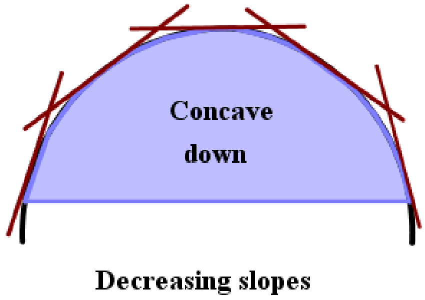
Example 4.4 Find the intervals on which $f(x) = x^3 - 3x$ is increasing and the intervals on which it is decreasing.



4.2 Concavity

Definition 14 If f is differentiable on an open interval, then f is said to be ***concave up*** on the open interval if $f'(x)$ is increasing on that interval, and f is said to be ***concave down*** on the open interval if $f'(x)$ is decreasing on that interval.



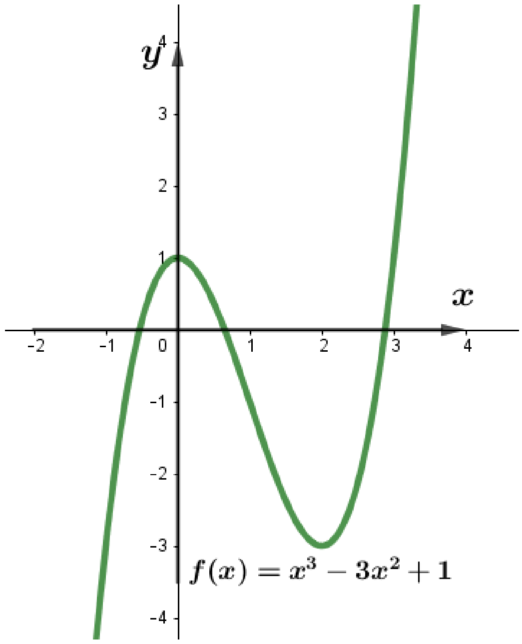


Theorem 4.3 Let f be twice differentiable on an open interval.

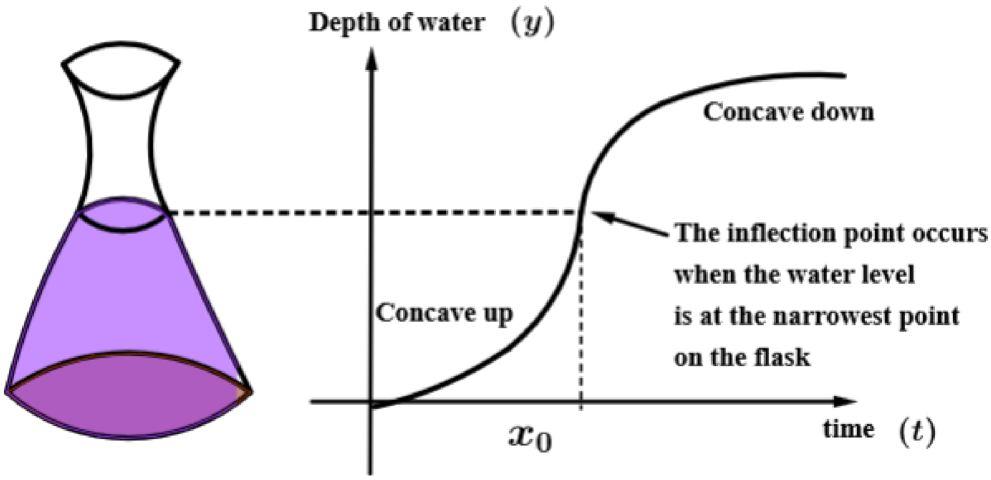
1. If $f''(x) > 0$ for every value of x in the open interval, then f is concave up on that interval.
2. If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval.

Definition 15 If f is continuous on an open interval containing a value x_0, and if f changes the direction of its concavity at the point $(x_0, f(x_0))$, then we say that f has an ***inflection point*** at x_0 and we call the point $(x_0, f(x_0))$ on the graph of f an inflection point of f.

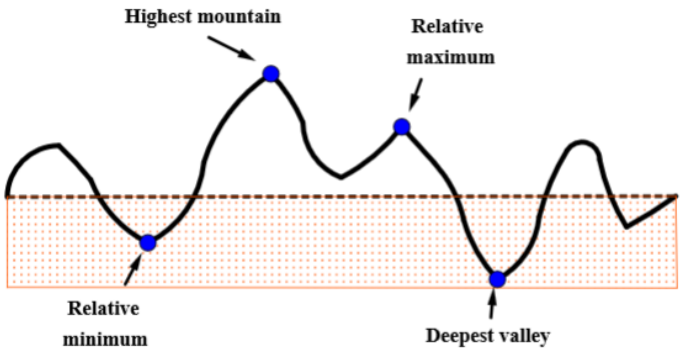
Example 4.5 Use the first and second derivatives of $f(x) = x^3 - 3x^2 + 1$ to determine where f is increasing, decreasing, concave up and concave down.



Inflection Points in Applications



4.3 Relative Maxima and Minima



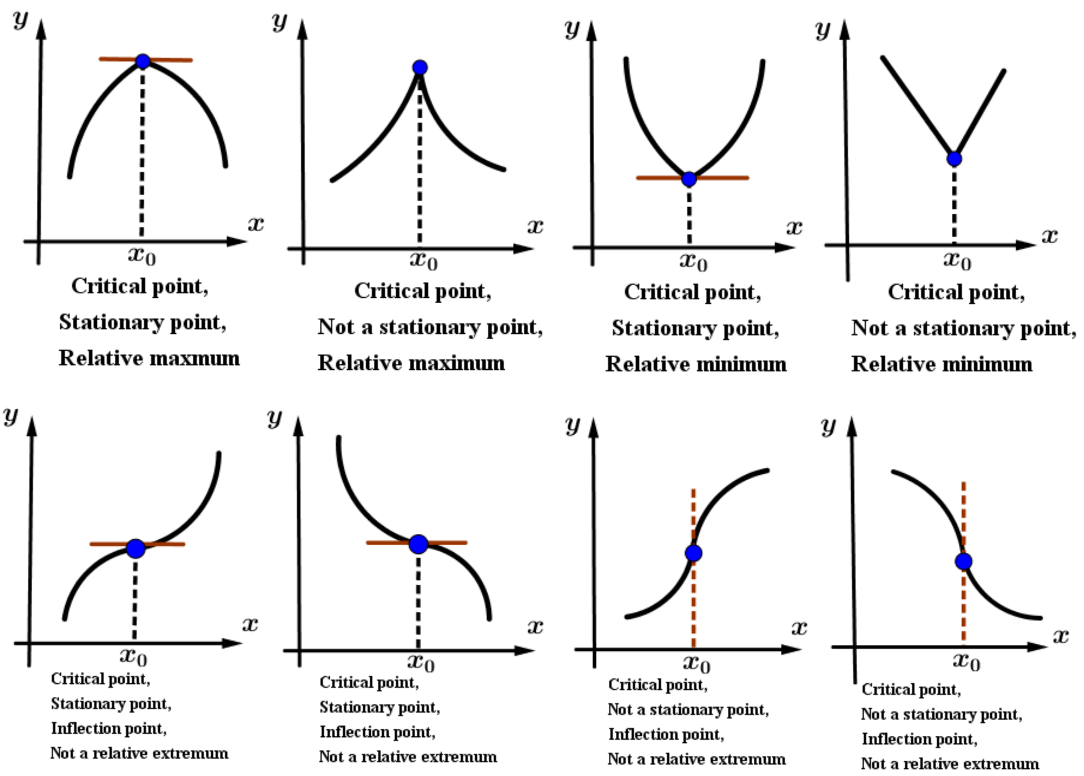
Definition 16 A function f is said to have a ***relative maximum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval.

Similarly, f is said to have a ***relative minimum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in the interval.

If f has either a relative maximum or a relative minimum at x_0, then f is said to have a ***relative extremum*** at x_0.

Theorem 4.4 Assume that f is differentiable on an open interval containing x_0. If f has a relative extremum at x_0, then $f'(x_0) = 0$.

4.4 First Derivative Test & Second Derivative Test



Theorem 4.5 Let x_0 be a critical point of the function f that is continuous on the open interval $[a, b]$ containing x_0. If f is differentiable on the interval, except possibly at x_0, then $f(x_0)$ can be classified as:

1. A relative minimum if $f'(x)$ changes from negative to positive at x_0.
2. A relative maximum if $f'(x)$ changes from positive to negative at x_0.
3. Neither a relative maximum nor a relative minimum if $f'(x)$ is positive on both sides of x_0 or negative on both sides of x_0.

Theorem 4.6 Suppose that f is twice differentiable at the point x_0 and $f'(x_0) = 0$.

1. If $f''(x_0) > 0$, then f has a relative minimum at x_0.
2. If $f''(x_0) < 0$, then f has a relative maximum at x_0.
3. If $f''(x_0) = 0$, then the test is inconclusive; that is f may have a relative maximum, a relative minimum or neither at x_0.

Example 4.6 Find the relative extrema of $f(x) = 3x^5 - 5x^3$.

Solution.

4.5 Analysis of Functions

Procedure Graphing Strategy

Step 1 : Find the domain, the x-intercepts and the y-intercept.

Step 2 : Find the vertical asymptotes.

Step 3 : Find the end behaviour.

Step 4 : Find $f'(x)$ and $f''(x)$.

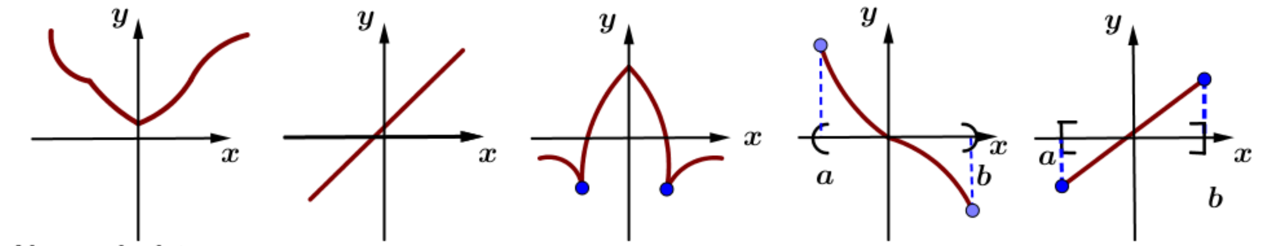
Step 5 : Sketch the graph of f.

Example 4.7 Sketch a graph of the equation $y = x^4 - 4x^3 + 10$.

Example 4.8 Sketch a graph of the equation $y = \dfrac{2x}{x-1}$.

4.6 Absolute Maxima and Minima

Definition 17 Consider an interval in the domain of a function f and a point x_0 in that interval. We say that f has an ***absolute maximum*** at x_0 if $f(x) \leq f(x_0)$ for all x in the interval, and we say that f has an ***absolute minimum*** at x_0 if $f(x_0) \leq f(x)$ for all x in the interval. We say that f has an ***absolute extremum*** at x_0 if it has either an absolute maximum or an absolute minimum at that point.



f has an absolute minimum but no absolute maximum on $(-\infty, \infty)$.

f has no absolute extreme on $(-\infty, \infty)$.

f has an absolute maximum and minimum on $(-\infty, \infty)$.

f has no absolute extrema on (a, b).

f has an absolute maximum and minimum on $[a, b]$.

Procedure for finding the absolute extremum on $[a, b]$

Step 1 : Find the critical points of f in (a, b).

Step 2 : Evaluate f at all the critical points and at the end points.

Step 3 : The largest of the values in step 2 is the absolute maximum and the smallest value is the absolute minimum.

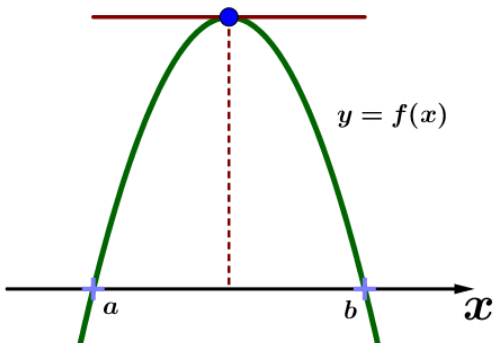
Example 4.9 Find the absolute extremum of $f(x) = 2x^{5/3} - 4x^{2/3}$ on the interval $[-1, 1]$.

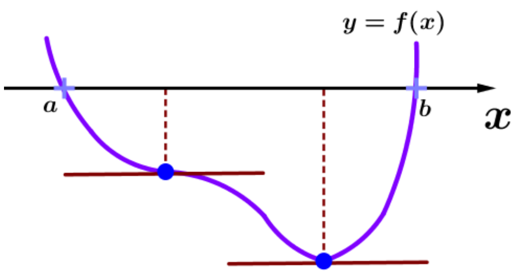
4.7 Applied Maximum and Minimum Problem

Example 4.10 An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. What size should the squares be to obtain a box with the largest volume?

4.8 Rolle's Theorem and Mean-Value Theorem

Theorem 4.7 (Rolle's Theorem) Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = 0$ and $f(b) = 0$, then there is at least one point c in the interval (a, b) such that $f'(c) = 0$.

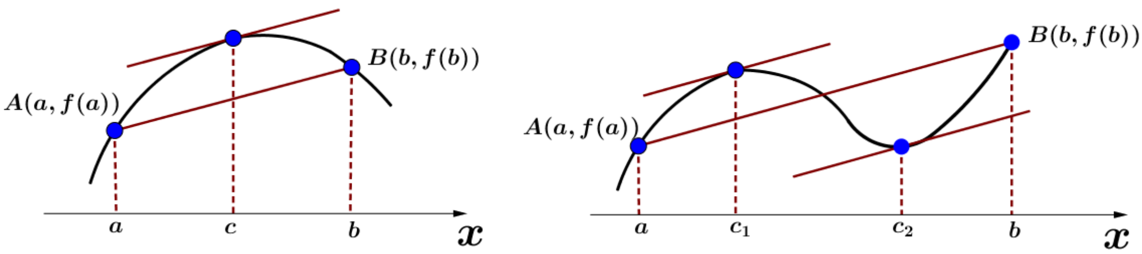




Example 4.11 Find the two x-intercepts of $f(x) = x^2 - 6x + 8$ and confirm that $f'(c) = 0$ at some point c between those intercepts.

Theorem 4.8 (Mean-Value Theorem) Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



The Mean-Value Theorem plays an important role in differential calculus since we use it to prove Theorem 4.1. Remark that Rolle's Theorem is the special case of the Mean-Value Theorem.

Example 4.12 Show that the function $f(x) = \frac{1}{2}x^3 + 2$ satisfies the hypotheses of the Mean-Value Theorem over the interval $[0, 2]$, and find all values of c in $(0, 2)$ at which the tangent line to the graph of f is parallel to the secant line joining the points $(0, f(0))$ and $(2, f(2))$.

Exercise 4

1. Fill in the short answer of the following questions:

   (a) A function f is increasing on (a, b) if ____________ whenever $a < x_1 < x_2 < b$.

   (b) A function f is decreasing on (a, b) if ____________ whenever $a < x_1 < x_2 < b$.

   (c) A function f is concave up on on (a, b) if $f'(x)$ is ____________ whenever $a < x < b$.

   (d) If $f''(a)$ exists and f has an inflection point at $x = a$, then $f''(a)$ ____________ .

2. Let $f(x) = 0.1(x^3 - 3x^2 - 9x)$. Then $f'(x) = 0.3(x + 1)(x - 3)$, $\quad f''(x) = 0.6(x - 1)$.

   (a) Solutions to $f'(x) = 0$ are $x =$ ____________ .

   (b) The function f is increasing (if any) on the interval(s) ____________ and is decreasing (if any) on the interval(s) ____________ .

   (c) The function f is concave down (if any) on the interval(s) ____________ and is concave up (if any) on the interval(s) ____________ .

   (d) ____________ is an inflection point on the graph of f.

3. In each part, sketch the graph of a function f with the stated properties, and discuss the signs of f' and f''.

   (a) The function f is concave up and increasing on the interval $(-\infty, +\infty)$.

   (b) The function f is concave down and increasing on the interval $(-\infty, +\infty)$.

   (c) The function f is concave up and decreasing on the interval $(-\infty, +\infty)$.

   (d) The function f is concave down and decreasing on the interval $(-\infty, +\infty)$.

4. In each part, sketch the graph of a function f with the stated properties.

(a) f is increasing on $(\infty, +\infty)$, has an inflection point at the origin and is concave up on $(0, +\infty)$.

(b) f is increasing on $(\infty, +\infty)$, has an inflection point at the origin and is concave down on $(0, +\infty)$.

(c) f is decreasing on $(\infty, +\infty)$, has an inflection point at the origin and is concave up on $(0, +\infty)$.

(d) f is decreasing on $(\infty, +\infty)$, has an inflection point at the origin and is concave down on $(0, +\infty)$.

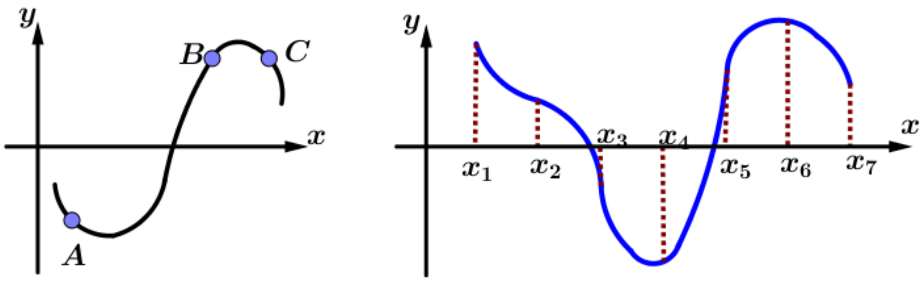


Figure 4.1: Accompanied Figures for problem 5 (left), problem 6 (right).

5. Use the graph of the equation $y = f(x)$ in the accompanying figure 4.1 (left) to find the signs of dy/dx and d^2y/dx^2 at the points A, B, and C.

6. In each part, use the graph of $y = f(x)$ in the accompanying figure to find the requested information.

   (a) Find the intervals on which f is increasing.

   (b) Find the intervals on which f is decreasing.

   (c) Find the open intervals on which f is concave up.

   (d) Find the open intervals on which f is concave down.

   (e) Find all values of x at which f has an inflection point.

   (f) Make a table that shows the signs of f' and f'' over the intervals $(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)$, and $(6, 7)$.

7. For problems (7a)-(7j), find:

   - the intervals on which f is increasing,
   - the intervals on which f is decreasing,

- the open intervals on which f is concave up,
- the open intervals on which f is concave down, and
- the x-coordinates of all inflection points.

(a) $f(x) = x^2 - 3x + 8$

(b) $f(x) = 5 - 4x-^2$

(c) $f(x) = (2x + 1)^3$

(d) $f(x) = (x^{2/3} - 1)^2$

(e) $f(x) = x^{4/3} - x1/3$

(f) $f(x) = \dfrac{x - 2}{(x^2 - x + 1)^2}$

(g) $f(x) = \dfrac{x}{x^2 + 2}$

(h) $f(x) = e^{-x^2/2}$

(i) $f(x) = xe^{x^2}$

(j) $f(x) = x^3 \ln x$

8. In each part, sketch a continuous curve $y = f(x)$ with the stated properties.

   (a) $f(2) = 4, f'(2) = 0, f''(x) < 0$ for all x

   (b) $f(2) = 4, f'(2) = 0, f''(x) > 0$ for $x < 2, f''(x) < 0$ for $x > 2$

   (c) $f(2) = 4, f''(x) > 0$ for $x \neq 2$ and $\lim_{x\to 2+} f'(x) = -\infty, \lim_{x\to 2-} f'(x) = +\infty$

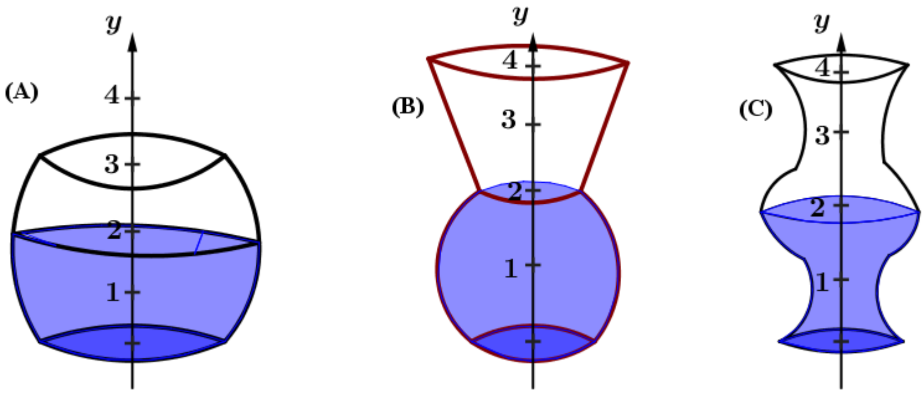


Figure 4.2: Accompanied Figures for problem 9.

9. Consider Fig. 4.2: Suppose that water is flowing at a constant rate into the container (A). Make a rough sketch of the graph of the water level y versus the time t of the container (A). Make sure that your sketch conveys where the each graph is concave up and concave down, and label the y-coordinates of the inflection points.

   Repeat the questions for containers (B) and (C) in Fig. 4.2.

10. In each part, sketch the graph of a continuous function f with the stated properties.

    (a) f is concave up on the interval $(-\infty, +\infty)$ and has exactly one relative extremum.

    (b) f is concave up on the interval $(-\infty, +\infty)$ and has no relative extrema.

(c) The function f has exactly two relative extrema on the interval $(-\infty, +\infty)$, and $f(x) \to +\infty$ as $x \to +\infty$.

(d) The function f has exactly two relative extrema on the interval $(-\infty, +\infty)$, and $f(x) \to -\infty$ as $x \to +\infty$.

11. Use both the first and second derivative tests to show that

    (a) $f(x) = 3x^2 6x + 1$ has a relative minimum at $x_0 = 1$

    (b) $f(x) = x^3 3x + 3$ has a relative minimum at $x = 1$ and a relative maximum at $x_0 = -1$.

    (c) $f(x) = \sin^2 x$ has a relative minimum at $x = 0$.

    (d) $f(x) = \tan^2 x$ has a relative minimum at $x = 0$.

12. Locate the critical points and identify which critical points are stationary points of the following function.

    (a) $f(x) = 4x^4 - 16x^2 - 17$

    (b) $f(x) = 3x^4 + 12x$

    (c) $f(x) = \dfrac{x+1}{x^2+3}$

    (d) $f(x) = x^2(x-1)^{2/3}$

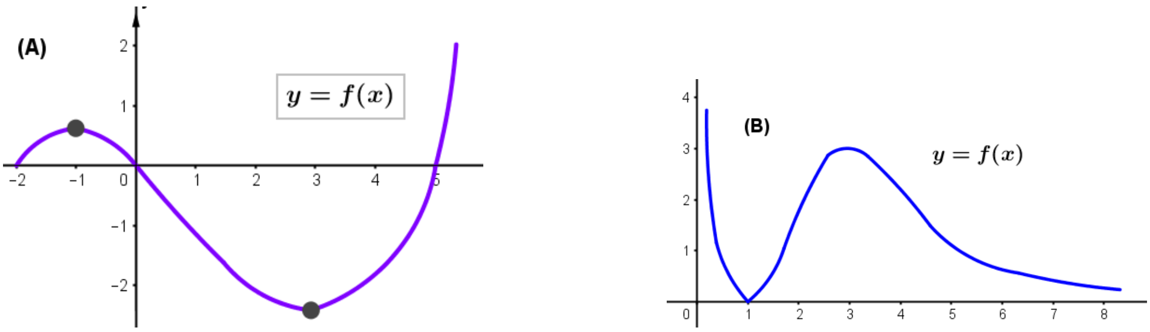


Figure 4.3: Accompanied Figures for problem 13.

13. Figure 4.3 (A): The graph of a function $f(x)$ is given in figure 4.3. Sketch graphs of $y = f'(x)$ and $y = f''(x)$. Repeat the same for Figure 4.3 (B).

14. Use the given derivative to find all critical points of f and at each critical point determine whether a relative maximum, relative minimum, or neither occurs. Assume in each case that f is continuous everywhere.

    (a) $f'(x) = x^2(x^3 - 5)$

    (b) $f'(x) = 4x^3 - 9x$

    (c) $f'(x) = \dfrac{2-3x}{\sqrt[3]{x+2}}$

    (d) $f'(x) = x^4(e^x - 3)$

15. Find the relative extrema using both first and second derivative tests.

(a) $f(x) = 1 + 8x - 3x^2$

(b) $f(x) = \sin 2x, \quad 0 < x < \pi$

(c) $f(x) = x^4 - 12x^3$

(d) $f(x) = e^x(x-3)$

16. Use any method to find the relative extrema of the function f.

(a) $f(x) = x^4 - 4x^3 + 4x^2$

(b) $f(x) = x(x-4)^3$

(c) $f(x) = \dfrac{x+3}{x-2}$

(d) $f(x) = 2x + 3x^{1/3}$

17. Let $f(x) = \dfrac{3(x+1)(x-3)}{(x+2)(x-4)}$. Given that $f'(x) = \dfrac{-30(x-1)}{(x+2)^2(x-4)^2}$, $f''(x) = \dfrac{90(x^2-2x+4)}{(x+2)^3(x-4)^3}$, determine the following properties of the graph of f.

(a) The x- and y-intercepts are ________________ .

(b) The vertical asymptotes are ________________ .

(c) The horizontal asymptote is ________________ .

(d) The graph is above the x-axis on the intervals ________________ .

(e) The graph is increasing on the intervals ________________ .

(f) The graph is concave up on the intervals ________________.

(g) The relative maximum point on the graph is ________________.

18. Give a graph of the rational function and label the coordinates of the stationary points and inflection points. Show the horizontal and vertical asymptotes and label them with their equations. Label point(s), if any, where the graph crosses a horizontal asymptote. Check your work with a graphing utility.

(a) $y = \dfrac{2x-6}{4-x}$

(b) $y = \dfrac{8}{x^2-9}$

(c) $y = \dfrac{x^2}{x^2-4}$

(d) $y = \dfrac{3(x+1)^2}{(x-1)^2}$

19. A rectangular plot of land is to be fenced off so that the area enclosed will be 400 ft^2. Let L be the length of fencing needed and x the length of one side of the rectangle. Show that $L = 2x + 800/x$ for x >0, and sketch the graph of L versus x for $x > 0$.

20. A box with a square base and open top is to be made from sheet metal so that its volume is 500 in^3. Let S be the area of the surface of the box and x the length of a side of the square base. Show that $S = x^2 + 2000/x$ for $x > 0$, and sketch the graph of S versus x for $x > 0$.

21. In each part, sketch the graph of a continuous function f with the stated properties on the interval $[0, 10]$.

    (a) f has an absolute minimum at $x = 0$ and an absolute maximum at $x = 10$.

    (b) f has an absolute minimum at $x = 2$ and an absolute maximum at $x = 7$.

    (c) f has relative minima at $x = 1$ and $x = 8$, has relative maxima at $x = 3$ and $x = 7$, has an absolute minimum at $x = 5$, and has an absolute maximum at $x = 10$.

22. Find the absolute maximum and minimum values of f on the given closed interval, and state where those values occur.

    (a) $f(x) = 4x^2 - 12x + 10; \quad [1, 2]$

    (b) $f(x) = 8x - x^2; \quad [0, 6]$

    (c) $f(x) = (x - 2)^3; \quad [1, 4]$

    (d) $f(x) = \dfrac{3x}{\sqrt{4x^2 + 1}}; \quad [-1, 1]$

    (e) $f(x) = x^2 - x - 2; \quad (-\infty, +\infty)$

    (f) $f(x) = 3 - 4x - 2x^2; \quad (-\infty, +\infty)$

    (g) $f(x) = \dfrac{x^2 + 1}{x + 1}; \quad (-5, -1)$

    (h) $f(x) = \dfrac{x - 2}{x + 1}; \quad (-1, 5]$

23. Determine whether the statement is true or false. Explain your answer.

    (a) If a function f is continuous on $[a, b]$, then f has an absolute maximum on $[a, b]$.

    (b) If a function f is continuous on (a, b), then f has an absolute minimum on (a, b).

    (c) If a function f has an absolute minimum on (a, b), then there is a critical point of f in (a, b).

    (d) If a function f is continuous on $[a, b]$ and f has no relative extreme values in (a, b), then the absolute maximum value of f exists and occurs either at $x = a$ or at $x = b$.

24. A positive number x and its reciprocal are added together. The smallest possible value of this sum is obtained by minimizing $f(x) =$ ________________ for x in the interval ________________.

25. An open box is to be made from a 20-inch by 32-inch piece of cardboard by cutting out x-inch by x-inch squares from the four corners and bending up the sides. The largest possible volume of the box is obtained by maximizing $V(x) =$ ________________ for x in the interval ________________.

26. Find a number in the closed interval $[\frac{1}{2}, \frac{3}{2}]$ such that the sum of the number and its reciprocal is (a) as small as possible, (b) as large as possible.

27. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10.

28. A rectangular area of 3200 ft^2 is to be fenced off. Two opposite sides will use fencing costing \$1 per foot and the remaining sides will use fencing costing \$2 per foot. Find the dimensions of the rectangle of least cost.

29. Show that among all rectangles with area A, the square has the minimum perimeter.

30. A rectangular page is to contain 42 square inches of printable area. The margins at the top and bottom of the page are each 1 inch, one side margin is 1 inch, and the other side margin is 2 inches. What should the dimensions of the page be so that the least amount of paper is used?

31. A closed rectangular container with a square base is to have a volume of 2000 cm^3. It costs twice as much per square centimeter for the top and bottom as it does for the sides. Find the dimensions of the container of least cost.

32. A closed cylindrical can is to have a surface area of S square units. Show that the can of maximum volume is achieved when the height is equal to the diameter of the base.

33. Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6 inches and height 10 inches.

34. A liquid form of antibiotic manufactured by a pharmaceutical firm is sold in bulk at a price of \$200 per unit. If the total production cost (in dollars) for x units is

$$C(x) = 500{,}000 + 80x + 0.003x^2$$

and if the production capacity of the firm is at most 30,000 units in a specified time, how many units of antibiotic must be manufactured and sold in that time to maximize the profit?